Wavelets

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Abstract. In this thesis, we discuss some problems in wavelet analysis. More precisely, the content of this paper is the following:

I. On dual wavelet tight frames

II. Some applications of projection operators in wavelets

III. A sufficient and necessary condition on Γ_0 for $T(\Gamma_0, M)$ to be a self-affine tiling

IV. Miscellaneous results on shift-invariant subspaces of $L^2(R^n)$.

In Part I, we present a complete description of dual wavelet tight frames and by using these results, we construct dual wavelet bases and dual wavelet tight frames in $L^2(R^n)$. In Part II, we obtain a criterion for dual wavelet bases which can be generated by an MRA. In Part III, we find a sufficient and necessary condition for $T(\Gamma_0, M)$ to be a self-affine tiling (a kind of special wavelets) which is convenient to apply. In the last Part IV, we show some results on shift-invariant space.

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Part 0: Introduction and Definitions

At first, we shall review some concepts and notations. Throughout this paper, we shall always let Γ be a lattice subgroup of $R^n(\Gamma := E\mathbb{Z}^n, E$ is an $n \times n$ nonsigular matrix) and M, an $n \times n$ real matrix, be an acceptable dilation for $\Gamma(M\Gamma \subset \Gamma$, and all the eigenvalues, λ_i, of M satisfy $|\lambda_i| > 1)$ with $m = |\det M|$. Then we let $B := M^{*-1}, \Gamma^* := 2\pi E^{*-1}\mathbb{Z}^n$ called the dual lattice of $\Gamma, S := 2\pi E^{*-1}[0,1)^n$ called the fundamental block of Γ^*. $|S|$ denotes the Lebesgue measure of S. For any $f, g \in L^2(R)$, we shall use the following notations:

$$\begin{aligned}
\widehat{f}(\xi) &:= \frac{1}{(2\pi)^{n/2}} \int_{R^n} f(t)e^{-i\xi t}dt, \\
f_{j,\gamma}(x) &:= m^{j/2} f(M^j x - \gamma), \quad j \in \mathbb{Z}, \gamma \in \Gamma, \\
\tau_\gamma f(x) &:= f(x-\gamma), \quad \gamma \in \Gamma, \\
f_{||x} &:= \{f(x+\gamma^*)\}_{\gamma^* \in \Gamma^*} \in l^2(\Gamma^*), \\
[f,g](\xi) &:= \sum_{\gamma^* \in \Gamma^*} f(\xi+\gamma^*)\overline{g(\xi+\gamma^*)}.
\end{aligned}$$

And notice that we have

$$\begin{aligned}
\langle f, g\rangle &= \langle \widehat{f}, \widehat{g}\rangle, \quad \forall f, g \in L^2(R^n), \\
f(x) &:= \frac{1}{(2\pi)^{n/2}} \int_{R^n} \widehat{f}(t)e^{ixt}dt, \quad \forall f \in L^2(R^n).
\end{aligned}$$

A sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(R^n)$ forms a multiresolution analysis (MRA) if

(i) $V_j \subset V_{j+1} \quad \forall j \in \mathbb{Z}$,

(ii) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$ and $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(R^n)$,

(iii) $f(x) \in V_j$ iff $f(Mx) \in V_{j+1} \quad j \in \mathbb{Z}$

(iv) there exists an element $\phi \in V_0$ such that $\{\phi(x-\gamma) : \gamma \in \Gamma\}$ is an orthonormal basis of V_0 (ϕ is called a scaling function).

Part I: On Dual Wavelet Tight Frames

§1. Introduction

Dual wavelet tight frames play a very important role in both the theory and application of wavelet analysis. How to consruct them, therefore, becomes a key and interesting problem (since dual wavelet bases are the special cases of dual wavelet tight frames). For simplicity, in this paper we call dual wavelet tight frames by dual tight frames. Although some results on dual tight frames appeared elsewhere by other authors in various forms, here we present a more complete, general and strict approach of this problem. To speak in detail, we first use two classes of equalities (these equalities appeared in Lemarié [6] and were known to Y. Meyer and A. Bonami, F. Soria & G. Weiss [2], C. K. Chui & X. Shi [3]) to characterize dual tight frames. Worthy of a mention is that we obtain a necessary and sufficient condition to characterize dual tight frames with the Fourier transforms of their wavelet functions having finite supports. Then by generating Lawton's results on tight frames (see Lawton [5]) to several dimensional cases and the method in G. V. Wellend & M. Lundberg [7] to construct wavelet bases with compact support to dual wavelet tight frames, we construct dual wavelet tight frames. Futhermore we can use one function to generate a tight frame in $L^2(R^n)$ and construct some new wavelet bases in $H^2(R)$.

In this paper we always let F denote a measurable subset of R^n such that $\chi_F(x) = \chi_{BF}(x)$, a.e. $x \in R^n$ and define $L^2_{BC}(F) := \{f \in L^2(R^n) : \widehat{f} \in L^2(F) \cap L^\infty(F)$ such that $\widehat{f}$ vanishes in a neighborhood of the origin and supp$\widehat{f}$ is included in a ball with finite radius.$\}$. We say that $f \in L^2(R^n)$ is (strictly) admissible if $\sum\limits_{j\in\mathbb{Z}} |\widehat{f}(B^j\xi)|^2 \in L^2_{\text{loc}}(R^n)(\in L^\infty(R^n))$.

We call that $\{\psi_i\}_{i=1}^d$ with dual functions $\{\widetilde{\psi}_i\}_{i=1}^d$ generates a dual tight frame in $\widehat{L^2(F)}$ ($\widehat{L^2(F)} := \{f \in L^2(R^n) : \widehat{f} \in L^2(F)\}$, and without further mention, we always assume $\psi_i, \widetilde{\psi}_i \in \widehat{L^2(F)}$) with the tight frame bound $C_0 \neq 0$, if $\forall \widehat{f} \in L^2(F)$,

$$C_0 f = \sum_{i=1}^{d}\sum_{j\in\mathbb{Z}}\sum_{\gamma\in\Gamma}\langle f, \psi_{i;j,\gamma}\rangle\widetilde{\psi}_{i;j,\gamma}, \quad \overline{C_0} f = \sum_{i=1}^{d}\sum_{j\in\mathbb{Z}}\sum_{\gamma\in\Gamma}\langle f, \widetilde{\psi}_{i;j,\gamma}\rangle\psi_{i;j,\gamma} \tag{1.1}$$

with the series converging in L^2, and there exists a constant $C > 0$ such that $\forall \widehat{f} \in L^2(F)$

$$C^{-1}\|f\|^2_{L^2} \le \sum_{i=1}^{d}\sum_{j\in\mathbb{Z}}\sum_{\gamma\in\Gamma}|\langle f, \psi_{i;j,\gamma}\rangle|^2 \le C\|f\|^2_{L^2}, \tag{1.2}$$

$$(1.3)\qquad C^{-1}\|f\|_{L^2}^2 \le \sum_{i=1}^{d}\sum_{j\in\mathbb{Z}}\sum_{\gamma\in\Gamma} |\langle f, \widetilde{\psi}_{i;j,\gamma}\rangle|^2 \le C\|f\|_{L^2}^2.$$

If in addition $C_0 = 1$ and there exists $C_1 > 0$ such that $\forall\{C_{i;j,\gamma}\} \in l^2(\{1 \le i \le d\} \times \mathbb{Z} \times \Gamma)$

$$(1.4)\qquad C_1^{-1}\|\{C_{i;j,\gamma}\}\|_{l^2} \le \|\sum_{i=1}^{d}\sum_{j\in\mathbb{Z}}\sum_{\gamma\in\Gamma} C_{i;j,\gamma}\psi_{i;j,\gamma}\|_{L^2} \le C_1\|\{C_{i;j,\gamma}\}\|_{l^2}$$

$$(1.5)\qquad C_1^{-1}\|\{C_{i;j,\gamma}\}\|_{l^2} \le \|\sum_{i=1}^{d}\sum_{j\in\mathbb{Z}}\sum_{\gamma\in\Gamma} C_{i;j,\gamma}\widetilde{\psi}_{i;j,\gamma}\|_{L^2} \le C_1\|\{C_{i;j,\gamma}\}\|_{l^2}$$

and $\langle \psi_{i;j,\gamma}, \widetilde{\psi}_{i1;j1,\gamma 1}\rangle = \delta_{i,i1}\delta_{j,j1}\delta_{\gamma,\gamma 1}, \quad 1 \le i, i1 \le d, \quad j, j1 \in \mathbb{Z}, \quad \gamma, \gamma 1 \in \Gamma$,we will say that $\{\psi_i\}_{i=1}^d$ with dual functions $\{\widetilde{\psi}_i\}_{i=1}^d$ generates a dual wavelet basis in $\widehat{L^2(F)}$. If $\{\psi_i\}_{i=1}^d$ satisfies only (1.4), we shall call $\{\psi_i\}_{i=1}^d$ generates a Riesz basis.

§2. Characterization of Dual Tight Frames

In this section let us first review and state some basic results.
Proposition 2.1 (C.K. Chui and X. Shi [3]). If $\psi \in L^2(R^n)$ satisfies

$$(2.1)\qquad \forall f \in L^2(R^n), \quad \sum_{j\in\mathbb{Z}}\sum_{\gamma\in\Gamma} |\langle f, \psi_{j,\gamma}\rangle|^2 \le C\|f\|_{L^2}^2$$

here $C > 0$ is a constant (we call that such ψ has an upper frame bound), then ψ is strictly admissible, i.e. $\sum_{j\in\mathbb{Z}} |\widehat{\psi}(B^j\xi)|^2 \in L^\infty$.
Proposition 2.2. $\{\psi_i\}_{i=1}^d$ with dual functions $\{\widetilde{\psi}_i\}_{i=1}^d$ generates a dual tight frame in $\widehat{L^2(F)}$ with the tight frame bound $C_0 \ne 0$ if and only if the following hold

$$(2.2)\qquad \forall \widehat{f} \in L^2(F), \quad \sum_{i=1}^{d}\sum_{j\in\mathbb{Z}}\sum_{\gamma\in\Gamma} (|\langle f, \psi_{i;j,\gamma}\rangle|^2 + |\langle f, \widetilde{\psi}_{i;j,\gamma}\rangle|^2) \le C\|f\|_{L^2}^2$$

$$(2.3)\qquad \forall \widehat{f}, \widehat{g} \in L^2(F), \quad C_0\langle f, g\rangle = \sum_{i=1}^{d}\sum_{j\in\mathbb{Z}}\sum_{\gamma\in\Gamma} \langle f, \psi_{i;j,\gamma}\rangle\langle \widetilde{\psi}_{i;j,\gamma}, g\rangle.$$

where $C > 0$ and $C_0 \ne 0$ are constants.
Proof. The necessity is obvious. Now we prove the sufficiency. By (2.2) and the

Cauchy-Schwartz inequality, the condition (2.3) implies (1.2) and (1.3). To complete the proof, it suffices to prove (1.1). $\forall N \in I\!N, f \in \widehat{L^2(F)}$,

$$\begin{aligned}
&\left\| C_0 f - \sum_{i=1}^{d} \sum_{|j|\le N} \sum_{\|\gamma\|\le N} \langle f, \psi_{i;j,\gamma}\rangle \widetilde{\psi}_{i;j,\gamma} \right\|_{L^2} \\
&= \sup_{\|g\|_{L^2}\le 1} \left| C_0\langle f, g\rangle - \sum_{i=1}^{d} \sum_{|j|\le N} \sum_{\|\gamma\|\le N} \langle f, \psi_{i;j,\gamma}\rangle \langle \widetilde{\psi}_{i;j,\gamma}, g\rangle \right| \\
&= \sup_{\|g\|_{L^2}\le 1} \left| \sum_{i=1}^{d} \sum_{\max(|j|,\|\gamma\|)>N} \langle f, \psi_{i;j,\gamma}\rangle \langle \widetilde{\psi}_{i;j,\gamma}, g\rangle \right| \\
&\le C \left(\sum_{i=1}^{d} \sum_{\max(|j|,\|\gamma\|)>N} |\langle f, \psi_{i;j,\gamma}\rangle| \right)^{1/2}
\end{aligned}$$

which converges to 0 as N converges to infinity. Thus the proof is finished. □

In the rest of this section, we shall present our main results of this paper. Using the following Theorems, we characterize dual tight frames by two classes of equalities.

Theorem 2.3. If $\{\psi_i\}_{i=1}^d$ with dual functions $\{\widetilde{\psi}_i\}_{i=1}^d$ satisfies the following conditions

$$\psi_i, \widetilde{\psi}_i, 1 \le i \le d \quad \text{are admissible}$$

$$\forall f \in L^2_{BC}(F), \quad \sum_{i=1}^{d} \sum_{j\in Z\!\!Z} \sum_{\gamma\in\Gamma} (|\langle f, \psi_{i;j,\gamma}\rangle|^2 + |\langle f, \widetilde{\psi}_{i;j,\gamma}\rangle|^2) < \infty,$$

$$\forall f, g \in L^2_{BC}(F), \quad C_0\langle f, g\rangle = \sum_{i=1}^{d} \sum_{j\in Z\!\!Z} \sum_{\gamma\in\Gamma} \langle f, \psi_{i;j,\gamma}\rangle \langle \widetilde{\psi}_{i;j,\gamma}, g\rangle \tag{2.4}$$

where $C_0 \ne 0$ is a constant. Then the following equalities hold

$$\sum_{i=1}^{d} \sum_{j\in Z\!\!Z} \overline{\widehat{\psi}_i}(B^j\xi) \widehat{\widetilde{\psi}_i}(B^j\xi) = C_0 |S|^{-1} \chi_F(\xi), \tag{2.5}$$

$$\forall \gamma_0^* \in \Gamma^* \backslash M^*\Gamma^*, \quad \sum_{i=1}^{d} \sum_{j=0}^{\infty} \overline{\widehat{\psi}_i}(M^{*j}\xi) \widehat{\widetilde{\psi}_i}(M^{*j}(\xi + \gamma_0^*)) = 0. \tag{2.6}$$

Proof. From the assumptions, by application of the Parseval equality and the polarization identity, we get

$$\forall f, g \in L^2_{BC}(F) \tag{2.7}$$

$$\sum_{i=1}^{d}\sum_{j\in\mathbb{Z}}\sum_{\gamma\in\Gamma}\langle f,\psi_{i;j,\gamma}\rangle\langle\widetilde{\psi}_{i;j,\gamma},g\rangle=\sum_{i=1}^{d}\sum_{j\in\mathbb{Z}}m^j|S|\int_S[\widehat{f}(M^{*j}\xi),\widehat{\psi}_i(\xi)][\widehat{\widetilde{\psi}}_i(\xi),\widehat{g}(M^{*j}\xi)]d\xi.$$

To simplify our argument, without loss of generality we assume $F=R^n$. For any fixed $\omega\in R^n\backslash 0$, there exists $\varepsilon_0>0$ such that $0\notin B_{2\varepsilon_0}(\omega)$ and $\forall i,j\in\mathbb{Z},i\neq j,|B^iB_{\varepsilon_0}(\omega)\cap B^jB_{\varepsilon_0}(\omega)|=0$ (here $B_{\varepsilon_0}(\omega)$ denotes a ball centered at ω with radius ε_0). Also for any $\varepsilon>0$, there exists j_0 such that $\forall\gamma^*\in\Gamma^*\backslash 0,j\geq j_0$, $|(B^jB_\varepsilon(\omega)+\gamma^*)\cap B^jB_\varepsilon(\omega)|=0$. Thus we let j_ε denote the mimium number of such j_0. Now we define

$$R(f,g,j_\varepsilon):=\sum_{i=1}^{d}\sum_{j\geq j_\varepsilon}m^j|S|\int_S[\widehat{f}(M^{*j}\xi),\widehat{\psi}_i(\xi)][\widehat{\widetilde{\psi}}_i(\xi),\widehat{g}(M^{*j}\xi)]d\xi,$$
$$L(f,g,j_\varepsilon):=\sum_{i=1}^{d}\sum_{j< j_\varepsilon}m^j|S|\int_S[\widehat{f}(M^{*j}\xi),\widehat{\psi}_i(\xi)][\widehat{\widetilde{\psi}}_i(\xi),\widehat{g}(M^{*j}\xi)]d\xi.$$

To prove (2.5), For any $0<\varepsilon<\varepsilon_0$, we select $\widehat{f}(\xi)=\widehat{g}(\xi)=\frac{1}{|B_\varepsilon(\omega)|^{1/2}}\chi_{B_\varepsilon(\omega)}(\xi)$. We now compute and estimate $R(f,g,j_\varepsilon)$ and $L(f,g,j_\varepsilon)$.

$$\begin{aligned}R(f,g,j_\varepsilon)&=\sum_{i=1}^{d}\sum_{j\geq j_\varepsilon}\frac{|S|}{|B_\varepsilon(\omega)|}\int_{R^n}\chi_{B_\varepsilon(\omega)}(\xi)\overline{\widehat{\psi}_i}(B^j\xi)\widehat{\widetilde{\psi}}_i(B^j\xi)d\xi\\&=\sum_{i=1}^{d}\sum_{j\geq j_\varepsilon}\frac{|S|}{|B_\varepsilon(\omega)|}\int_{B_\varepsilon(\omega)}\overline{\widehat{\psi}_i}(B^j\xi)\widehat{\widetilde{\psi}}_i(B^j\xi)d\xi.\end{aligned}$$

Since $\psi_i,\widetilde{\psi}_i$ are admissible, thus $\sum_{i=1}^{d}\sum_{j\in\mathbb{Z}}|\widehat{\psi}_i(B^j\xi)\widehat{\widetilde{\psi}}_i(B^j\xi)|\in L^2_{loc}$. By the Lebesgue's dominated convergence theorem, we have

$$R(f,g,j_\varepsilon)=\frac{|S|}{|B_\varepsilon(\omega)|}\int_{B_\varepsilon(\omega)}\sum_{i=1}^{d}\sum_{j\geq j_\varepsilon}\overline{\widehat{\psi}_i}(B^j\xi)\widehat{\widetilde{\psi}}_i(B^j\xi)d\xi. \tag{2.8}$$

On the other hand

$$\begin{aligned}|L(f,g,j_\varepsilon)|\leq\ &\sum_{i=1}^{d}\sum_{j<j_\varepsilon}\frac{m^j|S|}{|B_\varepsilon(\omega)|}\int_S[\chi_{B_\varepsilon(\omega)}(M^{*j}\xi),|\widehat{\psi}_i(\xi)|^2]^{1/2}\\&[\chi_{B_\varepsilon(\omega)}(M^{*j}\xi),|\widehat{\widetilde{\psi}}_i(\xi)|^2]^{1/2}\left[\sum_{\gamma^*\in\Gamma^*}\chi_{B_\varepsilon(\omega)}(M^{*j}(\xi+\gamma^*))\right]d\xi,\end{aligned}$$

Noting that $|B_\varepsilon(\omega)|=\lim_{j\to-\infty}m^j|S|\cdot\#\{\gamma^*\in\Gamma^*:|B_\varepsilon(\omega)\cap M^{*j}(S+\gamma^*)|\neq 0\}$. Thus

$$\begin{aligned}&\forall j\leq 0,\ \sum_{\gamma^*\in\Gamma^*}\chi_{B_\varepsilon(\omega)}(M^{*j}(\xi+\gamma^*))\\&\leq\#\{\gamma^*\in\Gamma^*:|B_\varepsilon(\omega)\cap M^{*j}(S+\gamma^*)|\neq 0\}\leq C_1\tfrac{|B_\varepsilon(\omega)|}{m^j|S|}\end{aligned} \tag{2.9}$$

here C_1 is a constant. Let $F_\varepsilon := B^{j_\varepsilon}\bigcup_{j<0} B^j B_\varepsilon(\omega)$, then

$$\begin{aligned}
|L(f,g,j_\varepsilon)| \leq\ & C_1 \sum_{i=1}^{d} \left(\sum_{j<j_\varepsilon} \int_{B^j B_\varepsilon(\omega)} |\widehat{\psi}_i(\xi)|^2 d\xi \right)^{1/2} \left(\sum_{j<j_\varepsilon} \int_{B^j B_\varepsilon(\omega)} |\widehat{\widetilde{\psi}}_i(\xi)|^2 d\xi \right)^{1/2} \\
=\ & C_1 \sum_{i=1}^{d} \left(\int_{F_\varepsilon} |\widehat{\psi}_i(\xi)|^2 d\xi \right)^{1/2} \left(\int_{F_\varepsilon} |\widehat{\widetilde{\psi}}_i(\xi)|^2 d\xi \right)^{1/2}
\end{aligned}$$

Note that $\inf\limits_{\xi \in \bigcup_{j<0} B^j B_{\varepsilon_0}(\omega)} \|\xi\| > 0$ and j_ε converges to negative infinity, as $\varepsilon \to 0$, we know $\inf\limits_{\xi \in F_\varepsilon} \|\xi\| \to +\infty$, as $\varepsilon \to 0$. Thus $\lim\limits_{\varepsilon\to 0} \int_{F_\varepsilon} (|\widehat{\psi}_i(\xi)|^2 + |\widehat{\widetilde{\psi}}_i(\xi)|^2) d\xi = 0$, which means that $\lim\limits_{\varepsilon\to 0} |L(f,g,j_\varepsilon)| = 0$. Since $C_0 = R(f,g,j_\varepsilon) + L(f,g,j_\varepsilon)$, by using Lebesgue's differentiability Theorem, we get $C_0 = \lim\limits_{\varepsilon\to 0} R(f,g,j_\varepsilon) + \lim\limits_{\varepsilon\to 0} L(f,g,j_\varepsilon) = \lim\limits_{\varepsilon\to 0} R(f,g,j_\varepsilon) = |S| \sum_{i=1}^{d} \sum_{j\in \mathbb{Z}} \overline{\widehat{\psi}_i}(B^j\xi) \widehat{\widetilde{\psi}}_i(B^j\xi)$.

To prove (2.6), select $\omega \in R^n \backslash \Gamma^*$ fixed. Let $\varepsilon_1 = \frac{1}{3} \min\limits_{\gamma^* \in \Gamma^* \backslash 0} \|\gamma^*\|$, then

$$\forall \gamma^* \in \Gamma^* \backslash 0, \quad |B_{\varepsilon_1}(\omega) \cap (B_{\varepsilon_1}(\omega) + \gamma^*)| = 0. \tag{3.0}$$

Let $\gamma_0^* \in \Gamma^* \backslash M^*\Gamma^*$ be fixed, then there exists $\varepsilon_2 > 0$ such that $0 \notin B_{2\varepsilon_2}(\omega) \cup B_{2\varepsilon_2}(\omega + \gamma_0^*)$ and $\forall i,j \in \mathbb{Z}, i \neq j, |B^i B_{\varepsilon_2}(\omega) \cap B^j B_{\varepsilon_2}(\omega)| = 0, |B^i B_{\varepsilon_2}(\omega+\gamma_0^*) \cap B^j B_{\varepsilon_2}(\omega + \gamma_0^*)| = 0$. As in the proof of (2.5), we let j_ε denote the minium integer such that $\forall j \geq j_\varepsilon, \gamma^* \in \Gamma^* \backslash 0, |(B^j B_\varepsilon(\omega) + \gamma^*) \cap B^j B_\varepsilon(\omega)| = 0$. Noticing that $j_\varepsilon \to -\infty$, as $\varepsilon \to 0$, we know that there exists $\varepsilon_0 > 0$ such that $\varepsilon_0 < \min(\varepsilon_1, \varepsilon_2)$ and $j_{\varepsilon_0} < 0$.

Now we prove (2.6). For any $0 < \varepsilon < \varepsilon_0$, we select $\widehat{f}(\xi) = \frac{1}{|B_\varepsilon(\omega)|^{1/2}} \chi_{B_\varepsilon(\omega)}(\xi), \widehat{g}(\xi) = \frac{1}{|B_\varepsilon(\omega)|^{1/2}} \chi_{B_\varepsilon(\omega+\gamma_0^*)}(\xi)$ and calculate $R(f,g,j_\varepsilon)$ and $L(f,g,j_\varepsilon)$.

By (3.0), we have

$$\forall j > 0, \quad [\chi_{B_\varepsilon(\omega)}(M^{*j}\xi), \widehat{\psi}_i(\xi)] \cdot [\chi_{B_\varepsilon(\omega+\gamma_0^*)}(M^{*j}\xi), \widehat{\widetilde{\psi}}_i(\xi)] = 0.$$

So

$$R(f,g,j_\varepsilon) = \sum_{i=1}^{d} \sum_{j_\varepsilon \leq j \leq 0} \frac{m^j |S|}{|B_\varepsilon(\omega)|} \int_S [\chi_{B_\varepsilon(\omega)}(M^{*j}\xi), \widehat{\psi}_i(\xi)][\widehat{\widetilde{\psi}}_i(\xi), \chi_{B_\varepsilon(\omega+\gamma_0^*)}(M^{*j}\xi)] d\xi,$$

Note that $\forall j \leq 0, [\chi_{B_\varepsilon(\omega+\gamma_0^*)}(M^{*j}\xi), \widehat{\widetilde{\psi}}_i(\xi)](\xi) = [\chi_{B\varepsilon(\omega)}(M^{*j}\xi), \widehat{\widetilde{\psi}}_i(\xi + B^j\gamma_0^*)](\xi)$ and (3.0) holds

$$\begin{aligned}
R(f,g,j_\varepsilon) =\ & \sum_{i=1}^{d} \sum_{j_\varepsilon \leq j \leq 0} \frac{m^j |S|}{|B_\varepsilon(\omega)|} \int_S [\overline{\widehat{\psi}_i}(\xi) \widehat{\widetilde{\psi}}_i(\xi + B^j\gamma_0^*), \chi_{B_\varepsilon(\omega)}(M^{*j}\xi)] d\xi \\
=\ & \sum_{i=1}^{d} \sum_{j_\varepsilon \leq j \leq 0} \frac{|S|}{|B_\varepsilon(\omega)|} \int_{B_\varepsilon(\omega)} \overline{\widehat{\psi}_i}(B^j\xi) \widehat{\widetilde{\psi}}_i(B^j(\xi + \gamma_0^*)) d\xi.
\end{aligned}$$

Since $\psi_i, \widetilde{\psi}_i$ are admissible, using Lebesgue's dominated convergence theorem and Lebesgue's differentiability theorem, noticing that $j_\varepsilon \to -\infty$, as $\varepsilon \to 0$, we have

$$\begin{aligned}\lim_{\varepsilon\to 0} R(f,g,j_\varepsilon) &= \lim_{\varepsilon\to 0}\frac{|S|}{|B_\varepsilon(\omega)|}\int_{B_\varepsilon(\omega)}\sum_{i=1}^{d}\sum_{j_\varepsilon\le j\le 0}\overline{\widehat{\psi}_i}(B^j\xi)\widehat{\widetilde{\psi}}_i(B^j(\xi+\gamma_0^*))d\xi\\ &= |S|\sum_{i=1}^{d}\sum_{j=0}^{\infty}\overline{\widehat{\psi}_i}(M^{*j}\xi)\widehat{\widetilde{\psi}}_i(M^{*j}(\xi+\gamma_0^*))\end{aligned}$$

Using the same method in proving (2.5), we also can prove that $\lim_{\varepsilon\to 0} L(f,g,j_\varepsilon) = 0$. By the assumption $R(f,g,j_\varepsilon) + L(f,g,j_\varepsilon) = 0$, we obtain (2.6). □

Remark. Given $\psi \in L^2(R^n)$, if there exist $\varepsilon > 0, \delta > 0, C > 0$ such that $|\widehat{\psi}(\xi)| \le C\|\xi\|^\varepsilon$, whenever $\|\xi\| < \delta$, then ψ is admissible. If in addition there exist $\varepsilon_1 > 0, C_1 > 0$ such that $|\widehat{\psi}(\xi)| \le C_1\|\xi\|^{-\varepsilon_1}$, then ψ is strict admissible. In the general case, when F is a measurable subset of R^n such that $F = BF$, in proving (2.5), we can select $\omega \in F\backslash 0$ with $\liminf_{\varepsilon\to 0}\frac{|F\cap B_\varepsilon(\omega)|}{|B_\varepsilon(\omega)|} = 1$ (noticing that for a.e. $\omega \in F$, this holds) and substitute $D_\varepsilon(\omega) := F \cap B_\varepsilon(\omega)$ for $B_\varepsilon(\omega)$. In proving (2.6), we still choose f, g as above and (2.4) still holds for such f and g.

Under some mild conditions, we prove the converse theorem of Theorem 2.3.

We say that $\psi \in L^2(R^n)$ satisfies condition (I) if there exist positive constants δ_1, δ_2 such that $\|\xi\|^{\delta_1}\widehat{\psi}(\xi) \in L^\infty$ and $\widehat{\psi} \in L^{2-\delta_2}(R^n)$.

Theorem 2.4. If $\psi_i, \widetilde{\psi}_i \in L^2, 1 \le i \le d$ are admissible, $[\widehat{\psi}_i, \widehat{\psi}_i] \in L^\infty, \widetilde{\psi}_i$ satisfy condition (I) and (2.5), (2.6) hold, then (2.4) holds.

Proof. For any $f, g \in L^2_{BC}(F)$, letting $h(\xi) := \sum_{i=1}^{d}\sum_{j\in\mathbb{Z}}\sum_{\gamma^*\in\Gamma^*}|\widehat{\widetilde{\psi}}_i(B^j\xi)\overline{\widehat{\psi}_i}(B^j\xi + \gamma^*)\overline{\widehat{g}}(\xi)\widehat{f}(\xi + M^{*j}\gamma^*)|$, we shall prove $h \in L^1$. Since $f, g \in L^2_{BC}(F)$, then there exists $j_0 \in \mathbb{N}$ such that $\forall\gamma^* \in \Gamma^*\backslash 0, j \ge j_0, \widehat{g}(\xi)\widehat{f}(\xi + M^{*j}\gamma^*) = 0$, a.e. $\xi \in R^n$.

Let $I_1 := \int_{R^n}\sum_{i=1}^{d}\sum_{j\in\mathbb{Z}}|\widehat{\widetilde{\psi}}_i(B^j\xi)\widehat{\psi}_i(B^j\xi)||\widehat{f}(\xi)\widehat{g}(\xi)|d\xi$ and

$$I_2 := \int_{R^n}\sum_{i=1}^{d}\sum_{j<j_0}\sum_{\gamma^*\in\Gamma^*\backslash 0}|\widehat{\widetilde{\psi}}_i(B^j\xi)\widehat{\psi}_i(B^j\xi+\gamma^*)\widehat{g}(\xi)\widehat{f}(\xi+M^{*j}\gamma^*)|d\xi.$$

since $\psi_i, \widetilde{\psi}_i$ are admissible and $f, g \in L^2_{BC}(F)$, then

$$I_1 \le \|\widehat{f}\|_{L^\infty}\|\widehat{g}\|_{L^\infty}\int_{\mathrm{supp}\widehat{g}}\sum_{j\in\mathbb{Z}}|\widehat{\widetilde{\psi}}_i(B^j\xi)\widehat{\psi}_i(B^j\xi)|d\xi < \infty.$$

Since $\mathrm{supp}\widehat{g} \subset \{\xi : C^{-1} \le \|\xi\| \le C\}$, then $\inf_{C^{-1}\le\|\xi\|\le C}\|B^j\xi\| \ge C^{-1}\|M^{*j}\|^{-1}$ and $\forall j < j_0, \sum_{\gamma^*\in\Gamma^*}\chi_{B^j\mathrm{supp}\widehat{g}}(\xi + \gamma^*) \le C_1 m^j$, where C_1 is a positive constant. By

$[\widehat{\psi}_i, \widehat{\psi}_i] \in L^\infty$ and $\widetilde{\psi}_i$ satisfying condition (I), we get

$$
\begin{aligned}
I_2 \leq\ & \sum_{i=1}^{d} \sum_{j<j_0} \|\widehat{g}\|_{L^\infty} m^j \int_{B^j \mathrm{supp}\widehat{g}} |\widehat{\widetilde{\psi}}_i(\xi)| [\widehat{f}(M^{*j}\xi), \widehat{f}(M^{*j}\xi)]^{1/2} [\widehat{\psi}_i, \widehat{\psi}_i]^{1/2} d\xi \\
\leq\ & \|\widehat{g}\|_{L^\infty} \left(\sum_{i=1}^{d} \|[\widehat{\psi}_i, \widehat{\psi}_i]\|_{L^\infty}^{1/2} \right) \sum_{i=1}^{d} \sum_{j<j_0} m^j \Big\{ \int_{B^j \mathrm{supp}\widehat{g}} |\widehat{\widetilde{\psi}}_i(\xi)|^{\delta_2} \\
& [\widehat{f}(M^{*j}\xi), \widehat{f}(M^{*j}\xi)] d\xi \Big\}^{1/2} \left[\int_{B^j \mathrm{supp}\widehat{g}} |\widehat{\widetilde{\psi}}(\xi)|^{2-\delta_2} d\xi \right]^{1/2} \\
& \leq C_2 \sum_{i=1}^{d} \sum_{j<j_0} m^j \Big\{ \int_{B^j \mathrm{supp}\widehat{g}} |\widehat{\widetilde{\psi}}_i(\xi)|^{\delta_2} [\widehat{f}(M^{*j}\xi), \widehat{f}(M^{*j}\xi)] d\xi \Big\}^{1/2},
\end{aligned}
$$

here $C_2 = \|\widehat{g}\|_{L^\infty} \left(\sum_{i=1}^{d} \|[\widehat{\psi}_i, \widehat{\psi}_i]\|_{L^\infty}^{1/2} \right) \left[\int_{R^n} |\widehat{\widetilde{\psi}}(\xi)|^{2-\delta_2} d\xi \right]^{1/2}$. Since

$$
\begin{aligned}
& \int_{B^i \mathrm{supp}\widehat{g}} |\widehat{\widetilde{\psi}}_i(\xi)|^{\delta_2} [\widehat{f}(M^{*j}\xi), \widehat{f}(M^{*j}\xi)] d\xi \\
& \leq \| \|\xi\|^{\delta_1} \widehat{\widetilde{\psi}}_i(\xi) \|_{L^\infty}^{\delta_2} C^{\delta_1\delta_2} \|M^{*j}\|^{\delta_1\delta_2} \int_{B^j \mathrm{supp}\widehat{g}} [\widehat{f}(M^{*j}\xi), \widehat{f}(M^{*j}\xi)] d\xi \\
& \leq C_3 m^{-2j} \|M^{*j}\|^{\delta_1\delta_2},
\end{aligned}
$$

here $C_3 = C_1 C^{\delta_1\delta_2} \max_{1\leq i\leq d} \| \|\xi\|^{\delta_1} \widehat{\widetilde{\psi}}_i(\xi) \|_{L^\infty}^{\delta_2} \|\widehat{f}\|_{L^2}^2$, then

$$
I_2 \leq C_2 \sum_{i=1}^{d} \sum_{j<j_0} m^j C_3^{1/2} m^{-j} \|M^{*j}\|^{\delta_1\delta_2/2} \leq C_2 C_3^{1/2} \sum_{i=1}^{d} \sum_{j<j_0} \|M^{*j}\|^{\delta_1\delta_2/2} < \infty.
$$

which means that $h \in L^1$. By (2.7) and Lebesgue's dominated convergence theorem, we obtain $\forall f, g \in L^2_{BC}(F)$

$$
\begin{aligned}
& \sum_{i=1}^{d} \sum_{j\in\mathbb{Z}} \sum_{\gamma\in\Gamma} \langle f, \psi_{i;j,\gamma}\rangle \langle \widetilde{\psi}_{i;j,\gamma}, g\rangle \\
& = \sum_{i=1}^{d} \sum_{j\in\mathbb{Z}} m^j |S| \int_S [\widehat{f}(M^{*j}\xi), \widehat{\psi}_i(\xi)][\widehat{\widetilde{\psi}}_i(\xi), \widehat{g}(M^{*j}\xi)] d\xi \\
& = |S| \int_{R^n} \sum_{i=1}^{d} \sum_{j\in\mathbb{Z}} \sum_{\gamma^*\in\Gamma^*} \widehat{\widetilde{\psi}}_i(B^j\xi) \overline{\widehat{\psi}}_i(B^j\xi + \gamma^*) \overline{\widehat{g}}(\xi) \widehat{f}(\xi + M^{*j}\gamma^*) d\xi \\
& = |S| \int_{R^n} \widehat{f}(\xi) \overline{\widehat{g}}(\xi) \sum_{i=1}^{d} \sum_{j\in\mathbb{Z}} \widehat{\widetilde{\psi}}_i(B^j\xi) \overline{\widehat{\psi}}_i(B^j\xi) d\xi + |S| \int_{R^n} \overline{\widehat{g}}(\xi) \sum_{j_0\in\mathbb{Z}} \sum_{\gamma_0^*\in\Gamma^*\setminus M^*\Gamma^*} \\
& \widehat{f}(\xi + M^{*j_0}\gamma_0^*) \sum_{j=0}^{\infty} \sum_{i=1}^{d} \widehat{\widetilde{\psi}}_i(M^{*j}(B^{j_0}\xi + \gamma_0^*)) \overline{\widehat{\psi}}_i(M^{*j}(B^{j_0}\xi)) d\xi.
\end{aligned}
$$

From (2.5), (2.6), we obtain (2.4).□

Remark. The condition (I) is not very strict. For example, if $|\widehat{\psi}(\xi)| \le C\|\xi\|^{-\frac{n}{2}-\varepsilon}$, a.e. $\xi \in R^n$, here C and ε are positive constants, then ψ satisfies condition (I). If $\widehat{\psi} \in L^\infty \cap L^2$ has compact support, it is obvious to see that ψ satisfies condition (I).

Proposition 2.5. If ψ has an upper frame bound, then $[\widehat{\psi}, \widehat{\psi}] \in L^\infty$.

Proof. $\forall \{C_\gamma\}_{\gamma\in\Gamma} \in l^2(\Gamma)$, we have

$$\left\| \sum_{\gamma\in\Gamma} C_\gamma \psi_{0,\gamma} \right\|_{L^2} = \sup_{\|g\|_{L^2}\le 1} \left| \sum_{\gamma\in\Gamma} C_\gamma \langle \psi_{0,\gamma}, g\rangle \right| \le C\|\{C_\gamma\}\|_{l^2}$$

which means $[\widehat{\psi}, \widehat{\psi}] \in L^\infty$. □

Proposition 2.6. If Y is a dense subset of $L^2(R^n)$ and $\psi \in L^2(R^n)$ and there exists a constant $C > 0$ such that $\forall f \in Y, \sum_{j\in\mathbb{Z}} \sum_{\gamma\in\Gamma} |\langle f, \psi_{j,\gamma}\rangle|^2 \le C\|f\|_{L^2}^2$, then ψ has an upper frame bound C.

Now we state our main results in this section.

Theorem 2.7. Given $\psi_i, \widetilde{\psi}_i \in L^2(R^n), 1 \le i \le d, \widetilde{\psi}_i$ (or ψ_i) satisfy condition (I), then $\{\psi_i\}_{i=1}^d$ with dual functions $\{\widetilde{\psi}_i\}_{i=1}^d$ generates a dual tight frame in $\widehat{L^2(F)}$ if and only if $\psi_i, \widetilde{\psi}_i, 1 \le i \le d$ have upper frame bounds and (2.5), (2.6) hold.

Proof. Combining Proposition 2.1, Theorem 2.3, Theorem 2.4, and Proposition 2.5 and Proposition 2.6, the result is obtained. □

Theorem 2.8. $\psi_i \in L^2(R^n), 1 \le i \le d$ satisfy condition (I), then $\{\psi_i\}_{i=1}^d$ generates a tight frame in $\widehat{L^2(F)}$ if and only if $[\widehat{\psi}_i, \widehat{\psi}_i] \in L^\infty, 1 \le i \le d$ and (2.5), (2.6) hold.

Using the results we have obtained, we now characterize (dual) wavelet bases.

Corollary 2.9. $\psi_i, \widetilde{\psi}_i \in L^2(R^n), 1 \le i \le d$. Assume that ψ_i (or $\widetilde{\psi}_i$) satisfy condition (I), then $\{\psi_i\}_{i=1}^d$ with dual functions $\{\widetilde{\psi}_i\}_{i=1}^d$ generates a dual wavelet basis of $\widehat{L^2(F)}$ if and only if the following hold

(i) (Dual orthogonality condition)

$$[\widehat{\psi_i}(\xi), \widehat{\widetilde{\psi}_j}(M^{*k}\xi)] = [\widehat{\widetilde{\psi}_i}(\xi), \widehat{\psi_j}(M^{*k}\xi)] = |S|^{-1}\delta_{i,j}\delta_{0,k}, \ \forall k \ge 0, 1 \le i, j \le d,$$

(ii) (Dual tight frame condition)

$\psi_i, \widetilde{\psi}_i, 1 \le i \le d$ have upper frame bounds and (2.5), (2.6) hold with $C_0 = 1$.

Corollary 2.10. $\psi_i \in L^2(R^n), 1 \le i \le d$ satisfy condition (I). Then $\{\psi_i\}_{i=1}^d$ generates a wavelet basis of $\widehat{L^2(F)}$ if and only if the following hold

(i) (Orthonormality condition)

$$[\widehat{\psi_i}(\xi), \widehat{\psi_j}(M^{*k}\xi)] = |S|^{-1}\delta_{i,j}\delta_{0,k}, \ \forall k \ge 0, k \in \mathbb{Z}, 1 \le i, j \le d,$$

(ii) (Tight frame condition)

$[\widehat{\psi}_i, \widehat{\psi}_i] \in L^\infty, 1 \le i \le d$ and (2.5), (2.6) hold with $\widetilde{\psi}_i = \psi_i, 1 \le i \le d$, and $C_0 = 1$.

Remark. From Corollary 2.9 and Corollary 2.10, it seems that to a great extent, the dual orthgonality condition and the dual tight frame condition are relatively independent. In fact, item (i) in Corollary 2.9 can be replaced by any one of the following

(i1) $\{\psi_i\}_{i=1}^d$(or $\{\widetilde{\psi}_i\}_{i=1}^d$) generates a Riesz basis.

(i2) If $\forall\{C_{i;j,\gamma}\}_{1\le i\le d, j\in \mathbb{Z},\gamma\in\Gamma}\in l^2$ such that $\sum_{i=1}^{d}\sum_{j\in\mathbb{Z}}\sum_{\gamma\in\Gamma} C_{i;j,\gamma}\psi_{i;j,\gamma}=0$ in L^2, then $C_{i;j,\gamma}=0,\ 1\le i\le d, j\in\mathbb{Z},\gamma\in\Gamma$.

(i3) $G: L^2(R^n)\mapsto l^2(\{1\le i\le d\}\times\mathbb{Z}\times\Gamma)$ is a surjection, here $G(f)_{i;j,\gamma}=\langle f,\psi_{i;j,\gamma}\rangle$.

(i4) $\{\psi_{i;j,\gamma}\}_{1\le i\le d, j\in\mathbb{Z},\gamma\in\Gamma}$ has a unique dual frame with the tight frame bound $C_0=1$.

We also note that item (i) in Corollary 2.10 can be replaced by $\|\psi_i\|_{L^2}=1, 1\le i\le d$. But we shall show that item (i) in Corollary 2.9 can not be replaced by $[\widehat{\psi}_i,\widehat{\widetilde{\psi}_j}]=|S|^{-1}\delta_{i,j}, 1\le i,j\le d$. In the case $n=1, M=2, F=R$, let $K=[-2\pi,,-\pi]\cup[\pi,2\pi], \widetilde{K}=[-2\pi,-\pi/2]\cup[\pi/2,2\pi]$ and $\widehat{\psi}(\xi)=\frac{1}{\sqrt{2\pi}}\chi_K(\xi), \widehat{\widetilde{\psi}}(\xi)=\frac{1}{\sqrt{2\pi}}\chi_{\widetilde{K}}(\xi)$. Using Theorem 2.8, we know that ψ (also $\widetilde{\psi}$) generates a tight frame in $L^2(R)$. Checking the conditions in Theorem 2.7, we know ψ with the dual function $\widetilde{\psi}$ generates a dual tight frame in $L^2(R)$ with the tight frame bound $C_0=1$ and $[\widehat{\psi},\widehat{\widetilde{\psi}}]=1/2\pi$. But ψ generates a wavelet basis, which means that item (4) does not hold.

Proposition 2.11 (also see [2]). $\{\psi_i\}_{i=1}^d$ with dual functions $\{\widetilde{\psi}_i\}_{i=1}^d$ generates a dual wavelet basis of $\widehat{L^2(F)}$ and $\widehat{\psi}_i,\widehat{\widetilde{\psi}}_i,\ 1\le i\le d$ are ε-hölder continuous at the origin (i.e. there exists $\delta>0,\varepsilon>0, C>0$ such that $|\widehat{\psi}(\xi)|\le C\|\xi\|^{\varepsilon}$, whenever $\|\xi\|<\delta$), if $\widehat{\psi_l}$ is compactly supported, then $\widehat{\psi_l}$ vanishes in a neighborhood of the origin.

Proof. By item (i) in corollary 2.9, since $\widehat{\psi}_l$, is compactly supported, then there exists j_0 large enough such that $\forall j>j_0, 1\le k\le d$, $\overline{\widehat{\psi}_l}(\xi)\widehat{\widetilde{\psi}}_k(B^j\xi)=\sum_{\gamma^*\in\Gamma^*}\overline{\widehat{\psi}_l}(\xi+M^{*j}\gamma^*)\widehat{\widetilde{\psi}}_k(B^j\xi+\gamma^*)=0,\ \forall\xi\in \mathrm{supp}\widehat{\psi}_l$. Since $\widehat{\psi}_k,\widehat{\widetilde{\psi}}_k$ are ε-hölder continuous at the origin, then there exists $\delta>0$ such that

$$|\sum_{k=1}^{d}\sum_{j\le j_0}\overline{\widehat{\psi}_k}(B^j\xi)\widehat{\widetilde{\psi}}_k(B^j\xi)|\le\frac{1}{2|S|},\ \forall\|\xi\|<\delta$$

which means that $\sum_{k=1}^{d}\sum_{j>j_0}\overline{\widehat{\psi}_k}(B^j\xi)\widehat{\widehat{\psi}_k}(B^j\xi)| \geq \frac{1}{2|S|}\chi_F,\ \forall\|\xi\|<\delta$. Thus

$$\frac{|\widehat{\psi}_l|}{2|S|} \leq |\sum_{k=1}^{d}\sum_{j>j_0}\overline{\widehat{\psi}_k}(B^j\xi)\widehat{\widehat{\psi}_k}(B^j\xi)\overline{\widehat{\psi}_l}| = 0,\ \forall\|\xi\|<\delta$$

which finishes the proof. □

Proposition 2.12. If ψ generates a wavelet basis in $L^2(R)$ and $\widehat{\psi}$ has compact support, let $l := \sup\{b-a:\ a\leq 0\leq b \text{ and } \widehat{\psi}(\xi)\chi_{[a,b]}(\xi)=0\}$ and $h := \inf\{b-a: a<b \text{ and } \mathrm{supp}\widehat{\psi}\subseteq[a,b]\}$, then $l\leq 2\pi$ and $h\geq 4\pi$, moreover, $l=2\pi$ (or $h=4\pi$) if and only if $|\widehat{\psi}(\xi)| = \frac{1}{\sqrt{2\pi}}\chi_{[2a-4\pi,a-2\pi]\cup[a,2a]}(\xi)$, for some $0<a<2\pi$.

Proof. If $l>0$, by definition of l, there exist a,b such that $a<0<b, b-a=l$ and $\widehat{\psi}(\xi)\chi_{[a,b]}(\xi)=0$. By Corollary 2.10, (2.5) gives us that $\sum_{j>0}|\widehat{\psi}(2^j\xi)|^2 = \frac{1}{2\pi},\ \forall\xi\in[a,b]$. So $\int_R\sum_{j>0}|\widehat{\psi}(2^j\xi)|^2d\xi \geq \frac{b-a}{2\pi}$. Since $\int_R\sum_{j>0}|\widehat{\psi}(2^j\xi)|^2d\xi = \sum_{j>0}2^{-j}\|\psi\|_{L^2} = 1$, we obtain $l=b-a\leq 2\pi$. If $l=2\pi$, then $\sum_{j>0}|\widehat{\psi}(2^j\xi)|^2 = \frac{1}{2\pi}\chi_{[a,b]}(\xi)$ which gives us that $|\widehat{\psi}(\xi) = \frac{1}{\sqrt{2\pi}}\chi_{[2a,a]\cup[b,2b]}(\xi)$, by $[\widehat{\psi},\widehat{\psi}]=1$, thus there must exist $0<a<2\pi$, such that $|\widehat{\psi}(\xi)| = \frac{1}{\sqrt{2\pi}}\chi_{[2a-4\pi,a-2\pi]\cup[a,2a]}(\xi)$

If $h<\infty$, by definition of h, there exist a,b such that $b-a=h$ and $\mathrm{supp}\widehat{\psi}\subseteq[a,b]$. By Corollary 2.10, (2.5) and $\mathrm{supp}\widehat{\psi}\subseteq[a,b]$ tell us that $\sum_{j>0}|\widehat{\psi}(2^j\xi)|^2 \leq \frac{1}{2\pi}\chi_{[a/2,b/2]}(\xi)$. Thus $\int_R\sum_{j>0}|\widehat{\psi}(2^j\xi)|^2d\xi \leq \frac{b-a}{4\pi}$. Since $\int_R\sum_{j>0}|\widehat{\psi}(2^j\xi)|^2d\xi = 1$, we get $h=b-a\geq 4\pi$. If $h=4\pi$, then $\sum_{j>0}|\widehat{\psi}(2^j\xi)|^2 = \frac{1}{2\pi}\chi_{[a/2,b/2]}(\xi)$ which gives us that $|\widehat{\psi}(\xi)| = \frac{1}{\sqrt{2\pi}}\chi_{[2a-4\pi,a-2\pi]\cup[a,2a]}(\xi)$ for some $0<a<2\pi$. □

Since how to check that $\psi\in L^2(R^n)$ has an upper frame bound is an important problem, to complete our approach, in the last of this section, we shall cite some results on frame bound.

Proposition 2.13 (I. Daubechies [4]). If $[\widehat{\psi_i},\widehat{\psi_i}]\in L^\infty,\ 1\leq i\leq d$ and ψ_i satisfies condition (I), then $\forall f\in L^2(R^n), \|f\|_{L^2}=1$,

$$essinf_{\xi\in R^n}\eta(\xi)-\theta \leq \frac{1}{|S|}\sum_{i=1}^{d}\sum_{j\in\mathbb{Z}}\sum_{\gamma\in\Gamma}|\langle f,\psi_{i;j,\gamma}\rangle|^2 \leq esssup_{\xi\in R^n}\eta(\xi)+\theta,$$

here $\eta(\xi) := \sum_{i=1}^{d}\sum_{j\in\mathbb{Z}}|\widehat{\psi_i}(B^j\xi)|^2$, $\theta = \sum_{\gamma_0^*\in\Gamma^*\backslash M^*\Gamma^*}(\beta(\gamma_0^*)\beta(-\gamma_0^*))^{1/2}$ and $\beta(\gamma_0^*) = \sup_{\xi\in R^n}\sum_{j_0\in\mathbb{Z}}\left|\sum_{i=1}^{d}\sum_{j=0}^{\infty}\overline{\widehat{\psi_i}}(M^{*j}(B^{j_0}\xi+\gamma_0^*))\overline{\widehat{\psi_i}}(M^{*j}(B^{j_0}\xi))\right|$.

Proposition 2.14(I.Daubechies [4]). $\psi\in L^2(R^n)$, if there exists $\delta>0$ such that $\sum_{j\in\mathbb{Z}}|\widehat{\psi}(B^j\xi)|^{2\delta}\in L^\infty$ and $\sum_{\gamma^*\in\Gamma^*}|\widehat{\psi}(\xi+\gamma^*)|^{2-2\delta}\in L^\infty$, then ψ has an upper frame bound.

§3. Construction of Dual Tight Frames

In this section, we shall generalize Lawton's results on tight frames in $L^2(R)$ to several dimensional cases. In the end of this section, we shall show another way to construct tight frames.

Theorem 3.1. Let $\Gamma_0^* = \{\gamma_i^*\}_{i=0}^{m-1}$ be a full collection of representatives of distinct cosets of $\Gamma^*/M^*\Gamma^*$ with $\gamma_0^* = 0$. $\widehat{\phi}(\xi) = p_0(B\xi)\widehat{\phi}(B\xi), \widehat{\widetilde{\phi}}(\xi) = \widetilde{p}_0(B\xi)\widehat{\widetilde{\phi}}(B\xi)$ with $p_0, \widetilde{p_0}$ periodic and bounded such that

$$\sum_{i=0}^{m-1} \overline{p}_0(\xi + B\gamma_i^*)\widetilde{p_0}(\xi + B\gamma_i^*) = 1. \tag{3.1}$$

Moreover we assume that $[\widehat{\phi}, \widehat{\phi}] \in L^\infty, \widetilde{\phi}$ satisfies condition (I), $\lim\limits_{||\xi||\to 0} \overline{\widehat{\phi}}(\xi)\widehat{\widetilde{\phi}}(\xi) = \overline{\widehat{\phi}}(0)\widehat{\widetilde{\phi}}(0)$, $\lim\limits_{||\xi||\to\infty} (|\overline{\widehat{\phi}}(\xi)| + |\widehat{\widetilde{\phi}}(\xi)|) = 0$, and there exist two $n \times n$ matrices

$$P(\xi) := \begin{pmatrix} p_0(\xi + B\gamma_0^*), & p_0(\xi + B\gamma_1^*), & \cdots, & p_0(\xi + B\gamma_{m-1}^*) \\ p_1(\xi + B\gamma_0^*), & p_1(\xi + B\gamma_1^*), & \cdots, & p_1(\xi + B\gamma_{m-1}^*) \\ \vdots & \vdots & \ddots & \vdots \\ p_{m-1}(\xi + B\gamma_0^*), & p_{m-1}(\xi + B\gamma_1^*), & \cdots, & p_{m-1}(\xi + B\gamma_{m-1}^*) \end{pmatrix}$$

$$\widetilde{P}(\xi) := \begin{pmatrix} \widetilde{p_0}(\xi + B\gamma_0^*), & \widetilde{p_0}(\xi + B\gamma_1^*), & \cdots, & \widetilde{p_0}(\xi + B\gamma_{m-1}^*) \\ \widetilde{p_1}(\xi + B\gamma_0^*), & \widetilde{p_1}(\xi + B\gamma_1^*), & \cdots, & \widetilde{p_1}(\xi + B\gamma_{m-1}^*) \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{p}_{m-1}(\xi + B\gamma_0^*), & \widetilde{p}_{m-1}(\xi + B\gamma_1^*), & \cdots, & \widetilde{p}_{m-1}(\xi + B\gamma_{m-1}^*) \end{pmatrix}$$

such that $\overline{P(\xi)}\widetilde{P^*}(\xi) = I$, $\sum\limits_{i=0}^{m-1} (|p_i(\xi)| + |\widetilde{p}_i(\xi)|) \in L^\infty$. Define $\widehat{\psi_i}(\xi) = p_i(B\xi)\widehat{\phi}(B\xi)$, $\widehat{\widetilde{\psi}_i}(\xi) = \widetilde{p}_i(B\xi)\ \widehat{\widetilde{\phi}}(B\xi), 1 \le i \le m-1$. Assume that $\psi_i, \widetilde{\psi}_i$ have upper frame bounds. Let $F = R^n$, then $\{\psi_i\}_{i=1}^{m-1}$ with dual functions $\{\widetilde{\psi}_i\}_{i=1}^{m-1}$ generates a dual tight frame in $L^2(R^n)$ with the tight frame bound $|S|\widehat{\phi}(0)\widehat{\widetilde{\phi}}(0)$.

Proof. From the assumptions, it is easy to see that $[\widehat{\psi}_i, \widehat{\psi}_i] \in L^\infty, \widetilde{\psi}_i$ satisfy condition (I) and $\psi_i, \widetilde{\psi}_i$ are admissible whenever $1 \le i \le m-1$. To apply Theorem 2.7, it suffices to check (2.5), (2.6).

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}} \sum_{i=1}^{m-1} \overline{\widehat{\psi}_i}(B^j\xi)\widehat{\widetilde{\psi}_i}(B^j\xi) = \sum_{j\in\mathbb{Z}} \sum_{i=1}^{m-1} \overline{p_i}(B^{j-1}\xi)\widetilde{p}_i(B^{j-1}\xi)\overline{\widehat{\phi}}(B^{j-1}\xi)\widehat{\widetilde{\phi}}(B^{j-1}\xi) \\
&= \sum_{j\in\mathbb{Z}} [\overline{\widehat{\phi}}(B^{j-1}\xi)\widehat{\widetilde{\phi}}(B^{j-1}\xi) - \overline{\widehat{\phi}}(B^{j-2}\xi)\widehat{\widetilde{\phi}}(B^{j-2}\xi)] \\
&= \lim_{j\to+\infty} \overline{\widehat{\phi}}(B^j\xi)\widehat{\widetilde{\phi}}(B^j\xi) - \lim_{j\to-\infty} \overline{\widehat{\phi}}(B^j\xi)\widehat{\widetilde{\phi}}(B^j\xi) = \overline{\widehat{\phi}}(0)\widehat{\widetilde{\phi}}(0).
\end{aligned}$$

which is the equality (2.5). For any $\gamma_0^* \in \Gamma^* \backslash M^*\Gamma^*$,

$$\sum_{i=1}^{m-1}\sum_{j=0}^{\infty}\overline{\widehat{\psi}_i}(M^{*j}\xi)\widehat{\widetilde{\psi}_i}(M^{*j}(\xi+\gamma_0^*))$$
$$=\sum_{i=1}^{m-1}\sum_{j=0}^{\infty}\overline{p_i}(BM^{*j}\xi)\overline{\widehat{\phi}}(BM^{*j}\xi),\widetilde{p_i}(BM^{*j}(\xi+\gamma_0^*))\widehat{\widetilde{\phi}}(BM^{*j}(\xi+\gamma_0^*))$$
$$=\sum_{i=1}^{m-1}\overline{p_i}(B\xi)\widetilde{p_i}(B(\xi+\gamma_0^*))\overline{\widehat{\phi}}(B\xi)\widehat{\widetilde{\phi}}(B(\xi+\gamma_0^*))$$
$$+\sum_{i=1}^{m-1}\sum_{j=0}^{\infty}\overline{p}_i(M^{*j}\xi)\widetilde{p_i}(M^{*j}\xi)\overline{\widehat{\phi}}(M^{*j}\xi)\widehat{\widetilde{\phi}}(M^{*j}(\xi+\gamma_0^*))$$

$$=\sum_{j=0}^{\infty}(\overline{\widehat{\phi}}(M^{*j}\xi)\widehat{\widetilde{\phi}}(M^{*j}(\xi+\gamma_0^*))-\overline{\widehat{\phi}}(M^{*j+1}\xi)\widehat{\widetilde{\phi}}(M^{*j+1}(\xi+\gamma_0^*))-\overline{\widehat{\phi}}(\xi)\widehat{\widetilde{\phi}}(\xi+\gamma_0^*)$$
$$=\lim_{j\to+\infty}\overline{\widehat{\phi}}(M^{*j}\xi)\widehat{\widetilde{\phi}}(M^{*j}(\xi+\gamma_0^*))=0.$$

which means (2.6) holds.

By using Theorem 2.7, the result is obtained. □

Lemma 3.2. $\phi \in L^2(R^n)$, and there exist positive conitants δ_1, δ_2 such that $(1+\|\xi\|)^{\delta_1}\widehat{\phi}(\xi) \in L^\infty, \sum|\widehat{\phi}(\xi+\gamma^*)|^{2-2\delta_2} \in L^\infty$. Let $p(\xi)$ be Γ^*-periodic and there exist constants $\delta > 0, C > 0$ such that $|p(\xi)| \le C\|\xi\|^\delta$. Define $\widehat{\psi}(\xi) := p(B\xi)\widehat{\phi}(B\xi)$, then ψ has an upper frame bound.

Proof. Checking the conditions in Proposition 2.14, we obtain the result. □

Remark. In the special case when $\phi = \widetilde{\phi}, P(\xi) = \widetilde{P}(\xi)$, without the assumption that $\psi_i, \widetilde{\psi}_i, 1 \le i \le m-1$ have upper frame bounds, Theorem 3.1 still holds. Theorem 3.1 is also true for $\{\phi_i\}_{i=1}^d$ with dual functions $\{\widetilde{\phi}_i\}_{i=1}^d$ under the same conditions as stated above.

Now we construct Γ^*-periodic functions $p_0, \widetilde{p}_0$ such that (3.1) holds.

Lemma 3.3 (see G. V. Wellend & M. Lundberg [7]).

Let $q_N(X) = q_N(X_1, X_2, \cdots, X_{m-1}) = \sum_{0\le j\le (N-1)e_0}\binom{N-1+|j|}{j}X^j$, here $e_0 = (1,1,\cdots,1), |j| = \sum_{i=1}^{m-1} j_i, \binom{N-1+|j|}{j} = \prod_{i=1}^{m-1}\binom{N-1+|j|}{j_i}$ and $\alpha \le \beta, \alpha, \beta \in \mathbb{Z}^{m-1}$ iff $\alpha_i \le \beta_i, \forall 1 \le i \le m-1$. If $\sum_{j=0}^{m-1} X_j = 1$, then $\sum_{j=0}^{m-1} X_j^N q_N(\widehat{X}_j) = 1$, here $\widehat{X_j} = (X_0, \cdots, X_{j-1}, X_{j+1}, \cdots, X_{m-1})$.

Proposition 3.4. If $p_0, \widetilde{p}_0$ are Γ^*-periodic and satisfy

$$\sum_{i=0}^{m-1}\overline{p}_0(\xi+B\gamma_i^*)\widetilde{p}_0(\xi+B\gamma_i^*) = 1.$$

Let $p(\xi) = q_N(\overline{p}_0(\xi + B\gamma_1^*)\widetilde{p}_0(\xi + B\gamma_1^*), \cdots, \overline{p_0}(\xi + B\gamma_{m-1}^*)\widetilde{p}_0(\xi + B\gamma_{m-1}^*))$, then $\sum_{i=0}^{m-1} \overline{p}_0^N(\xi + B\gamma_i^*)\widetilde{p}_0^N(\xi + B\gamma_i^*)p(\xi + B\gamma_i^*) = 1$.

Remark. By using these results above, we can construct new dual tight frames from known dual wavelet bases, especially those wavelets whose Fourier transforms have compact support.

Now we present another way to construct tight frames.

Theorem 3.5. Let $\eta \in L^2(R^n)$ with supp$\widehat{\eta} \subseteq \{\xi \in R^n : \|\xi\| \le C\}$ for a constant $C > 0$. Assuming $l(\xi) := \sum_{j\in\mathbb{Z}} |\widehat{\eta}(B^j\xi)|^2 < \infty$, a.e. $\xi \in R^n$, we define $\widehat{\eta}_1(\xi) := \begin{cases} \widehat{\eta}(\xi)/l(\xi), & \text{when } l(\xi) \neq 0, \\ 0 & \text{when } l(\xi) = 0. \end{cases}$ If $k \in \mathbb{Z}$ satisfies $\sup_{\xi_1,\xi_2\in \text{supp}\widehat{\eta}} \|\xi_1 - \xi_2\| < \inf_{\gamma^*\in\Gamma^*\setminus 0} \|B^k\gamma^*\|$, then $\widehat{\psi}(\xi) := \widehat{\eta}_1(B^k\xi)$ generates a tight frame in the space $\{f \in L^2(R^n) : \text{supp}\widehat{f} \subseteq \text{supp}l(\xi)\}$.

Proof. Since $\widehat{\eta}_1 \le 1$ with supp$\widehat{\eta}_1 \subseteq \{\xi \in R^n : \|\xi\| \le C\}$, then $\psi \in L^2(R^n)$ and satisfies condition (I). It is easy to verify (2.5),(2.6). Applying Theorem 2.8, we obtain the result. □

Remark. By using this result, we can easily construct a function ψ with $\widehat{\psi} \in C_0^\infty(R^n)$ generating a tight frame, but $\widehat{\psi}(\xi) \neq 0, \forall\xi \neq 0$ in a neighborhood of the origin.

It is interesting to construct wavelet bases by using characteristic functions. This result is as follows.

Corollary 3.6. If $K_i, 1 \le i \le d$ are measurable subsets of R^n and can be included in a ball with finite radius. If the following hold

(i) $\sum_{i=1}^{d}\sum_{j\in\mathbb{Z}} \chi_{K_i}(B^j\xi) = \chi_F$

(ii) $\sum_{\gamma\in\Gamma^*} \chi_{K_i}(\xi + \gamma^*) = 1, \quad \forall 1 \le i \le d$

here F is a measurable subset of R^n, then $\{\frac{1}{|S|^{1/2}}\widehat{\chi_{K_i}}\}_{i=1}^d$ generates a wavelet basis in $\widehat{L^2(F)}$. Moreover this wavelet basis can be derived from a Multiresultion Analysis (MRA) with one scaling function if and only if $d = m - 1$ and

$$\left|\left(\bigcup_{j=1}^{\infty} B^jK\right) \cap \left(\bigcup_{\gamma^*\in\Gamma^*} (K + M^*\gamma^*)\right)\right| = 0,$$

here $K = \bigcup_{i=1}^{d} K_i$.

When $n = 1, M = 2, \Gamma = \mathbb{Z}$, using Corollary 3.6, we will give examples of such wavelets in $L^2(R)$ and $H^2(R) := \{f \in L^2(R) : \text{supp}\widehat{f} \subseteq [0, \infty)]$.

Example 1. Let $K = [a, 2a] \cup [2a - 4\pi, a - 2\pi]$, when $0 < a < 2\pi$. Such wavelets can be generated by an MRA

Example 2. Let $K = [a, b] \cup [2^s b, 2^{s+1} a] \cup [-2c, -c]$, here $a = \frac{2\pi k}{2^{s+1}-1}, b = \frac{4\pi k - 2\pi}{2^{s+1}-1}, c = 2\pi k - 2^s b$, when $1 < k < 2^s, k, s \in I\!N$. When k is odd, such wavelets can be generated by an MRA; When k is even, they can't be generated by an MRA with one scaling function.

Example 3. K_1 denotes $\left[\frac{2^s \pi}{2^{s+1}-1}, \pi\right] \cup \left[2^s \pi, \frac{2^{2s+1}\pi}{2^{s+1}-1}\right]$, let $K = K_1 \cup -K_1$ here $s \in I\!N$. When $s \geq 2$, such wavelets can't be generated by an MRA with one scaling function. In the case $s = 2$, this is the Journé's counterexample.

All the three examples above are wavelets in $L^2(R)$, now we give out such wavelets in $H^2(R)$.

Example 4. $K = [a, b] \cup [2^s b, 2^{s+1} a]$, when $a = \frac{2\pi + 2\pi k}{2^{s+1}-1}, b = \frac{2\pi k}{2^s - 1}$, for any $k < 2^{s+1} - 2, k, s \in I\!N$. Thus we construct other orthonormal wavelets for $H^2(R)$ which are different from $\frac{1}{\sqrt{2\pi}} \widehat{\chi}_{[2\pi, 4\pi]}$ (the only known example of a wavelet basis of $H^2(R)$ before, see [1] and [2]). Also we know that when k is odd. Such wavelets can't be generated by an MRA.

Remark. The Lemarié and Meyer wavelets and the wavelets constructed in Theorem 3.1 in [2] are just the modified wavelets with so called bell functions of the wavelets in Example 1 by letting $a = \pi$. There also exists such modification to construct smooth wavelets for any $0 < a < 2\pi$ by using the same method.

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Part II: Some Applications of Projection Operators in Wavelets

§1. Definitions and Results

In I.Daubechies [4], the author put forth such a question that if one imposes some smoothness conditions on the Fourier transform of the wavelet function $\psi \in L^2(R)$, does ψ must be derived from an MRA? In [6] and [7], P.G.Lemarié gave a satisfactory positive solution to this question. The main ingredient in the recipe of his proof is a new formula of the projection operator P_0. In this paper, we shall again obtain this formula under weaker conditions and give a complete description of $\dim J_{V_0}(x)$, which means that we have a criterion for a dual wavelet basis which can be generated by an MRA . More interesting is that whether a wavelet basis can be generated by an MRA with one scaling function is totally determined by the support of the Fourier transforms of their wavelet functions. So it seems that Daubechies' question whether ψ can be drived from an MRA is not related to the smoothness of $\hat{\psi}$ but the support of $\hat{\psi}$.

In order to state our results more clearly, we shall recall some notations and concepts in advance.

If Φ is a subset of $L^2(R^n)$, we let $S^0(\Phi) := Span\{\phi(x-\gamma) : \phi \in \Phi, \gamma \in \Gamma\}$ with Span denoting the closed linear span. If V_0 is a Γ-shift-invariant subspace of $L^2(R^n)$ (V_0 is a closed linear subspace in $L^2(R^n)$ and $f(x) \in V_0$ iff $f(x-\gamma) \in V_0, \ \forall \gamma \in \Gamma$), we let J_{V_0} denote the range function of V_0 (see [3]), that is,

$$J_{V_0} : \quad S \longrightarrow l^2(\Gamma^*), \qquad J_{V_0}(x) := Span\{\widehat{\phi}_{\|x} : \quad \phi \in \Phi\}$$

where Φ is a countable set and $V_0 = S^0(\Phi)$.

Now we state our main results in this paper.

Theorem 1. *If $\{\psi_i\}_{i=1}^d$ with dual functions $\{\tilde{\psi}_i\}_{i=1}^d$ generates a dual wavelet basis in $\widehat{L^2(F)}$ and $\tilde{\psi}_i$ satisfy condition* (I)*, let $V_0 = S^0(\{\psi_i(M^{-j}x)\}_{1\le i\le d, j\in \boldsymbol{N}})$, then $\dim J_{V_0}(\xi) = |S| \sum_{j<0} \sum_{i=1}^d [\hat{\psi}_i(B^j\xi), \hat{\tilde{\psi}}_i(B^j\xi)]$.*

From Theorem 1, we obtain the following interesting result.

Corollary 2. *If $\{\psi_i\}_{i=1}^d$ generates a wavelet basis in $\widehat{L^2(R)}$ and ψ_i satisfy condition* (I)*, then $\{\psi_i\}_{i=1}^d$ can be derived from an MRA with one scaling function if and only if $d = m-1$ and one of the following properties holds*

(i) *$\dim J_{V_0}(\xi) = 1, \quad a.e.\ \xi \in S$, here $V_0 = S^0(\{\psi_i(M^{-j}x)\}_{1\le i\le d, j\in \boldsymbol{N}})$,*

(ii) $\sum_{j<0} \sum_{i=1}^d [\hat{\psi}_i(B^j\xi), \hat{\psi}_i(B^j\xi)] = |S|^{-1}$,

(iii) $\cup_{\gamma^*\in\Gamma^*}(F_0+\gamma^*) = R^n$, *here $F_0 = \cup_{j<0} \cup_{i=1}^d M^{*j} supp\hat{\psi}_i$.*

Another application of Theorem 1 is the following Theorem 3.

Theorem 3. *If $\{2^{j/2}\psi(2^jx-k)\}_{j\in\boldsymbol{Z},k\in\boldsymbol{Z}}$ is a wavelet basis in $\widehat{L^2(F)}$ such that ψ satisfies condition* (I) *and $\sum_{k\in\boldsymbol{Z}} |\hat{\psi}(2\xi+4\pi k)|^2 \ne 0$, a.e. $\xi \in R$, then $supp\hat{\psi} = F$.*

One consequence of this result is the following.

Corollary 4. *There exists no wavelet basis$\{2^{j/2}\psi(2^jx-k)\}_{j\in\boldsymbol{Z},k\in\boldsymbol{Z}}$ in $\widehat{L^2(F)}$ such that $\hat{\psi}$ is compactly supported and $\sum_{k\in\boldsymbol{Z}} |\hat{\psi}(2\xi+4\pi k)|^2 \ne 0$, a.e. $\xi \in R$.*

§2. Projection Operators in Wavelets

In this section, we shall define projection operators $Q_j, \tilde{Q}_j$ and $P_0, \tilde{P}_0$ derived from a dual wavelet tight frame and then rewrite P_0 in another very useful formula which was first introduced in Lemarié [6].

Assuming that $\psi_i, \tilde{\psi}_i, \quad 1 \le i \le d$ have upper frame bounds, we define projection operators $Q_j, \tilde{Q}_j$ and $P_0, \tilde{P}_0$ as follows:

$$\forall f \in L^2(R), \quad Q_j f = \sum_{i=1}^{d} \sum_{\gamma \in \Gamma} \langle f, \tilde{\psi}_{i;j,\gamma} \rangle \psi_{i;j,\gamma}, \quad \tilde{Q}_j f = \sum_{i=1}^{d} \sum_{\gamma \in \Gamma} \langle f, \psi_{i;j,\gamma} \rangle \tilde{\psi}_{i;j,\gamma},$$
$$P_0 = \sum_{j<0} Q_j, \qquad \tilde{P}_0 = \sum_{j<0} \tilde{Q}_j.$$

Note that $\tilde{P}_0 = P_0^*$, $\tilde{Q}_j = Q_j^*$, where P_0^*, Q_j^* are the conjugate operators of P_0, Q_j respectively.

Lemma 2.1. *ψ has an upper frame bound C if and only if ψ has an upper Riesz bound C, i.e., $\forall \{C_{j,\gamma}\} \in l^2(\boldsymbol{Z} \times \Gamma)$, $\quad \| \sum_{j \in \boldsymbol{Z}} \sum_{\gamma \in \Gamma} C_{j,\gamma} \psi_{j,\gamma} \|_{L^2} \le C \| \{C_{j,\gamma}\} \|_{l^2}$.*

Corollary 2.2. *If ψ has an upper frame bound, then $[\hat{\psi}, \hat{\psi}] \in L^\infty$.*

From the following Proposition 2.3, we will see that $Q_j, \tilde{Q}_j$ and $P_0, \tilde{P}_0$ are well-defined.

Proposition 2.3. *If $\psi_i, \tilde{\psi}_i, 1 \le i \le d$ have upper frame bounds, then $\forall -\infty \le j_1 < j_2 \le +\infty, \sum_{j_1 < j \le j_2} Q_j$ are unifromly bounded, moreover, $\forall j \in \boldsymbol{Z}, f \in L^2(R^n)$*

$$\widehat{Q_j f}(\xi) = |S| \sum_{i=1}^{d} \hat{\psi}_i(B^j \xi) \sum_{\gamma^* \in \Gamma^*} \hat{f}(\xi + M^{*j} \gamma^*) \bar{\hat{\tilde{\psi}}}_i(B^j \xi + \gamma^*). \tag{2.1}$$

Proof. By applying Lemma 2.1, the results are obvious. ■

Proposition 2.4. *If $\psi_i, \tilde{\psi}_i, 1 \le i \le d$ have upper frame bounds and $\tilde{\psi}_i$ satisfy condition* (I)*, then for any $f \in L^2_{BC}(R^n)$, there exists constants $\delta > 0, C > 0$ (C is determined by $\psi_i, \tilde{\psi}_i, 1 \le i \le d, \|\hat{f}\|_{L^\infty}\|, \|\hat{f}\|_{L^2}$ and $supp\hat{f}$) such that $\forall j < 0, \|Q_j f\|_{L^2} \le C \|M^{*j}\|^\delta$.*

Proof. Since $\psi_i, \tilde{\psi}_i$ have upper frame bounds, by Corollary 2.2, $[\hat{\psi}_i, \hat{\psi}_i] + [\hat{\tilde{\psi}}_i, \hat{\tilde{\psi}}_i] \in L^\infty$. By (2.1), for any $f \in L^2_{BC}(F)$,

$$\|Q_j f\|_{L^2} = \|\widehat{Q_j f}\|_{L^2} \le |S| \sum_{i=1}^{d} \|\hat{\psi}_i(B^j \xi) \sum_{\gamma^* \in \Gamma^*} \hat{f}(\xi + M{*}j \gamma^*) \bar{\hat{\tilde{\psi}}}_i(B^j \xi + \gamma^*)\|_{L^2}.$$

Since $\tilde{\psi}_i$ satisfy condition (I) and $f \in L^2_{BC}(F)$, then there exist positive constants δ_1, C_1, C_2 such that $\forall 1 \le i \le d$, $|\hat{\psi}_i(\xi)| \le C_1 \|\xi\|^{-\delta_1}$ and $\inf_{\xi \in B^j supp \hat{f}} \|\xi\| \ge C_2 \|M^{*j}\|^{-1}$. Thus

$$|\hat{\psi}_i(\xi)| \le C_1 C_2^{-\delta_1} \|M^{*j}\|^{\delta_1}, \quad \forall\ \xi \in B^j supp\hat{f}, \quad 1 \le i \le d \tag{2.2}$$

$$\int_{R^n} |\hat{\psi}_i(B^j\xi)|^2 |[\hat{f}(M^{*j}\xi), \hat{\tilde{\psi}}_i(\xi)](B^j\xi)|^2 d\xi$$

$$= m^j \int_{R^n} |\hat{\psi}_i(\xi)|^2 |[\hat{f}(M^{*j}\xi), \hat{\tilde{\psi}}_i(\xi)]|^2 d\xi = m^j \int_S [\hat{\psi}_i, \hat{\psi}_i] |[\hat{f}(M^{*j}\xi), \hat{\tilde{\psi}}_i(\xi)]|^2 d\xi$$

$$\leq m^j \|[\hat{\psi}_i, \hat{\psi}_i]\|_{L^\infty} \int_{R^n} |\hat{f}(M^{*j}\xi)\hat{\tilde{\psi}}_i(\xi)| [\hat{f}(M^{*j}\xi), \hat{f}(M^{*j}\xi)]^{1/2} [\hat{\tilde{\psi}}_i, \hat{\tilde{\psi}}_i]^{1/2} d\xi$$

$$\leq m^j \|\hat{f}\|_{L^\infty} \|[\hat{\psi}_i, \hat{\psi}_i]\|_{L^\infty} \|[\hat{\tilde{\psi}}_i, \hat{\tilde{\psi}}_i]\|_{L^\infty}^{1/2} \int_{B^j supp \hat{f}} |\hat{\tilde{\psi}}_i(\xi)| [\hat{f}(M^{*j}\xi), \hat{f}(M^{*j}\xi)]^{1/2} d\xi$$

$$\leq C_3 m^j \left(\int_{B^j supp \hat{f}} |\hat{\tilde{\psi}}_i(\xi)|^{\delta_2} [\hat{f}(M^{*j}\xi), \hat{f}(M^{*j}\xi)] d\xi \right)^{1/2} \left(\int_{R^n} |\hat{\tilde{\psi}}_i(\xi)|^{2-\delta_2} d\xi \right)^{1/2}$$

$$\leq C_4 \|M^{*j}\|^{\delta_1\delta_2/2} m^j \left(\int_{B^j supp \hat{f}} [\hat{f}(M^{*j}(\xi), \hat{f}(M^{*j}\xi)] d\xi \right)^{1/2}$$

here

$$C_3 = \|\hat{f}\|_{L^\infty} \max_{1\leq i\leq d} (\|\hat{\psi}_i, \hat{\psi}_i]\|_{L^\infty} \|[\hat{\tilde{\psi}}_i, \hat{\tilde{\psi}}_i]\|_{L^\infty}^{1/2}),$$

$$C_4 = C_3 C_1^{\delta_2/2} C_2^{-\delta_1\delta_2/2} \max_{1\leq i\leq d} \left(\int_{R^n} |\hat{\tilde{\psi}}_i(\xi)|^{2-\delta_2} d\xi \right)^{1/2},$$

and in the last inequality we used (2.2).

Since $f \in L^2_{BC}(F)$, then there exists a constant $C_5 > 0$ such that

$$\forall j < 0, \quad \sum_{\gamma^*\in\Gamma^*} \chi_{B^j supp \hat{f}}(\xi + \gamma^*) \leq C_5 m^{-j},$$

So

$$\|\widehat{\psi}_i(B^j\xi) \sum_{\gamma^*\in\Gamma^*} \hat{f}(\xi + M^{*j}\gamma^*) \overline{\hat{\tilde{\psi}}_i}(B^j\xi + \gamma^*)\|_{L^2}^2$$

$$\leq C_4 C_5^{1/2} \|M^{*j}\|^{\delta_1\delta_2/2} m^{j/2} \left(\int_{R^n} |\hat{f}(M^{*j}\xi)|^2 d\xi \right)^{1/2}$$

$$= C_4 C_5^{1/2} \|\hat{f}\|_{L^2} \|M^{*j}\|^{\delta_1\delta_2/2}.$$

Thus $\forall j < 0$, $\|Q_j f\|_{L^2} \leq d|S| C_4^{1/2} C_5^{1/4} \|\hat{f}\|_{L^2}^{1/2} \|M^{*j}\|^{\delta_1\delta_2/4}$, which completes the proof. ■

Let Γ_0 be a full collection of representatives of distinct cosets of $\Gamma/M\Gamma$, for any $j \geq 0$, define $\Gamma_j := \Gamma_0 + M\Gamma_0 + \cdots + M^{j-1}\Gamma_0$. It is clear that Γ_j is a full collection of representatives of distinct cosets of $\Gamma/M^j\Gamma$. For any $j < 0$, define $Q_j^0 := m^j \sum_{\gamma\in\Gamma_{-j}} \tau_\gamma Q_j \tau_{-\gamma}$. and $G_j := \sum_{j\leq k<0} Q_k^0$. Note that

$$\widehat{Q_j^0 f}(\xi) = |S| \sum_{i=1}^d \widehat{\psi}_i(B^j\xi)[\hat{f}(\xi), \widehat{\tilde{\psi}}(B^j\xi)](\xi) \tag{2.3.}$$

Theorem 2.5. *If $\psi_i, \tilde{\psi}_i, 1 \le i \le d$ have upper frame bounds and $\tilde{\psi}_i$ satisfy condition* (I) *and $\tau_\gamma P_0 \tau_{-\gamma} = P_0$, $\forall \gamma \in \Gamma$, then $\forall f \in L^2(R^n), \lim_{j\to-\infty} \|P_0 f - G_j f\|_{L^2} = 0$. i.e., $\widehat{P_0 f}(\xi) = |S| \sum_{j<0} \sum_{i=1}^d \hat{\psi}_i(B^j\xi)[\hat{f}(\xi), \widehat{\tilde{\psi}}_i(B^j\xi)](\xi)$ with the series converging in $L^2(R^n)$.*

Proof. At first, we shall prove that $\forall f \in L^2_{BC}(R^n)$, $\lim_{j\to-\infty} \|P_0 - G_j f\|_{L^2} = 0$. By the assumptions, we know that $\forall j_0 < 0$,

$$\begin{aligned} P_0 &= m^{j_0} \sum_{\gamma\in\Gamma_{-j_0}} \tau_\gamma P_0 \tau_{-\gamma} \\ &= m^{j_0} \sum_{\gamma\in\Gamma_{-j_0}} \sum_{j_0\le j<0} \tau_\gamma Q_j \tau_{-\gamma} + m^{j_0} \sum_{\gamma\in\Gamma_{-j_0}} \sum_{j<j_0} \tau_\gamma Q_j \tau_{-\gamma} \\ &= \sum_{j_0\le j<0} m^j \sum_{\gamma\in\Gamma_{-j}} \tau_\gamma Q_j \tau_{-\gamma} + m^{j_0} \sum_{\gamma\in\Gamma_{-j_0}} \sum_{j<j_0} \tau_\gamma Q_j \tau_{-\gamma} \\ &= G_{j_0} + m^{j_0} \sum_{\gamma\in\Gamma_{-j_0}} \sum_{j<j_0} \tau_\gamma Q_j \tau_{-\gamma}. \end{aligned}$$

For any $f \in L^2_{BC}(R^n)$, since $\tilde{\psi}_i$ satisfy condition (I), by Proposition 2.4, there exist constants $C > 0, \delta > 0$ such that $\forall j < 0, \gamma \in \Gamma$, $\|Q_j \tau_\gamma f\|_{L^2} \le C\|M^{*j}\|^\delta$. Thus

$$\|P_0 f - G_{j_0} f\|_{L^2} = \|m^{j_0} \sum_{\gamma\in\Gamma_{-j_0}} \sum_{j<j_0} \tau_\gamma Q_j \tau_{-\gamma} f\|_{L^2} \le C \sum_{j<j_0} \|M^{*j}\|^\delta,$$

which means that $\lim_{j_0\to-\infty} \|P_0 f - G_{j_0} f\|_{L^2} = 0$.

Since $L^2_{BC}(R^n)$ is dense in $L^2(R^n)$, for any $f \in L^2(R^n)$, there exists $f_k \in L^2_{BC}(R^n)$, $k \in \mathbb{N}$ such that $\lim_{k\to\infty} \|f_k - f\|_{L^2} = 0$. By Proposition 2.3, It is clear that $\sup_{j<0} \|G_j\| < \infty$. Thus we get $\forall j_0 < 0, k \in \mathbb{N}$,

$$\begin{aligned} \|P_0 f - G_{j_0} f\|_{L^2} &\le \|P_0 f - P_0 f_k\|_{L^2} + \|P_0 f_k - G_{j_0} f_k\|_{L^2} + \|G_{j_0} f_k - G_{j_0} f\| \\ &\le \|P_0 f_k - G_{j_0} f_k\|_{L^2} + \|f_k - f\|_{L^2}(\|P_0\| + \sup_{j_0<0} \|G_{j_0}\|) \end{aligned}$$

which gives us that $\lim_{j_0\to-\infty} \|P_0 f - G_{j_0} f\|_{L^2} = 0$. ■

Corollary 2.6. *If $\{\psi_i\}_{i=1}^d$ with dual functions $\{\tilde{\psi}_i\}_{i=1}^d$ generates a dual tight frame in $\widehat{L^2(F)}$ and $\tilde{\psi}_i$ satisfy conditoin* (I)*, then $P_0 = \sum_{j<0} Q_j^0$.*

§3. Proofs of the Main Results

Lemma 3.1(see [3]). *if V_0 is a Γ-shift-invariant subspace of $L^2(R^n)$, then there exist $\phi_k, k \in \mathbb{N}$ such that $V_0 = S^0(\{\phi_k\}_{k\in\mathbb{N}})$ and $[\hat{\phi}_i, \hat{\phi}_j] = 0, \forall i \ne j$ and $[\hat{\phi}_i, \hat{\phi}_i](\xi) = 0$ or 1, $\forall \xi \in R^n$, that is, $V_0 = \oplus_{k\in\mathbb{N}} S^0(\phi_k)$ and $[\hat{\phi}_k, \hat{\phi}_k](\xi) = 1$, whenever $\xi \in supp[\hat{\phi}_k, \hat{\phi}_k]$, $k \in \mathbb{N}$, here $\oplus$ denoting the orthogonal sum.*

From this Lemma, we know that $\dim J_{V_0}(\xi) = \sum_{k\in\mathbb{N}} [\hat{\phi}_k, \hat{\phi}_k](\xi)$.

Proof of Theorem 1. Since V_0 is a Γ-shift-invariant subspace of $L^2(R^n)$, by Lemma 3.1, there exist $\phi_k,\ k \in \mathbb{N}$ such that $V_0 = \oplus S^0(\phi_k)$ and $\dim J_{V_0}(\xi) = \sum_{k\in\mathbb{N}}[\hat{\phi}_k, \hat{\phi}_k](\xi)$. It is easy to verify that $\forall f \in V_0,\ \hat{f}(\xi) = \sum_{k\in\mathbb{N}}[\hat{f}, \hat{\phi}_k]\hat{\phi}_k(\xi)$. Thus $\forall j < 0,\ 1 \le i \le d$,

$$\hat{\psi}(B^j\xi) = \sum_{k\in\mathbb{N}}[\hat{\psi}(B^j\xi), \hat{\phi}_k(\xi)]\hat{\phi}_k(\xi).$$

On the other hand, since

$$\begin{aligned}
&\int_S \sum_{j<0}\sum_{k\in\mathbb{N}} |[\hat{\psi}_i(B^j\xi), \hat{\phi}_k(\xi)][\hat{\phi}_k(\xi), \hat{\tilde{\psi}}_i(B^j\xi)]|d\xi \\
&\le \left(\int_S \sum_{j<0}\sum_{k\in\mathbb{N}} |[\hat{\psi}_i(B^j\xi), \hat{\phi}_k(\xi)]|^2 d\xi\right)^{1/2} \left(\int_S \sum_{j<0}\sum_{k\in\mathbb{N}} |[\hat{\phi}_k(\xi), \hat{\tilde{\psi}}_i(B^j\xi)]|^2 d\xi\right)^{1/2} \\
&\le \left(\int_S \sum_{j<0}[\hat{\psi}_i(B^j\xi), \hat{\psi}_i(B^j\xi)]d\xi\right)^{1/2} \left(\int_S \sum_{j<0}[\hat{\tilde{\psi}}_i(B^j\xi), \hat{\tilde{\psi}}_i(B^j\xi)]d\xi\right)^{1/2} \\
&= (m-1)^{-1}\|\psi_i\|_{L^2}\|\tilde{\psi}_i\|_{L^2} < \infty,
\end{aligned}$$

then by Theorem 2.5

$$\begin{aligned}
\sum_{i=1}^d \sum_{j<0}[\hat{\psi}_i(B^j\xi), \hat{\tilde{\psi}}_i(B^j\xi)] &= \sum_{i=1}^d \sum_{j<0}\sum_{k\in\mathbb{N}}[\hat{\psi}_i(B^j\xi), \hat{\phi}_k(\xi)][\hat{\phi}_k(\xi), \hat{\tilde{\psi}}_i(B^j\xi)] \\
&= \sum_{k\in\mathbb{N}}\sum_{i=1}^d \sum_{j<0}[\hat{\psi}_i(B^j\xi), \hat{\phi}_k(\xi)][\hat{\phi}_k(\xi), \hat{\tilde{\psi}}_i(B^j\xi)] \\
&= |S|^{-1}\sum_{k\in\mathbb{N}}[\hat{\phi}_k(\xi), \hat{\phi}_k(\xi)] = |S|^{-1} dim J_{V_0}(\xi)
\end{aligned}$$

which completes the proof. ∎

Proof of Corollary 2. Since $\int_S \sum_{j<0}\sum_{i=1}^{m-1}[\hat{\psi}_i(B^j\xi), \hat{\psi}_i(B^j\xi)]d\xi = 1$, by using Theorem 1, it is easy to verify that item (i), (ii), (iii) are equivalent.

Proof of Theorem 3. Since $\sum_{k\in\mathbb{Z}} |\hat{\psi}(2\xi+4\pi k)|^2 \ne 0$, a.e. $\xi \in R$, by Corollary 2, it gives us that ψ can be derived from an MRA with one scaling function ϕ such that $[\hat{\phi}, \hat{\phi}] = |S|^{-1}$ and $\hat{\phi}(\xi) = p_0(\xi/2)\hat{\phi}(\xi/2)$ and $\hat{\psi}(\xi) = p_1(\xi/2)\hat{\phi}(\xi/2)$ where $p_0,\ p_1$ are Γ^* periodic and $\begin{pmatrix} p_0(\xi) & p_0(\xi+\pi) \\ p_1(\xi) & p_1(\xi+\pi)\end{pmatrix}$ is a unitary matrix. By the assumption, $\sum_{k\in\mathbb{Z}} |\hat{\psi}(2\xi + 4\pi k)|^2 \ne 0$, we have that $p_1(\xi) \ne 0$, a.e. $\xi \in R$ which means that $p_0(\xi) \ne 0$, a.e. $\xi \in R$ and $supp\hat{\psi} = 2supp\hat{\phi}$. But $p_0(\xi) \ne 0$ means that $supp\hat{\phi} = 2supp\hat{\phi}$ which follows that $supp\hat{\phi} = F$. Thus $supp\hat{\psi} = 2supp\hat{\phi} = 2F = F$. ∎

§4. Other Applications of Theorem 1

In this section, we generalize the wavelets which were introduced by Lemarié and Meyer (also see [2]).

Theorem 4.1. *for any fixed* $0 < a < 2\pi,\ 0 < \varepsilon \leq \frac{1}{3}\min(a, 2\pi - a)$, *if* $0 \leq b(\xi) \in L^2(R)$ *satisfies the following conditions*

(i) $b(a+\xi) = b(a - 2\pi - \xi),\ b(2a+\xi) = b(2a - 4\pi - \xi),\ \forall \xi \in [-\varepsilon, \varepsilon]$

(ii) $b(a+\xi) = b(2a - 2\xi),\ b^2(a+\xi) + b^2(a-\xi) = 1,\ \forall \xi \in [-\varepsilon, \varepsilon]$

(iii) $b(\xi) = 1,\ \forall \xi \in [a+\varepsilon, 2a - 2\varepsilon] \cup [2a - 4\pi + 2\varepsilon, a - 2\pi - \varepsilon]$

and $\varphi(\xi)$ *is a measurable function such that* $\varphi(a+\xi) - \varphi(2a+2\xi) - \varphi(a - 2\pi + \xi) + \varphi(2(a - 2\pi + \xi)) = (2l+1)\pi$, *for some* $l \in \boldsymbol{Z}$ *and* $\xi \in [-\varepsilon, \varepsilon]$. *Let* $\hat{\omega}(\xi) := |S|^{-1/2} e^{i\varphi(\xi)} b(\xi)$, *then* $\{2^{j/2}\omega(2^j x - k)\}_{j \in \boldsymbol{Z}, k \in \boldsymbol{Z}}$ *is a wavelet basis and can be derived from an MRA with one scaling function.*

Proof. For the proof of the first assertion, please see B.Han [5]. By Corollary 2, noting that $\sum_{k \in \boldsymbol{Z}} \sum_{j>0} b^2(2^j(\xi + 2\pi k)) = 1$, we see that the second assertion is true. ■

In [1], Auscher has shown that if one assumes some smoothness for the function $\hat{\psi}$, then ψ cannot generate an wavelet basis for $H^2(R) := \{f \in L^2(R) :\ supp\hat{f} \subseteq [0, +\infty)\}$. We now use Theorem 1 to prove it. A precise version of Auscher's theorem is the following:

Theorem 4.2. *There does not exist a wavelet function* $\psi \in H^2(R)$ *such that* $\hat{\psi} \in C^1(R)$ *and* $|\hat{\psi}(\xi)| + |\hat{\psi}'(\xi)| \leq C|\xi|^{-\alpha}$ *for* $\xi \geq 1, \alpha > 1/2$.

Proof. we use proof by contradiction. If there exists such a wavelet function ψ, it is clear that ψ satisfies condition (I). By Theorem 1, we have $\sum_{j>0}[\hat{\psi}(2^j\xi), \hat{\psi}(2^j\xi)] = \dim J_{V_0}(\xi)$. But Lemma 3 in P.Auscher [1] says that $\sum_{j>0}[\hat{\psi}(2^j\xi), \hat{\psi}(2^j\xi)] = \frac{3\pi - \xi}{2}$ which means that $\dim J_{V_0}(\xi) = \frac{3\pi-\xi}{2}$. Thus we get a contradiction. ■

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Part III: A Sufficient and Necessary Condition on Γ_0 for $T(\Gamma_0, M)$ to Be a Self-Affine Tiling

§1. Introduction

Self-affine tilings are the fundamental blocks to construct scaling functions and wavelet bases (see [s]). The main task to construct a self-affine tiling is that given an acceptable dilation M for a lattice subtroup Γ of R^n, how to find Γ_0, a full collection of representatives of distinct cosets of $\Gamma/M\Gamma$, satisfying $\chi_{T(\Gamma_0,M)}(x) = \sum_{\gamma_0\in\Gamma_0} \chi_{T(\gamma_0,M)}(Mx-\gamma_0)$ and $\sum_{\gamma\in\Gamma} \chi_{T(\Gamma_0,M)}(x+\gamma) = 1$ (Here $T(\Gamma_0, M)$ is defined in the following text). The ordinary method is to check Cohin's condition (see [GM]). But this condition is ususlly not very easy to apply. In this paper we present a necessary and sufficient condition on Γ_0 for $T(\Gamma_0, M)$ to be a self-affine tiling. It is much easier to check this condition than to check Cohen's condition. Moreover, by using this result, we construct wavelet basis which basis which have exponential decay and high regularity with Frobenius matrix M as its acceptable dilation and $\boldsymbol{Z}^n$ as its lattice subgroup of R^n.

At first, we define some concepts and recall some basic facts on tilings and lattice subgroups of R^n.

Let Γ_0 be a full collection of representatives of distinct cosets of $\Gamma/M\Gamma$ with $0 \in \Gamma_0$. We define $T(\Gamma_0, M)$ by

$$T(\Gamma_0, M) = \{x \in R : x = \sum_{j=1}^{\infty} M^{-j}\gamma_j, \gamma_j \in \Gamma_0\}.$$

To understand lattice subgroups and self-affine tilings, we first state some basic facts.

Lemma 1.1. *If G is a discrete additive subgroup of R^n, then there exists an $n\times n$ matrix A satisfying $G = A\boldsymbol{Z}^n$.*

Due to the somewhat long and detailed analysis, we state this results without proof. The key is to prove that the least number of generators of G is less than $n+1$.

Given a measurable set S, χ_S denotes its characteristic function and $|S|$ denotes its Lebesgue measure.

Q is a measurable set, we say Q is a self-affine tiling if $\chi_Q(x) = \sum_{\gamma_0\in\Gamma_0} \chi_Q\ (x-\gamma_0)$ and $\sum_{\gamma\in\Gamma}(x+\gamma) = 1$.

Lemma 1.2(see [JM]). *Let A be an acceptable dilation for Γ, $\phi \not\equiv 0$, and $\phi \in L'(R^n) \cap L^2(R^n)$ satisfying*

$$\phi(x) = \sum_{\gamma\Gamma} b_\gamma \phi(Ax-\gamma), \tag{1.1}$$

$$\sum_{\gamma\in\Gamma} |b_\gamma| < \infty \tag{1.2}$$

then $\hat{\phi}(\gamma^*) = 0\ \forall\gamma^* \in \Gamma^*\backslash 0$ *and* $\sum_{\gamma\in\Gamma}\phi(x-\gamma) = \hat{\phi}(0)$

Proof. Let B denote A^{*-1}. Applying the Fourier transform to both sides of (1.1), we have

$$\hat{\phi}(\xi) = b(B\xi)\hat{\phi}(B\xi).$$

Here $b(\xi) = (\det A)^{-1}\sum_{\gamma\in\Gamma} b_\gamma e^{-i\xi\gamma}$. (1.2) and $\phi \in L'(R^n)\cap L^2(R^n)$ mean that $b(\xi)$ and $\hat{\phi}(\xi)$ are continuous and bounded functions.

If $|b(0)| < 1$, then there exists ϵ satisfying $\epsilon > 0$ and $|b(0)| + \epsilon < 1$. Given $\xi \in R^n$, there exists N, a large enough natural number, satisfying $\forall s \geq N$, *and* $s \in \mathbb{N}$, then $|\hat{\phi}(\xi)| = \left|\prod_{j=1}^{s} b(B^j\xi)\right| |\hat{\phi}(B^s\xi)| \leq C\prod_{j=1}^{s}(|b(0)|+\epsilon) = C(|b(0)|+\epsilon)^s \to 0, as \quad s \to +\infty$ (Here C is determined by $\xi, b(\xi)$ and $\hat{\phi}$), but $\phi \equiv 0$ is contradictary to the given conditions. So $|b(0)| \geq 1$.

Let $\xi = (A^*)^s\gamma^*, s \in \mathbb{N}$ and $\gamma^* \in \Gamma^*\backslash 0$ then

$$\begin{aligned}\hat{\phi}(\xi) = \hat{\phi}(A^{*s}\gamma^*) &= \hat{\phi}(B^s\xi)\prod_{j=1}^{s} b(B^j\xi)\\ &= \hat{\phi}(A^{*-s}A^{*s}\gamma^*)\prod_{j=1}^{s} b(A^{*-j}A^{*s}\gamma^*)\\ &= \hat{\phi}(\gamma^*)\prod_{j=1}^{s} b(A^{*s-j}\gamma^*)\\ &= \hat{\phi}(\gamma^*)\prod_{j=1}^{s} b(0) = \hat{\phi}(\gamma^*)b^s(0)\end{aligned}$$

that is, $|\hat{\phi}(\gamma^*)| = |b(0)|^{-s}|\hat{\phi}(\xi)| \leq |\hat{\phi}((A^*)^s\gamma^*)|$. A is an acceptable dilation and $\gamma^* \neq 0$, so $\|(A^*)^s\gamma^*\| \to +\infty$, as $s \to +\infty$. Since $\phi \in L'(R^n)$, by using Riemann-Lebesglle lemma, we know $\lim_{s\to+\infty}\hat{\phi}(A^{*s}\gamma^*) = 0$, so $\hat{\phi}(\gamma^*) = 0\ \forall\ \gamma^* \in \Gamma^*\backslash 0$. $\sum_{\gamma\in\Gamma}\phi(\xi-\gamma) = \hat{\phi}(0)$ is obtained from the equality, $\sum_{\gamma\in\Gamma}\phi(x-\gamma) = \sum_{\gamma^*\in\Gamma^*}\hat{\phi}(\gamma^*)e^{i\gamma^* x}$. ■

Lemma 1.3. $|T(\Gamma_0, M)|/|\det E| \in \mathbb{N}$, *and* $\sum_{\gamma\in\Gamma}\chi_{T(\Gamma_0,M)}(x+\gamma) = |T(\Gamma_0, M)|/|\det E|$. *Here* $\Gamma = E\mathbf{Z}^n$.

Proof. From Lemma 1.2, we can easily obtain Lemma 1.3.

Lemma 1.4(see [GM]). *The set* $T(\Gamma_0, M)$ *has the following properties.*

1) $T(\Gamma_0, M)$ *is a compact set.*

2) $MT(\Gamma_0, M) = \cup_{\gamma_0\in\Gamma_0}(T(\Gamma_0, M)+\gamma_0)$

$$|T(\Gamma_0, M)+\gamma_1)\cap(T(\Gamma_0, M)+\gamma_2)| = 0 \quad \forall\gamma_1, \gamma_2 \in \Gamma_0 \quad and \quad \gamma_1 \neq \gamma_2.$$

3) $R^n = \cup_{\gamma\in\Gamma}(T+\gamma)$.

Let $\{A_i\}$ *be a measurable subset family of* $R^n, i \in \mathbb{N}$. *If* $|A_i \cap A_j| = \delta_{ij}|A_j|, \forall i, j \in \mathbb{N}$, *we let* $\sqcup_{i=1}^{\infty}A_i$ *denote* $\cup_{i=1}^{\infty}A_i$. *Thus 2) is equivalent to*

2)' $MT(\Gamma_0, M) = \sqcup_{\gamma_0\in\Gamma_0}(T(\Gamma_0, M)+\gamma_0)$.

Lemma 1.5(see [GM]). *Suppose that* Q *is a measurable subset of* R^n *such that*

$$\cup_{\gamma\in\Gamma}(Q+\gamma) = R^n$$

then the following are equivalent

1) $\sqcup_{\gamma\in\Gamma}(Q+\gamma)=R^n$,
2) $|Q\cap(Q+\gamma)|=0$, *whenever* γ *is a nonzero element in* Γ.
3) $|Q|=|\det E|$,
4) Q is a self-affine tiling.

§2. Main Results

We let Γ_1 denote $\langle\Gamma_0, M\Gamma_0,\cdots\rangle$, the additive group generated by $\{M^i\Gamma_0\}_{i=0}^{\infty}$. Let $\Gamma_{-1}:=\{\gamma_1-\gamma_2:\gamma_1\in\Gamma_0 \quad and \quad \gamma_2\in\Gamma_0\}$, $\Gamma_2:=\Gamma_0+M\Gamma_0+M^2\Gamma_0+\cdots$ and $\Gamma_3:=\{\gamma_1-\gamma_2:\gamma_1\in\Gamma_2 \; and \; \gamma_2\in\Gamma_2\}$.

Lemma 2.1 $|T(\Gamma_0,M)|/(|\det E|\cdot|\Gamma/\Gamma_1|)\in\mathbb{N}$. *Here* $|\Gamma/\Gamma_1|$ *is the number of cosets of* Γ/Γ_1.

Proof. $\Gamma_1\subseteq\Gamma$ means that Γ_1 is a discrete additive subgroup of R^n. Lemma 1.1 says that there exists an $n\times n$ matrix E_1 satisfying $\Gamma_1=E_1\boldsymbol{Z}^n$ and $|\det E_1|=|\det E|\cdot|\Gamma/\Gamma_1|$. The assumption, Γ_0 is a full collection of representatives of distinct cosets of $\Gamma/M\Gamma$, also means that Γ_0 is a full collect of representatives of distinct cosets of $\Gamma_1/M\Gamma_1$, by applying Lemma 1.3, we obtain $|T(\Gamma_0,M)|/|\det E_1|\in\mathbb{N}$, that is, $|T(\Gamma_0,M)|/(|\det E|\cdot|\Gamma/\Gamma_1|)\in\mathbb{N}$. ■

Remark. If $T(\Gamma_0,M)$ is a self-affine tiling for Γ, then $\Gamma_1=\Gamma$.

Lemma 2.2. *Let* T *denote* $T(\Gamma_0,M)$, *and* $\overset{\circ}{T}$ *denotes the interior of* T, *then* $\overset{\circ}{T}$ *is a nonempty set, and* $|\overset{\circ}{T}|=|T|$, *and also there exists* $k_0\in\Gamma$ *and* $N\in\mathbb{N}$, $\forall s_0\in\mathbb{N}$ *satisfying* $R^n=\cup_{s=s_0}^{\infty}M^s(M^NT-k_0)$.

Proof. From Lemma 1.4, we know $R^n=\cup_{\gamma\in\Gamma}(T+\gamma), T$ is a compact set and R^n is a complete space, by the Baire category theorem, $\overset{\circ}{T}$ is nonempty. It means $M^N\overset{\circ}{T}$ contains a large ball, if N is large enough. Then there exists $k_0\in\Gamma$ satisfying $k_0\in M^N\overset{\circ}{T}$, that is, M^NT-k_0 contains a neighborhood of the origin, then $\forall s_0\in\mathbb{N}$

$$(2.1)\quad R^n=\cup_{s=s_0}^{\infty}M^s(M^NT-k_0)$$
$$=\cup_{s=s_0}^{\infty}(\sqcup_{\substack{\gamma_i\in\Gamma_0\\0\le i\le N+s-1}}(T+\gamma_0+\cdots+M^{N+s-1}\gamma_{N+s-1}-M^sk_0)),$$

From this equality, we know that there exist $s\in\mathbb{N}$ and $\eta\in W_s=\{\gamma:\gamma=\gamma_0+M\gamma_1+\cdots+M^{N+s-1},\gamma_i\in\Gamma_0, 0\le i\le\gamma_{N+s-1}\}$ such that $T+\eta-M^sk_0$ is included in the interior of $M^s(M^NT-k_0)$.

Note that the interior of $T+\eta-M^sk_0=(T+\eta-M^sk_0)\setminus\sqcup_{\substack{\gamma\in W_s\\\gamma\ne\eta}}(T+\gamma-M^sk_0)$, namely, $\overset{\circ}{T}=T\setminus\cup_{\substack{\gamma\in W_s\\\gamma\ne\eta}}(T+\gamma-\eta)$.

Therefore $|T\setminus\overset{\circ}{T}|\le\sum_{\substack{\gamma\in W_s\\\gamma\ne\eta}}|T\cap(T+\gamma-\eta)|=\sum_{\substack{\gamma\in W_s\\\gamma\ne\eta}}|(T+\eta)\cap(T+\gamma)|=0$, then we have $|\overset{\circ}{T}|=|T|$. ■

Remark. We can choose $s_i\to+\infty$, as $i\to+\infty$ satisfying $\{\Gamma_0+\cdots+M^{s_i+N-1}\,\Gamma_0-M^{s_i}k_0\}\subseteq\{\Gamma_0+\cdots+M^{N+s_{i*+1}-1}\gamma_{N+s_{i+1}-1}-M^{s_{i+1}}k_0\}$ $\forall i\in\mathbb{N}$.

The following Theorem is already implicitly obtained in [6], in order to complete our approach, we state it here explicitly.

Theorem 2.3. *$k_0 \in \Gamma, N \in \mathbb{N}$ then $|T(\Gamma_0, M)| = |\det E|$ and $M^{-N}k_0 \in \overset{\circ}{T}(\Gamma_0, M)$ if and only if*

$$\cup_{s=1}^{\infty}\{\Gamma_0 + \cdots + M^{s+N-1}\Gamma_0 - M^s k_0\} = \Gamma. \tag{2.2}$$

Proof. Assume $|T(\Gamma_0, M)| = |\det E|$. By using Lemma 1.5, we get

$$\sqcup_{\gamma\in\Gamma}(T(\Gamma_0, M) + \gamma) = R^n. \tag{2.3}$$

From (2.1) and (2.3), we have (2.2)

If $\cup_{s=1}^{\infty}\{\Gamma_0 + \cdots + M^{s+N-1}\Gamma_0 - M^s k_0\} = \Gamma$. To prove $|T(\Gamma_0, M)| = |\det E|$, it suffices to prove that

$$\forall \eta_1, \eta_2 \in \Gamma \quad and \quad \eta_1 \neq \eta_2 \quad then \quad |(T + \eta_1) \cap (T + \eta_2)| = 0.$$

From (2.2), there exists $*s \in \mathbb{N}$ satisfying $\eta_1, \eta_2 \in \{\Gamma_0 + \cdots + M^{N+s-1}\Gamma_0 - M^s k_0\}$. Note

$$M^s(M^N T - k_0) = \sqcup_{\substack{\gamma_i\in\Gamma_0 \\ 0\leq i\leq N+s-1}} (T + \gamma_0 + \cdots + M^{N+s-1}\gamma_{s+N-1} - M^s k_0).$$

We have

$$|(T + \eta_1) \cap (T + \eta_2)| = 0.$$

Let B_r denote the ball of radius r centered at the origin. Because $\#\{\gamma \in \Gamma : B_1 \cap (T + \gamma) \neq \phi\} < \infty$, there exists $s \in \mathbb{N}$ satisfying

$$B_1 \subseteq \sqcup_{\substack{\gamma_i\in\Gamma_0 \\ 0\leq i\leq s+N-1}} (T + \gamma_0 + \cdots + M^{s+N-1}\gamma_{s+N-1} - M^s k_0) = M^s(M^N T - k_0),$$

then $M^{-s-N}B_1 \subseteq T - M^{-N}k_0$, that is, $M^{-N}k_0 \in \overset{\circ}{T}$. ■

Corollary 2.4. *$|T(\Gamma_0, M)|$ is a self-affine tiling and $0 \in \overset{\circ}{T}$ if and only if $\Gamma_0 + M\Gamma_0 + M^2\Gamma_0 + \cdots = \Gamma$.*

Our main result in this paper is the following theorem.

Theorem 2.5 *$|T(\Gamma_0, M)| = |\det E|$ if and only if $\Gamma_3 = \Gamma$.*

Proof. If $\Gamma_3 = \Gamma$, to prove $|T(\Gamma_0, M)| = |\det E|$, by Lemma 1.5, it suffices to prove that $\forall \gamma \in \Gamma_3\backslash 0$ then $|T \cap (T + \gamma)| = 0$. From the definition of Γ_3, we know $\forall \gamma \in \Gamma_3 = \Gamma$, $\gamma = \eta_1 - \eta_2, \eta_1 \in \Gamma_2$ and $\eta_2 \in \Gamma_2$, so there exists $s \in \mathbb{N}$ satisfying $\eta_1, \eta_2 \in \{\Gamma_0 + M\Gamma_0 + \cdots + M^{s-1}\Gamma_0\}$.Note that $\forall\ s \in \mathbb{N}, M^s T = \sqcup_{\substack{\gamma_i\in\Gamma_0 \\ 0\leq i\leq s-1}} (T + \gamma_0 + \cdots + M^{s-1}\gamma_{s-1})$, we have $|(T + \eta_1) \cap (T + \eta_2)| = 0$, that is, $|T \cap (T + \eta_1 - \eta_2)| = 0$. So $|T \cap (T + \gamma)| = 0$.

Now prove the converse. By (2.1) and item 1) of Lemma 1.5, we know that there exists $s_0 \in \mathbb{N}, \forall s \geq s_0$ satisfying $0 \in \{\Gamma_0 + \cdots + M^{N+s-1}\Gamma_0 - M^s k_0\}$, we have $M^s k_0 \in \Gamma_2$. Observe that $\forall s \in \mathbb{N}$ and $s \geq s_0, \Gamma_0 + \cdots + M^{N+s-1}\Gamma_0 - M^s k_0 \subseteq \Gamma_3$, we have

$$\begin{aligned} R^n &= \cup_{s=s_0}^{\infty}(T + \Gamma_0 + \cdots + M^{N+s-1}\Gamma_0 - M^s k_0) \\ &\subseteq \cup_{\gamma\in\Gamma_3}(T + \gamma) \subseteq \sqcup_{\gamma\in\Gamma}(T + r) = R^n. \end{aligned}$$

As thus we get $\cup_{\gamma\in\Gamma_3}(T+\gamma)=\sqcup_{\gamma\in\Gamma}(T+\gamma)$, which means that $\Gamma_3=\Gamma$. ■
Corollary 2.6. *If Γ_3 is an additive group (that is, $\Gamma_3=\Gamma_1$), then $|T(\Gamma_0,M)|=|\det E|\cdot|\Gamma/\Gamma_1|$ and $\sum_{\gamma_1\in\Gamma_1}\chi_{T(\Gamma_0,M)}(x+\gamma_1)=1$.*

In fact, by using the following lemma, it is easy to check $\Gamma_1=\Gamma_3$.

Lemma 2.6. $\Gamma_3=\Gamma_1 \Leftrightarrow \Gamma_1\cap T(\Gamma_0,M)\subseteq\Gamma_3$.

Proof. It suffices to prove that when $\Gamma_1\cap T(\Gamma_0,M)\subseteq\Gamma_3$, we have also $\Gamma_1\subset\Gamma_3$. Let $\gamma\in\Gamma_1$, $\forall s\in \mathbb{N}$, let $\gamma=\gamma_0+M\gamma_1+\cdots+M^{s-1}\gamma_{s-1}+M^s\tilde{\gamma}_s$. Here $\gamma_i\in\Gamma_0, 0\le i\le s-1, \tilde{\gamma}_s\in\Gamma_1$. Then $\tilde{\gamma}_s=M^{-s}\gamma-(M^{-s}\gamma_0+\cdots+M^{-1}\gamma_{s-1})$. Observe that $\lim_{s\to+\infty}\|M^{-s}\gamma\|=0$ and $M^{-s}\gamma_0+\cdots+M^{-1}\gamma_{s-1}\in T(\Gamma_0,M)$ and $T(\Gamma_0,M)$ is a compact set, so there exists a subsequence $\{s_i\}_{i=1}^{\infty}$ satisfying $\lim\limits_{t\to+\infty}\tilde{\gamma}_{s_i}$ exists and $\lim_{i\to\infty}\tilde{\gamma}_{s_i}\in T(\Gamma_0,M)\cap\Gamma_1$. Because Γ is a discrete additive subgroup of R^n, there exists i_0 satisfying $\forall i\ge i_0, \tilde{\gamma}_{s_i}=\tilde{\gamma}_{s_{i_0}}$. This means $\tilde{\gamma}_{s_{i_0}}=\lim_{i\to\infty}\tilde{\gamma}_{s_i}\in T(\Gamma_0,M)\cap\Gamma_1\subseteq\Gamma_3$. So $\gamma=\gamma_0+\cdots+M^{s_{i_0}}\tilde{\gamma}_{s_{i_0}}\in\Gamma_3$. ■

Corollary 2.7. *Denote $r_0=(\sum_{i=1}^{\infty}\|M^{-i}\|)\max_{\gamma_0\in\Gamma_0}\|\gamma_0\|$, if $\Gamma_1\cap B_{r_0}\subseteq\Gamma_3$, then $\Gamma_3=\Gamma_1$.*

Proof. By $T(\Gamma_0,M)\subseteq B_{r_0}$, the result is obtained.

Remark. In fact, for any Γ_0, the equality $\cup_{\gamma\in\Gamma_3}(T(\Gamma_0,M)+\gamma)=R^n$ is always true. There are results showing that when $n=1,\Gamma_3=\Gamma_1$ is always true, so we guess that for general $n\in \mathbb{N},\Gamma_3=\Gamma_1$ is always true. If this guess is right, from Corollary 2.6, the constructure of $T(\Gamma_0,M)$ is clearly understood.

Example. The 'twin dragons' tiling of R^2(where $\Gamma=\boldsymbol{Z}^2$, $M=\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$ and $\Gamma_0=0,(1,0)^T$) can be easily shown to be a self-affine tilingby using Corollary 2.7. In fact, we have $r_0=\sqrt{2}+1$, and $\boldsymbol{Z}^2\cap B_{r_0}\subseteq\{\Gamma_{-1}+M\Gamma_{-1}+M^2\Gamma_{-1}+M^3\Gamma_{-1}+M^4\Gamma_{-1}\}$, by using Corollary 2.7, $T(\Gamma_0,M)$is a self-affine tiling of R^2.

§3. Wavelet Bases on R^n with β-Exponential Decay

Throughout this section, $\forall f\in L^2(R^n)$, we use the following notation,

$$f_{j,\gamma}(x):=m^{j/2}f(M^jx-\gamma),\quad j\in\boldsymbol{Z},\quad \gamma\in\Gamma.$$

Let $e_0:=(1,1,\cdots,1).\forall\alpha,\beta\in\boldsymbol{Z}^n,\ \alpha=(\alpha_1,\cdots,\alpha_n),\ \beta=(\beta_1,\cdots,\beta_n)$, we mean $\alpha\le\beta$ if $\forall 1\le i\le n,\ \alpha_i\le\beta_i$.

In this section we let $\Gamma=\boldsymbol{Z}^n$ and M be the following $n\times n$ Frobenius matrix.

$$M:=\begin{pmatrix} 0 & 0 & \cdots & & 2\\ 1 & 0 & \cdots & & 0\\ 0 & 1 & \cdots & & 0\\ \cdots & \cdots & \cdots & & \cdots\\ 0 & 0 & \cdots & 1 & 0\end{pmatrix}.$$

Note that $|\lambda I-M|=\lambda^n-2$, then M is an acceptable dilation for $\boldsymbol{Z}^n$. Choose $\Gamma_0=\{0,(1,0,\cdots,0)^T\}$, it is easy to verify that $\Gamma_2=\Gamma_0+M\Gamma_0+\cdots=\{(\gamma_1,\gamma_2,\cdots,\gamma_n)\in\boldsymbol{Z}^n:\gamma_i\ge 0,\forall 1\le i\le n\}$. It follows that $\Gamma_3=\{\gamma_1-\gamma_2:\gamma_1\in\Gamma_2\quad and\quad \gamma_2\in\Gamma_2\}=\boldsymbol{Z}^n$, by applying theorem 2.5, we have $T(\Gamma_0,M)$ is a self-affine tiling of R^n. In fact, $T(\Gamma_0,M)=[0,1]^n$.

Given $\beta \in \boldsymbol{Z}^n, 0 \leq \beta$, we say that $f \in L^2(R^n)$ is a β-exponential decay function if there exists $\rho > 0$ such that

$$D_\beta f \in C^0(R^n), and \ |D_\gamma f(x)| \leq C_\gamma e^{-\rho|x|} \quad \forall \gamma \in \boldsymbol{Z}^n \ and \ 0 \leq \gamma \leq \beta.$$

Here $x = (x_1, \cdots, x_n)$ and $|x| = \sum_{i=1}^n |x_i|$.

We mean that $f \in L^2(R^n)$ has β vanishing moments (here $0 \leq \beta$) if

$$\int_{R^n} x^\alpha f(x)dx = 0, \ \forall 0 \leq \alpha \leq \beta.$$

We say that a multiresolution analysis with scaling function φ and associated wavelet basis $\{\psi_{i;j,\gamma}\}_{1\leq i\leq m-1, j\in\boldsymbol{Z}, \gamma\in\Gamma}$ has β-exponential decay if φ and $\{\psi\}_{i=1}^{m-1}$ are β-exponential decay functions. Let

$$\eta(x) = \frac{1}{|\det E|^{1/2}} \chi_{T(\Gamma_0, M)}(x) = \chi_{[0,1]^n}(x) \quad and \quad \hat{\phi}_N = (\hat{\eta})^N, \forall N \in \boldsymbol{N}.$$

By the definition of β-exponential decay functions and $\hat{\phi}_N(\xi_1, \cdots, \xi_n) = \prod_{j=1}^n \left(\frac{1-e^{-i\xi_j}}{i\xi_j}\right)^N$, it is easy to prove that ϕ_N is an $(N-2)e_0$-exponential decay function.

Note that $[\hat{\eta}, \hat{\eta}](\xi) = \sum_{\gamma\in\boldsymbol{Z}^n} |\hat{\eta}(\xi+2\pi\gamma)|^2 = 1$ and η is a compactly supported function, then ϕ_N satisfying $0 < C_1 \leq [\hat{\phi}_N, \hat{\phi}_N](\xi) \leq C_2 < \infty$. C_1, C_2 are real constants.Now we let

$$\hat{\varphi}_N = \frac{\hat{\phi}_N}{[\hat{\phi}_N, \hat{\phi}_N]^{1/2}} \quad and \quad \forall j \in \boldsymbol{Z}, \ V_j = \overline{\operatorname{span}\{\hat{\varphi}_N(M^j x - \gamma)\}_{\gamma\in\boldsymbol{Z}^n}}.$$

It is easy to prove that $\{V_j\}_{j\in\boldsymbol{Z}^n}$ associated with $\boldsymbol{Z}^n$ and M is a multiresolution analysis with the scaling function φ_N.

Now we construct wavelet basis. Note that $\hat{\varphi}_N(\xi) = m(M^{*-1}\xi)\hat{\varphi}_N(M^{*-1}\xi)$ and $|m(\xi)|^2 + |m(\xi + (\pi, 0, \cdots, 0)^T)|^2 = 1$. Here $m(\xi)$ is $2\pi Z^n$ periodic. Let

$$\hat{\psi}_N(\xi) = e^{-i\xi_n/2}\overline{m(M^{*-1}\xi + (\pi, 0, \cdots, 0)^T)}\hat{\varphi}_N(M^{*-1}\xi).$$

By using the Theorem 1 in [4], we can easily obtain that ψ_N has $(N-1)e_0$ vanishing moments. Due to the following Lamma 3.1, we can prove that φ_N and ψ_N are $(N-2)e_0$-exponential decay functions.

Lemma 3.1. *If $p(z_1, z_2, \cdots, z_n)$ is a holomorphic function in Ω_t (here $0 < t < 1$ and $\Omega_t = \{(z_1, z_2, \cdots, z_n) \in \mathbb{C}^n : t < |z_i| < t^{-1}, 1 \leq i \leq n\}$), and $f(x)$ is a β-exponential decay function, define*

$$\hat{g}(\xi) := p(e^{i\xi_1}, \cdots, e^{i\xi_n})\hat{f}(\xi), \quad \forall \xi = (\xi_1, \cdots, \xi_n) \in R^n$$

then $g(x)$ is a β-exponential decay function.

Proof. Since $p(z_1, \cdots, z_n)$ is an holomorphic function in Ω_t and Ω_t is a Reinhardt domain, then $p(z_1, \cdots, z_n)$, in Ω_t,has a Laurant series (see [5])

$$p(z_1, \cdots, z_n) = \sum_{\gamma\in\boldsymbol{Z}^n} a_\gamma z^\gamma \quad \forall z \in \Omega_t$$

and

$$a_\gamma = \omega^{-\gamma}(2\pi)^{-n}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} p(\omega_1 e^{i\theta_1},\cdots,\omega_n e^{i\theta_n}e^{-i(\gamma_1\theta_1+\cdots+\gamma_n\theta_n)}d\theta_1\cdots d\theta_n$$
$$= t^{1/2|\gamma|}(2\pi)^{-n}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} p(\omega_1 e^{i\theta_1,\cdots,\omega_n}e^{i\theta_n}e^{-i(\gamma_1\theta_1+\cdots+\gamma_n\theta_n)}d\theta_1\cdots d\theta_n.$$

Here $z^\gamma = z_1^{\gamma_1}\cdots z_n^{\gamma_n}, \omega = (\omega_1,\cdots,\omega_n)$, $\omega_i = t^{-1/2\mathrm{sgn}\gamma_i}$ (note that $\omega\in\Omega_t$).

Thus $|a_\gamma|\le C_1 t^{1/2|\gamma|} \quad \forall\gamma\in\boldsymbol{Z}^n$. Meanwhile $f(x)$ is a β-exponential decay function, so there exists $\rho>0$ such that

$$|D_\gamma f(x)|\le C_2 e^{-\rho|x|} \quad \forall 0\le\gamma\le\beta.$$

By the definition of $g(x)$, we have

$$g(x) = \sum_{\gamma\in\boldsymbol{Z}^n} a_\gamma f(x+\gamma).$$

Thus $\forall 0\le\alpha\le\beta$,

$$|D_\alpha g(x)|\le \sum_{\gamma\in\boldsymbol{Z}^n}|a_\gamma|\cdot|D_\alpha f(x+\gamma)|\le\sum_{\gamma\in\boldsymbol{Z}^n}C_1C_2 t^{1/2|\gamma|}e^{-\rho|x+\gamma|}$$
$$= C_1C_2\sum_{\gamma\in\boldsymbol{Z}^n}e^{-p|x+\gamma|+1/2|\gamma|\ln t}$$
$$= C_1C_2\left(\sum_{\gamma_1\in\boldsymbol{Z}}e^{-\rho|x_1+\gamma_1|+1/2|\gamma_1|\ln t}\right)\cdots\left(\sum_{\gamma_n\in\boldsymbol{Z}}e^{-\rho|x_n+\gamma_n|+1/2|\gamma_n|\ln t}\right).$$

Let $\rho_1 = \min(\rho, -1/2\ln t)$, then $\forall 1\le i\le n$

$$\rho|x_i+\gamma_i| - 1/2|\gamma_i|\ln t \ge \rho_1(|x_i+\gamma_i|+|\gamma_i|)\ge\rho_1\sqrt{(x_i+\gamma_i)^2+\gamma_i^2}$$
$$= \rho_1\sqrt{1/2[x_i^2+(x_i+2\gamma_i)^2]} = \frac{\rho_1}{\sqrt{2}}\sqrt{x_i^2+(x_i+2\gamma_i)^2}$$
$$\ge \frac{\rho_1}{2}(|x_i|+|x_i+2\gamma_i|),$$

thus, we get

$$\sum_{\gamma_i\in\boldsymbol{Z}}e^{-\rho|x_i+\gamma_i|+1/2|\gamma_i|\ln t}\le\sum_{\gamma_i\in\boldsymbol{Z}}e^{-1/2\rho_1(|x_i|+|x_i+2\gamma_i|)}\le C_3 e^{-1/2\rho_1|x_i|}.$$

So

$$|D_\alpha g(x)|\le C_1C_2C_3^n\prod_{i=1}^n e^{-1/2\rho_1|x_i|}\le C_4 e^{-1/2\rho_1|x|}$$

which means that $g(x)$ is a β-exponential decay function. ■

Since ϕ_N has compact support, thus $[\hat{\phi}_N, \hat{\phi}_N]$ is a polynomial. By using Lemma 3.1, φ_N and ψ_N are $(N-2)e_0$-exponential decay functions. ψ_N generates wavelet basis of $L^2(R^n)$, that is, $\{2^{j/2}\psi_N(M^j x - \gamma)\}_{j\in \mathbf{Z}, \gamma \in \mathbf{Z}^n}$ are orthonormal basis of $L^2(R^n)$. Notice that when $n = 1$, then $M = 2$, this is the case in [4].

By using the same method, for all $d \in \mathbb{N}$, we can choose M and construct $\{\psi_i\}_{i=1}^{d-1}$ generating wavelet basis of $L^2(R^n)$. This can be done as follows.

Let $\Gamma = \mathbf{Z}^n$ and M be the following $n \times n$ Frobenius matrix,

$$M = \begin{pmatrix} 0 & 0 & \cdots & & d \\ 1 & 0 & \cdots & & 0 \\ 0 & 1 & \cdots & & 0 \\ \cdots & \cdots & \cdots & & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

Choose $\Gamma_0 = \left\{0, (1, 0, \cdots, 0)^T, \cdots, (d-1, 0, \cdots, 0)^T\right\}$.

It is easy to check the condition in theorem 2.5 (in fact $T(\Gamma_0, M) = [0, 1]^n$). Following the method used in [6], we can construct $\{\psi\}_{i=1}^{d-1}$ which are $(N-2)e_0$-exponential decay functions and generate wavelets basis of $L^2(R^n)$.

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Part IV: Miscellaneous Results on Shift-Invariant subspaces of $L^2(R^n)$

§1. Definitions and Results

This paper follows the line developed by C.de Boor, R.A. Devore and A. Ron in [2]. We mainly treat the finitely generated shift-invariant subspaces in $L^2(R^n)$.

If Φ is a subset of $L^2(R^n)$, we let $\#\Phi$ denote the number of elements in Φ and define

$$S^j(\Phi) := \text{Span}\{\phi(x - 2^{-j}\gamma) : \gamma \in \boldsymbol{Z}^n, \phi \in \Phi\}$$

with Span denoting the closed linear span.

A closed linear subspace V_0 of $L^2(R^n)$ is called shift-invariant if for any $f \in V_0$, then $f(x-\gamma) \in V_0$, $\forall \gamma \in \boldsymbol{Z}^n$. If V_0 is a shift-invariant space, we let $\text{len}(V_0) := \min \#\{\Phi : V_0 = S^0(\Phi)\}$.

Now we state our main results in this paper.

Theorem 1. *If* $f_i \in L^2(R^n)$, $1 \le i \le d$, *then* $\sup_{j\in\boldsymbol{Z}} len(S^0(\{f_i(2^j x)\}_{1\le i\le d})) = dim\ Span\{f_1(x), \cdots, f_d(x)\}$.

The following result which is associated with multiresolution analysis is first obtained by R.Q. Jia and Z.W. Shen in [3]. But here we present another way to prove it.

Theorem 2. *If* Φ *is a subset of* $L^2(R^n)$ *such that* $\#\Phi < \infty$, *then* $\cap_{j\in\boldsymbol{Z}} S^j(\{\phi(2^j x) : \phi \in \Phi\}) = \{0\}$.

If V is a shift-invariant space, then let $E(f, V)$ denote the distance between f and V. In [1], C.de Boor, R.A. Devore and A.Ron obtain an important inequality, i.e., $E(f, S^0(P_V g)) \le E(f, V) + 2E(f, S^0(g))$, $\forall f, g \in L^2(R^n)$ where V is a shift-invariant subspace of $L^2(R^n)$ and P_V denotes the orthogonal projection into V. By using this inequality, they solved a long standing question in the area of spline theory, namely under what circumstances the approximation power of a local finitely generated shift-invariant space V (that is, there exist finite number of functions with compact support which generate V) is already realized by one of its local PSI subspace (that is, such space can be generated by one function with compact support). In this paper, we shall show the following better consequence in a very simple way.

Theorem 3. *If* V *is a shift-invariant subspace of* $L^2(R^n)$, *then for any* $f, g \in L^2(R^n)$,

$$E^2(f, S^0(P_V g)) \le E^2(f, V) + E^2(f, S^0(g)).$$

§2. Proofs of the Results

To prove Theorem 1, let us first prove the following lemma.

Lemma 2.1. *If* $\Phi_k := \{\phi_{k,i}\}_{1\le i\le d}$, $k \in \boldsymbol{Z}$ *are subsets of* $L^2(R^n)$, *then* $dim \cap_{k\in\boldsymbol{Z}} S^k(\Phi_k) \le d$.

Proof. We shall show that for any $f_i \in \cap_{k\in\boldsymbol{Z}} S^k(\Phi_k)$, $1 \le i \le d+1$, then f_i, $1 \le i \le$

$d+1$ are linearly dependent. Let $\vec{f} := (f_1, \cdots, f_{d+1})^T$ and $\vec{\hat{f}} := (\hat{f}_1, \cdots, \hat{f}_{d+1})^T$. Since $f_i \in S^k(\Phi_k)$, then there exists a $2\pi\boldsymbol{Z}^n$ periodic $(d+1)\times d$ measurable matrix T_k such that

$$\vec{\hat{f}}(x) = T_k(e^{-i2^{-k}x}) \begin{pmatrix} \hat{\phi}_{k,i}(x) \\ \vdots \\ \hat{\phi}_{k,d}(x) \end{pmatrix}$$

which means that $\forall k \in \boldsymbol{Z}, \dim \mathrm{Span}\{\vec{\hat{f}}(x + 2\pi 2^k\gamma) : \gamma \in \boldsymbol{Z}^n\} \leq d \quad$ a.e. $x \in R^n$.

The above inequality gives us that $\dim \mathrm{Span}\{\vec{\hat{f}}(x + 2\pi 2^k\gamma) : \ \gamma \in \boldsymbol{Z}^n, k \in \boldsymbol{Z}\} \leq d$, a.e. $x \in R^n$(Since if $\dim \mathrm{Span}\{\vec{\hat{f}}(x + 2\pi 2^k\gamma) : \ \gamma \in \boldsymbol{Z}^n, k \in \boldsymbol{Z}\} > d$ when x is in a positive measurable subset of R^n, then there must exist $\vec{\hat{f}}(x + 2\pi 2^{k_i}\gamma_i),\ 1 \leq i \leq d+1$ such that they are linearly independent when x is in a smaller positive measurable subset of R^n. Let $k = \min(k_1, \cdots, k_{d+1})$, then $\dim \mathrm{Span}\{\vec{\hat{f}}(x + 2\pi 2^k 2^{k_i - k}\gamma_i)\}_{1 \leq i \leq d+1} = d+1$ when x is in this subset. This is a contradiction to the above inequality). Define a matrix $M(x) = (\cdots \vec{\hat{f}}(x + 2\pi 2^k\gamma) \cdots)_{\gamma \in \boldsymbol{Z}^n, k \in \boldsymbol{Z}}$. We now show that there exists a measurable vector $\vec{t}(x) = (t_1(x), \cdots, t_{d+1}(x))^T$ such that $\sum_{i=1}^{d+1} |t_i(x)|^2 = 1$ and $\vec{t}(x)^T M(x) = 0$, a.e. $x \in R^n$.

Since the set of all square submatrice of $M(x)$ is countable, let us denote its elements by $M_1(x), M_2(x), \cdots$. Now we construct $\vec{t}(x)$ in three steps.

Step 1. Letting $\sigma_0 := \{x \in R^n : \ M(x) = 0\}$, we define $\vec{t}(x) := (1, 0, \cdots, 0)^T$ when $x \in \sigma_0$. Note that if $f_i \not\equiv 0$ for some $1 \leq i \leq d+1$, then $|\sigma_0| = 0$ with $|\sigma_0|$ denoting its Lebesgue measure of σ_0.

Step 2. Let $\sigma_1 := \{x \in R^n : \ \det M_1(x) \neq 0$ and all the determinants of the bordered square matrice of $M_1(x)$ are zero at $x\}$. If $|\sigma_1| = 0$, we go to the next step, otherwise, by $\dim \mathrm{Span}\{\vec{\hat{f}}(x + 2\pi 2^k\gamma) : \ \gamma \in \boldsymbol{Z}^n, k \in \boldsymbol{Z}\} \leq d$, we have $\mathrm{ord}(M_1(x)) < d+1$. Since there are exist two reversible transformations G_1 and G_2, one on rows and the other on columns, such that

$$G_1 M(x) G_2 = \begin{pmatrix} A(x) & M_1(x) & B(x) \\ C(x) & D(x) & E(x) \end{pmatrix}.$$

Let $l_1 := \mathrm{ord}(M_1)$ and define

$$H_1(x) := \begin{pmatrix} I_{l_1} & 0 \\ -D(x)M_1^{-1}(x) & I_{d+1-l_1} \end{pmatrix}$$

then

$$H_1(x)G_1M(x)G_2 = \begin{pmatrix} A(x) & M_1(x) & B(x) \\ C(x) - D(x)M_1^{-1}(x)A(x) & 0 & E(x) - D(x)M_1^{-1}(x)B(x) \end{pmatrix}$$

By the definition of σ_1, we have that $\forall x \in \sigma_1,\ C(x) - D(x)M_1^{-1}(x)A(x) = 0$ and $E(x) - D(x)M_1^{-1}(x)B(x) = 0$. Thus if we define $\vec{t_0}(x)^T = (0, \cdots, 0, 1)H_1(x)G_1$, $\forall x \in \sigma_1$, then, by $l < d+1$, we have

$$\vec{t_0}(x)^T M(x) G_2 = (0, \cdots, 0, 1) \begin{pmatrix} A(x) & M_1(x) & B(x) \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

So we define $\vec{t}(x) = \frac{\vec{t}_0(x)}{\|\vec{t}_0(x)\|}$, $\forall x \in \sigma_1$ with the norm denoting the ordinary Euclidean norm in R^{d+1}.

Step 3. By induction, we can define $\vec{t}(x)$ on $\sigma_l = \{x \in R^n : \det M_l(x) \neq 0$ and $x \notin \cup_{i=0}^{l-1}\sigma_i$ and all the determinants of the bordered square matrice of $M_l(x)$ are zero at $x\}$. If $|\sigma_l| = 0$, we go to the next step to define $\vec{t}(x)$ on σ_{l+1}, otherwise we using the same method in step 2 to define $\vec{t}(x)$ on σ_l.

For any $x_0 \in R^n$, if $M(x_0) = 0$, then $x_0 \in \sigma_0$, otherwise there must exist $l \in \mathbb{N}$ such that $\det M_l(x_0) \neq 0$ and all the determinants of the bordered square matrice at x_0 are zero. Choose the minimum number l_0 of such l, then $x_0 \in \sigma_{l_0}$ which means that $\vec{t}(x)$ is well defined on R^n. Since $\vec{t}(x)^T M(x) = 0$ a.e. $x \in R^n$, then $< \vec{t}(x), \vec{\hat{f}}(x + 2\pi 2^k\gamma) >= 0$ $\forall k \in \boldsymbol{Z}, \gamma \in \boldsymbol{Z}^n$ a.e. $x \in R^n$. For any $y \in R^n$ there exist subsequences $k_j, \gamma_j, j \in N$ such that $\lim_{j\to\infty} 2\pi 2^{k_j}\gamma_j = y$. So

$$\begin{aligned}\int_R |\langle \vec{t}(x), \vec{\hat{f}}(x+y)\rangle|^2 &= \int_{R^n} |\langle \vec{t}(x), \vec{\hat{f}}(x+y) - \vec{\hat{f}}(x + 2\pi 2^{k_j}\gamma_j)\rangle|^2 dx \\ &\leq \sum_{i=1}^{d+1} \int_{R^n} |\hat{f}_i(x+y) - \hat{f}_i(x + 2\pi 2^{k_j}\gamma_j)|^2 dx\end{aligned}$$

letting j converge to $+\infty$, we have $\int_{R^n} |\langle \vec{t}(x), \vec{\hat{f}}(x+y)\rangle|^2 dx = 0$. Thus $\int_{R^n} dy \int_{R^n} |\langle \vec{t}(x), \vec{\hat{f}}(x+y)\rangle|^2 dx = 0$ which means that for a.e. $x_0 \in R^n$, such that $\langle \vec{t}(x_0), \vec{\hat{f}}(x_0+y)\rangle = 0$, a.e. $y \in R^n$ that is, f_i, $1 \leq i \leq d+1$ are linearly dependent. Thus we complete the proof. ■

Proof of Theorem 1. It is clear that

$$\sup_{j\in\boldsymbol{Z}} \mathrm{len}(S^0(\{f_i(2^j x)\}_{1\leq i\leq d})) \leq \dim \mathrm{Span}\{f_i(x)\}_{1\leq i\leq d}.$$

If $l = \sup_{j\in\boldsymbol{Z}} \mathrm{len}(S^0(\{f_i(2^j x)\}_{1\leq i\leq d})) < d$ and $\dim \mathrm{Span}\{f_i(x)\}_{1\leq i\leq d} = d$, then $S^0(\{f_i(2^j x)\}_{1\leq i\leq d}) = S^0(\Phi_j)$ with $\#\Phi_j \leq l$. Thus for any $1 \leq i \leq d$,

$$f_i(x) = S^0(\Phi_j)(2^{-j}x) = S^{-j}(\Phi_j(2^{-j}x)).$$

So $f_i \in \cap_{j\in\boldsymbol{Z}} S^{-j}(\Phi_j(2^{-j}x))$ for any $1 \leq i \leq d$. By Lemma 2.1, $\dim \mathrm{Span}\{f_i\}_{1\leq i\leq d}$ $\leq \dim \cap_{j\in\boldsymbol{Z}} S^{-j}(\Phi_j(2^{-j}x)) \leq l < d$. This is a contradiction. Thus we have that $\sup_{j\in\boldsymbol{Z}} \mathrm{len}(S^0(\{f_i(2^j x)\}_{1\leq i\leq d}) = \dim \mathrm{Span}\{f_i(x)\}_{1\leq i\leq d}$. ■

Remark. In the case $d = 1$, Lemma 2.1 is proved in [2] in a different way. Note that Theorem 1 also implies Lemma 2.1

Lemma 2.2. *For any $f \in L^2(R^n)$, if $f \not\equiv 0$, then*

$$dim\ Span\{f(2^j x)\}_{j\in\boldsymbol{Z}} = lenS^0(\{f(2^j x)\}_{j\in\boldsymbol{Z}}) = +\infty.$$

Proof. If the dimension of $V = \mathrm{Span}\{f(2^j x)\}_{j\in\boldsymbol{Z}}$ is finite, then define an operator $P : V \to V$, $Pg(x) = g(2x), \forall g \in V$, we know that P must have an eigenvalue λ and a nonzero eigenvector $h(x)$ such that $Ph(x) = \lambda h(x)$, i.e. $h(2x) = \lambda h(x)$. By $\|h(2x)\|_{L^2} = \|\lambda h(x)\|_{L^2}$, we get $|\lambda| = 2^{n/2}$. So $\forall\ l > 0$, letting $B_l(0)$ denote the ball centered at the origin with radius l,

$$\int_{B_l(0)} |\lambda|^2 |h(x)|^2 dx = \int_{B_l(0)} |h(2x)|^2 dx = \int_{B_{2l}(0)} 2^n |h(x)|^2 dx.$$

which means that $h \equiv 0$. This is a contradiction. So dim Span$\{f(2^j x)\}_{j\in\boldsymbol{Z}} = +\infty$. By Theorem 1, we have len$S^0(\{f(2^j x)\}_{j\in\boldsymbol{Z}}) = +\infty$. ■

Proof of Theorem 2. For any $f \in \cap_{j\in\boldsymbol{Z}} S^j(\{\phi(2^j x) : \phi \in \Phi\})$, then $f(2^{-j}x) \in S^0(\Phi), \forall j \in \boldsymbol{Z}$. Thus $S^0(\{f(2^j x)\}_{j\in\boldsymbol{Z}}) \subseteq S^0(\Phi)$, which means that len$S^0(\{f(2^j x)\}_{j\in\boldsymbol{Z}} \leq$ len$S^0(\Phi) < \infty$. By Lemma 2.2, we have $f = 0$. So $\cap_{j\in\boldsymbol{Z}} S^j(\{\phi(2^j x) : \phi \in \Phi\}) = \{0\}$. ■

Note that Theorem 2 also implies Lemma 2.2.

Proof of Theorem 3. Let W be the orthogonal complement of $S^0(V, g)$ in $L^2(R^n)$ and $S^0(P_V g)^\perp \cap V$ be the orthogonal complement of $S^0(P_V g)$ in V. It is easy to see that $L^2(R^n) = S^0(P_V g) \oplus S^0(g - P_V g) \oplus (S^0(P_V g)^\perp \cap V) \oplus W$ with $\oplus$ denoting the orthogonal sum. Thus for any $f \in L^2(R^n), f = f_1 + f_2 + f_3 + f_4$ where f_1, f_2, f_3, f_4 are in $S^0(P_V g)$, $S^0(g - P_V g)$, $S^0(P_V g)^\perp \cap V, W$ respectively. Then

$$E^2(f, S^0(P_V g)) = \|f_2\|_{L^2}^2 + \|f_3\|_{L^2}^2 + \|f_4\|_{L^2}^2$$
$$E^2(f, V) = \|f_2\|_{L^2}^2 + \|f_4\|_{L^2}^2$$
$$E^2(f, S^0(g)) \geq E^2(f, S(Pg) \oplus S(g - P_V g)) = \|f_3\|_{L^2}^2 + \|f_4\|_{L^2}^2$$

Thus

$$E^2(f, V) + E^2(f, S^0(g))$$
$$\geq \|f_2\|_{L^2}^2 + \|f_3\|_{L^2}^2 + 2\|f_4\|_{L^2}^2 \geq E^2(f, S^0(P_V g)). \quad ■$$

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