

# ORBIFOLDS AND COSETS OF MINIMAL $\mathcal{W}$ -ALGEBRAS

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ABSTRACT. Let  $\mathfrak{g}$  be a simple, finite-dimensional Lie (super)algebra, equipped with an embedding of  $\mathfrak{sl}_2$  inducing the minimal gradation on  $\mathfrak{g}$ . The corresponding minimal  $\mathcal{W}$ -algebra  $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$  introduced by Kac and Wakimoto has strong generators in weights  $1, 2, \frac{3}{2}$ , and all operator product expansions are known explicitly. The weight one subspace generates an affine vertex (super)algebra  $\mathcal{A}(\mathfrak{g}^{\natural})$  where  $\mathfrak{g}^{\natural} \subset \mathfrak{g}$  denotes the centralizer of  $\mathfrak{sl}_2$ . Therefore  $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$  has an action of a connected Lie group  $G_0^{\natural}$  with Lie algebra  $\mathfrak{g}_0^{\natural}$ , where  $\mathfrak{g}_0^{\natural}$  denotes the even part of  $\mathfrak{g}^{\natural}$ . We show that for any reductive subgroup  $G \subset G_0^{\natural}$ , and for any reductive Lie algebra  $\mathfrak{g}' \subset \mathfrak{g}_0^{\natural}$ , the orbifold  $\mathcal{O}^k = \mathcal{W}^k(\mathfrak{g}, e_{-\theta})^G$  and the coset  $\mathcal{C}^k = \text{Com}(\mathcal{A}(\mathfrak{g}'), \mathcal{W}^k(\mathfrak{g}, e_{-\theta}))$  are strongly finitely generated for generic values of  $k$ . Here  $\mathcal{A}(\mathfrak{g}') \subset \mathcal{A}(\mathfrak{g}^{\natural})$  denotes the affine vertex algebra associated to  $\mathfrak{g}'$ . In the case  $\mathfrak{g} = \mathfrak{sl}_n$  and  $\mathfrak{g}' = \mathfrak{g}^{\natural} = \mathfrak{gl}_{n-2}$ , we show that  $\mathcal{C}^k$  is of type  $\mathcal{W}(2, 3, \dots, n^2 - 2)$  for generic values of  $k$ . Similarly, for  $\mathfrak{g} = \mathfrak{sp}_{2n}$  and  $\mathfrak{g}' = \mathfrak{g}^{\natural} = \mathfrak{sp}_{2n-2}$ , we show that  $\mathcal{C}^k$  is generically of type  $\mathcal{W}(2, 4, \dots, 2n^2 + 2n - 2)$ . Finally, we use these results to study cosets of the simple quotient  $\mathcal{W}_k(\mathfrak{g}, e_{-\theta})$  for nongeneric values of  $k$ .

## 1. INTRODUCTION

### **todo: Some general intro**

There are three closely related motivations to study cosets of minimal  $\mathcal{W}$ -algebras: Firstly a better understanding of the connections and coincidences between different types of  $\mathcal{W}$ -algebras; secondly improving and generalizing the theory of cosets of affine VOAs inside larger structures as well as employing the findings for representation theory and thirdly finding new  $C_2$ -cofinite but non-rational VOAs. In the following we will outline these three aspects.

**The zoo of  $\mathcal{W}$ -algebras. To do: explain on known results and how simple quotients of  $\mathcal{W}$ -algebras with associated variety of small dimension are rare.**

**The theory of cosets of affine VOAs inside larger structures.** Let  $\mathcal{A}^k$  be a family of vertex operator algebras parameterized by the parameter  $k$  in  $\mathbb{C}$ . We require  $\mathcal{A}^k$  to contain a universal affine VOA  $V^k(\mathfrak{g})$  of some reductive Lie algebra  $\mathfrak{g}$  at level  $k$  as sub VOA. A natural question is what is the structure of the commutant or coset sub VOA

$$\mathcal{C}^k := \text{Com}(V^k(\mathfrak{g}), \mathcal{A}^k).$$

This problem can be formulated and answered using the language of deformable family of VOAs. A deformable family  $\mathcal{A}$  of VOAs associated to  $\mathcal{A}^k$  is roughly speaking a VOA over the ring  $F_K$ . Here  $F_K$  are rational functions in a variable  $\kappa$  of degree at most zero and

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with possible poles only at the subset  $K \subset \mathbb{C}$ . The subset  $K$  is at most countable. One then requires that  $\mathcal{A}/(\kappa - k) \cong \mathcal{A}^k$ . This notion is due to [CLI, CLII] and it allows to answer the question of strong generators of  $\mathcal{C}^k$  for generic  $k$  by passing to the limit  $k \rightarrow \infty$  and identifying the limit with an orbifold problem that can often be solved using the orbifold theory of free field algebras [LI, LII, LIII, LIV, CLIII].

Presently this framework requires  $\mathcal{A}^k$  to be a tensor product of affine VOAs and free field algebras and the first objective of this project is to also incorporate minimal  $\mathcal{W}$ - (super) algebras.

In general we believe that the set of non-generic points is small and doesnot contain any positive rational numbers. Non-generic points are complex numbers  $k$  such that  $\mathcal{C}^k \not\cong \mathcal{C}/(\kappa - k)$ . Unfortunately this is difficult to establish and so a second objective is to determine this set of non-generic points in two families.

Determining explicit strong generating sets of families of orbifold and coset VOAs is the VOA analog of Weyl's invariant theory for the classical groups [W]. Finding results have the benefit that they might support various conjectures of the physics literature. Especially conjectured coset realizations of the principal  $\mathcal{W}$ -super algebras of  $\mathfrak{sl}_n$ ,  $\mathfrak{osp}(2n+1|2n)$ ,  $\mathfrak{sl}(n+1|n)$  are the VOAs of the higher spin gravity to CFT correspondence [] that has received a lot of attention recently. We plan to prove some of these conjectures in the near future.

We think of answering the strong generator question also as a tool to conjecture coincidences between simple quotients of special (usually positive and rational) values of  $k$ . The underlying idea is that if the associated variety of  $\mathcal{A}_k$  is fairly low-dimensional then the same is true for  $\mathcal{C}_k$  but VOAs of a given type with low-dimensional associated variety are rare. If one then in addition has a pattern of coincidences of central charges then we take this as enough information to formulate a conjecture. Such conjectures can then either be proven by direct OPE computation or by proving a uniqueness theorem for OPE-algebra. For VOAs of minimal  $\mathcal{W}$ -algebra we are indeed able to state such a uniqueness theorem and hence if we can prove that  $\mathcal{C}_k \otimes L_k(\mathfrak{g})$  allows for a VOA extension that is of minimal  $\mathcal{W}$ -algebra type that is simple then this provides a method to prove coincidences. This is one of the many purposes to establish an effective theory of VOA extensions [?, ?]. Furthermore, once having  $\mathcal{A}_k$  as an extension of  $\mathcal{C}_k \otimes L_k(\mathfrak{g})$  one can use the representation theory of VOA extensions to understand the one of  $\mathcal{A}_k$  provided the one of  $\mathcal{C}_k \otimes L_k(\mathfrak{g})$  is known.

We thus think of the theory of affine VOAs inside larger structure to have the following stepwise procedure to understand exceptional simple quotients:

- (Step 1) Find the generic strong generating set of a deformable family of coset VOAs;
- (Step 2) Use the findings together with coincidences of central charges to conjecture coincidences;
- (Step 3) Use either the computer or the theory of VOA extensions to prove the coincidence;
- (Step 4) If the representation theory of either  $\mathcal{A}_k$  or  $\mathcal{C}_k \otimes L_k(\mathfrak{g})$  is sufficiently well understood then use the theory of VOA extensions to understand (at least parts) of the representation theory of the other VOA.

The idea of Step 3 has succesfully be employed in the case of the rational Bershadsky-Polyakov algebras [ArI] as extensions of lattice VOAs times appropriate simple rational

type  $A$  principal  $\mathcal{W}$ -algebras [ACL]. The proof used the theory of simple current extensions [?]. The idea of Step 4 works impressively well for  $L_k(\mathfrak{osp}(1|2))$  as a fairly non-trivial extension of  $L_k(\mathfrak{sl}_2)$  times a rational Virasoro VOA []. Much easier accessible examples are problems involving cosets of lattice VOAs [CKLR, ?].

**Constructing new  $C_2$ -cofinite but non-rational VOAs.** The representation category of grading restricted modules of a simple, CFT-type and  $C_2$ -cofinite VOA that is its own contragredient dual is expected to be a log-modular tensor category [?, ?]. Similar to its rational cousin this comes with a close and powerful relation of the modularity of (pseudo)-trace functions [?] and (logarithmic)-Hopf links [?, ?]. In order to understand this relation better and to arrive at a conjecture one might eventually be able to prove one needs examples of interesting  $C_2$ -cofinite but non-rational VOAs.

We will now outline how minimal  $\mathcal{W}$ -algebra cosets provide suitable candidates for the construction of new  $C_2$ -cofinite VOAs.

Consider  $\mathcal{W}_k(\mathfrak{g})$  with  $\mathfrak{g}^\sharp = \mathfrak{gl}_1 \oplus \mathfrak{a}$  for some Lie algebra  $\mathfrak{a}$ . Let  $\mathcal{V}_{k'}(\mathfrak{g}^\sharp)$  the affine subVOA of  $\mathcal{W}_k(\mathfrak{g})$ . Then the coset satisfies

$$\mathcal{C}_k = \text{Com}(\mathcal{V}_{k'}(\mathfrak{g}^\sharp), \mathcal{W}_k(\mathfrak{g})) = \text{Com}(\mathcal{H}(1), \text{Com}(\mathcal{V}_{k'}(\mathfrak{a}), \mathcal{W}_k(\mathfrak{g}))).$$

Assume that the tensor category of grading restricted modules of  $\mathcal{C}_k$  is a vertex tensor category in the sense of [?]. Then the following statements of [CKLR, ?, ?] hold:

(1)

$$\text{Com}(\mathcal{V}_{k'}(\mathfrak{a}), \mathcal{W}_k(\mathfrak{g})) = \bigoplus_{\lambda \in L} \mathcal{F}_\lambda \otimes \mathcal{C}_\lambda$$

where  $L$  is a rank one lattice with corresponding generalized lattice VOA

$$V_L = \bigoplus_{\lambda \in L} \mathcal{F}_\lambda$$

the  $\mathcal{F}_\lambda$  are Fock modules of the rank one Heisenberg VOA  $\mathcal{H}(1)$ . the  $\mathcal{C}_\lambda$  are simple currents and above decomposition is as  $\mathcal{H}(1) \otimes \mathcal{C}_k$ -modules.

(2) Let  $N \subset L$  be an even sublattice so that  $V_N = \bigoplus_{\lambda \in N} \mathcal{F}_\lambda$  is a lattice VOA, then

$$\mathcal{D}_k = \bigoplus_{\lambda \in N} \mathcal{C}_\lambda$$

is a VOA.

(3) Every module of  $\mathcal{D}_k$  is an induced module and an indecomposable module  $X$  of  $\mathcal{C}_k$  lifts to a  $\mathcal{D}_k$ -module if and only if there exists  $\lambda \in N'$  such that  $\mathcal{F}_\lambda \otimes X$  is contained in a  $\text{Com}(\mathcal{V}_{k'}(\mathfrak{a}), \mathcal{W}_k(\mathfrak{g}))$ -module.

Sloppily phrased only very few modules of  $\mathcal{C}_k$  lift to  $\mathcal{D}_k$ -modules. We conjecture

**Conjecture 1.1.** *Let  $\mathcal{W}_k(\mathfrak{g})$  be a minimal  $\mathcal{W}$ -super algebra with one-dimensional associated variety and let  $\mathcal{C}_k$  be the coset with the weight one affine sub VOA. Then there is an infinite order simple current extension  $\mathcal{D}_k$  of  $\mathcal{C}_k$  with zero-dimensional associated variety.*

It is difficult to show that the complete category of grading restricted modules of a VOA is a vertex tensor category and an alternative route for this conjecture might be to even prove  $C_2$ -cofiniteness. For some progress on this idea in the context of admissible level  $L_k(\mathfrak{sl}_2)$  parafermions see [?].

**Results. todo: state the results and say how they can be used for both constructing potentially new  $C_2$ -cofinite VOAs and understanding the representation theory of few but interesting rational minimal  $W$ -algebras.**

## 2. MINIMAL $\mathcal{W}$ -ALGEBRAS

Let  $\mathfrak{g}$  be a simple, finite-dimensional Lie (super)algebra. We recall some basic properties of the minimal  $\mathcal{W}$ -algebra  $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$  which was introduced by Kac and Wakimoto in [KWI]. First, suppose that  $\{e, f, h\}$  is an  $\mathfrak{sl}_2$ -triple in  $\mathfrak{g}$  inducing the minimal  $\frac{1}{2}\mathbb{Z}$ -gradation

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1.$$

Here  $\mathfrak{g}_{-1} = \mathbb{C}f$  and  $\mathfrak{g}_1 = \mathbb{C}e$ , and the above decomposition is the eigenvalue decomposition with respect to  $\text{ad}(h)$  where  $h = [e, f]$ . Define  $\mathfrak{g}^\natural$  to be the centralizer of  $\{e, f, h\}$  in  $\mathfrak{g}$ . Recall that  $\mathfrak{g}_{-1/2}$  and  $\mathfrak{g}_{1/2}$  are isomorphic as  $\mathfrak{g}^\natural$ -modules, and that  $\mathfrak{g}_{1/2}$  possesses a nondegenerate, skew-supersymmetric bilinear form  $\langle \cdot, \cdot \rangle_{\text{ne}}$  given by

$$\langle a, b \rangle_{\text{ne}} = (f|[a, b]),$$

where  $(\cdot|\cdot)$  denotes the bilinear form on  $\mathfrak{g}$ .

By Theorem 5.1 of [KWI],  $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$  is strongly generated by a Virasoro field  $L$ , primary weight one fields  $J^a$ ,  $a \in \mathfrak{g}^\natural$ , which generate an affine vertex (super)algebra of  $\mathfrak{g}^\natural$ , and primary weight  $\frac{3}{2}$  fields  $G^u$ ,  $u \in \mathfrak{g}_{-1/2}$ . Moreover, all OPE relations are given explicitly in this theorem. A number of well-known vertex algebras can be realized in this way, such as the Virasoro algebra,  $N = 2$  superconformal algebra, small  $N = 4$  superconformal algebra, Bershadsky-Polyakov algebra, etc.

The even part  $\mathfrak{g}_0^\natural \subset \mathfrak{g}^\natural$  is reductive and  $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$  decomposes into a sum of finite-dimensional  $\mathfrak{g}_0^\natural$ -modules. Therefore the action of  $\mathfrak{g}_0^\natural$  lifts to an action of a connected Lie group  $G_0^\natural$  on  $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$  by automorphisms. In this paper, we are interested in the structure of orbifolds  $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})^G$  where  $G \subset G_0^\natural$  is a reductive subgroup, and cosets  $\text{Com}(\mathcal{A}(\mathfrak{g}'), \mathcal{W}^k(\mathfrak{g}, e_{-\theta}))$  where  $\mathcal{A}(\mathfrak{g}') \subset V^{k'}(\mathfrak{g}^\natural)$  is the affine vertex algebra corresponding to a reductive Lie algebra  $\mathfrak{g}' \subset \mathfrak{g}^\natural$ .

**Uniqueness of minimal  $\mathcal{W}$ -algebras.** Let  $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$  be a universal minimal  $\mathcal{W}$ -algebra with affine subalgebra  $V^{k'}(\mathfrak{g}^\natural)$ , as above. Let  $\mathcal{A}_k$  be another vertex algebra with the following properties:

- (1) The central charges of  $\mathcal{W}^k(\mathfrak{g})$  and  $\mathcal{A}_k$  coincide
- (2) The full affine subVOA of  $\mathcal{A}_k$  is a homomorphic image of  $V^{k'}(\mathfrak{g}^\natural)$ .
- (3)  $\mathcal{A}_k$  is strongly generated by the Virasoro field, the weight one and the weight  $3/2$  fields and the number of weight  $3/2$  strong generators of  $\mathcal{A}_k$  and  $\mathcal{W}^k(\mathfrak{g})$  are the same and have the same OPE algebra with the weight one fields (i.e. they carry the same representation under the action of the zero-modes of the weight one fields).

I then claim that the OPE algebra of  $\mathcal{A}_k$  and  $\mathcal{W}^k(\mathfrak{g})$  differ only by fields that are null for  $V^{k'}(\mathfrak{g}^\natural)$ . A field is called null if the OPE with no field contains the identity.

The reason is as follows. Let  $X, Y$  be two dimension  $3/2$  fields of  $\mathcal{W}^k(\mathfrak{g})$  and  $\tilde{X}, \tilde{Y}$  the corresponding fields of  $\mathcal{A}_k$ . We can normalize the dimension  $3/2$  fields of  $\mathcal{A}_k$  such that the

coefficient coming with the identity coincides with the corresponding OPE coefficient of  $X, Y$ , that is

$$X(z)Y(w) \sim (z-w)^{-3}a(X, Y) + \dots \quad \rightarrow \quad \tilde{X}(z)\tilde{Y}(w) \sim (z-w)^{-3}a(X, Y) + \dots$$

Now, in the OPE of  $X, Y$  the Virasoro field as well as dimension one and dimension two fields of  $V_{k'}(\mathfrak{g}^\sharp)$  appear in addition to the constant term. We restrict to  $c \neq 0$  so that the Virasoro field is not null. The set of null fields is a subspace of such fields. Extend a basis of null fields to a basis of dimension one and dimension two fields of  $V_{k'}(\mathfrak{g}^\sharp)$ . Let  $K_i, i \in I$  for some finite index set  $I$  denote the basis element that are not null and let  $J_i$  be dual elements that is fields in  $V_{k'}(\mathfrak{g}^\sharp)$  that have the property that the OPE of  $K_i$  with  $J_j$  contains the identity if and only if  $i = j$ . Fields that are not null cannot be fields of a proper ideal as the identity would be in such an ideal. Let  $\tilde{K}_i$  and  $\tilde{J}_i$  be the homomorphic images of  $K_i$  and  $J_i$  in the affine subVOA of  $\mathcal{A}_k$ .

Let  $\lambda_i$  be the coefficient with which  $K_i$  appears in the OPE of  $X$  with  $Y$  and let  $\lambda_i + \mu_i$  be the coefficient with which  $\tilde{K}_i$  appears in the OPE of  $\tilde{X}$  with  $\tilde{Y}$ . The Jacobi identity for the modes of  $X, Y, J_i$  differs from the Jacobi identity for the same modes of  $\tilde{X}, \tilde{Y}, \tilde{J}_i$  by  $\mu_i$  times a non-zero constant coefficient. This last argument is word by word analogous to the final argument of the proof of Lemma 8.2 of [ACL].

We have thus proven

**Theorem 2.1.** *Let  $\mathcal{W}^k(\mathfrak{g}) \mathcal{A}_k$  as above then the OPE algebra of the strong generators of  $\mathcal{W}_k(\mathfrak{g})$  and  $\mathcal{A}_k$  coincides up to null fields. In other words  $\mathcal{A}_k$  is a homomorphic image of  $\mathcal{W}^k(\mathfrak{g})$ . In particular if  $\mathcal{A}_k$  is simple then  $\mathcal{A}_k \cong \mathcal{W}_k(\mathfrak{g})$ .*

**Weak increasing filtrations.** Recall from [Lil] that a *good increasing filtration* on a vertex algebra  $\mathcal{A}$  is a  $\mathbb{Z}_{\geq 0}$ -filtration

$$(2.1) \quad \mathcal{A}_{(0)} \subset \mathcal{A}_{(1)} \subset \mathcal{A}_{(2)} \subset \dots, \quad \mathcal{A} = \bigcup_{d \geq 0} \mathcal{A}_{(d)}$$

such that  $\mathcal{A}_{(0)} = \mathbb{C}$ , and for all  $a \in \mathcal{A}_{(r)}, b \in \mathcal{A}_{(s)}$ , we have

$$(2.2) \quad a \circ_n b \in \begin{cases} \mathcal{A}_{(r+s)} & n < 0 \\ \mathcal{A}_{(r+s-1)} & n \geq 0 \end{cases}.$$

We set  $\mathcal{A}_{(-1)} = \{0\}$ , and we say that  $a(z) \in \mathcal{A}_{(d)} \setminus \mathcal{A}_{(d-1)}$  has degree  $d$ . The key property of such filtrations is that the associated graded algebra  $\text{gr}(\mathcal{A}) = \bigoplus_{d \geq 0} \mathcal{A}_{(d)} / \mathcal{A}_{(d-1)}$  is a  $\mathbb{Z}_{\geq 0}$ -graded associative, (super)commutative algebra with a unit 1 under a product induced by the Wick product on  $\mathcal{A}$ . For  $r \geq 1$  we have the projection

$$(2.3) \quad \varphi_r : \mathcal{A}_{(r)} \rightarrow \mathcal{A}_{(r)} / \mathcal{A}_{(r-1)} \subset \text{gr}(\mathcal{A}).$$

The assignment  $\mathcal{A} \mapsto \text{gr}(\mathcal{A})$  is a functor from  $\mathcal{R}$  to the category of  $\mathbb{Z}_{\geq 0}$ -graded (super)commutative rings with a differential  $\partial$  of degree zero, which we call  $\partial$ -rings. A  $\partial$ -ring is just an *abelian* vertex algebra, that is, a vertex algebra  $\mathcal{V}$  in which  $[a(z), b(w)] = 0$  for all  $a, b \in \mathcal{V}$ . A  $\partial$ -ring  $\mathcal{A}$  is said to be generated by a set  $\{a_i | i \in I\}$  if  $\{\partial^k a_i | i \in I, k \geq 0\}$  generates  $\mathcal{A}$  as a ring. The key feature of  $\mathcal{R}$  is the following reconstruction property. Let  $\mathcal{A}$  be a vertex algebra in  $\mathcal{R}$  and let  $\{a_i | i \in I\}$  be a set of generators for  $\text{gr}(\mathcal{A})$  as a  $\partial$ -ring, where  $a_i$  is homogeneous of degree  $d_i$ . If  $a_i(z) \in \mathcal{A}_{(d_i)}$  are elements satisfying  $\varphi_{d_i}(a_i(z)) = a_i$ , then  $\mathcal{A}$  is strongly generated as a vertex algebra by  $\{a_i(z) | i \in I\}$ .

In [ACL], a *weak increasing filtration* on a vertex algebra  $\mathcal{A}$  was defined to be a  $\mathbb{Z}_{\geq 0}$ -filtration

$$(2.4) \quad \mathcal{A}_{(0)} \subset \mathcal{A}_{(1)} \subset \mathcal{A}_{(2)} \subset \cdots, \quad \mathcal{A} = \bigcup_{d \geq 0} \mathcal{A}_{(d)}$$

such that for  $a \in \mathcal{A}_{(r)}$ ,  $b \in \mathcal{A}_{(s)}$ , we have

$$(2.5) \quad a \circ_n b \in \mathcal{A}_{r+s}, \quad n \in \mathbb{Z}.$$

This condition guarantees that  $\text{gr}(\mathcal{A}) = \bigoplus_{d \geq 0} \mathcal{A}_{(d)} / \mathcal{A}_{(d-1)}$  is a vertex algebra, but it is no longer abelian in general. As above, a strong generating set for  $\text{gr}(\mathcal{A})$  consisting of homogeneous elements always gives rise to a strong generating set for  $\mathcal{A}$ .

We define an increasing filtration

$$\mathcal{W}_{(0)}^k \subset \mathcal{W}_{(1)}^k \subset \cdots$$

on  $\mathcal{W}^k = \mathcal{W}^k(\mathfrak{g}, e_{-\theta})$  as follows:  $\mathcal{W}_{(-1)}^k = \{0\}$ , and  $\mathcal{W}_{(r)}^k$  is spanned by iterated Wick products of  $T, J^a, G^u$  and their derivatives, such that at most  $r$  of the fields  $G^u$  and their derivatives appear. It is clear from the defining OPE relations that this is a weak increasing filtration. Note that  $\mathcal{W}_{(0)}^k \cong \text{Vir} \ltimes V^{k'}(\mathfrak{g}^{\natural})$  where  $\text{Vir}$  denotes the Virasoro vertex algebra with generator  $T$ . Note also that the associated graded algebra

$$\text{gr}(\mathcal{W}^l) = \bigoplus_{d \geq 0} \mathcal{W}_{(d)}^k / \mathcal{W}_{(d-1)}^k$$

is not abelian since it contains  $\mathcal{W}_{(0)}$  as a subalgebra, but  $G^u(z)G^v(w) \sim 0$  in this algebra for all  $u, v$ .

### 3. DEFORMATIONS AND LIMITS OF MINIMAL $\mathcal{W}$ -ALGEBRAS

In this section, we shall adapt the methods of our previous papers on orbifolds and cosets of free field and affine vertex algebras to the setting of minimal  $\mathcal{W}$ -algebras.

First, recall the notion of a *deformable family* of vertex algebras [CLI]. Let  $K \subset \mathbb{C}$  be a subset which is at most countable, and let  $F_K$  denote the  $\mathbb{C}$ -algebra of rational functions in a formal variable  $\kappa$  of the form  $\frac{p(\kappa)}{q(\kappa)}$  where  $\deg(p) \leq \deg(q)$  and the roots of  $q$  lie in  $K$ . A *deformable family* will be a free  $F_K$ -module  $\mathcal{B}$  with the structure of a vertex algebra with coefficients in  $F_K$ . Vertex algebras over  $F_K$  are defined in the same way as ordinary vertex algebras over  $\mathbb{C}$ . We assume that  $\mathcal{B}$  possesses a  $\mathbb{Z}_{\geq 0}$ -grading  $\mathcal{B} = \bigoplus_{m \geq 0} \mathcal{B}[m]$  by conformal weight where each  $\mathcal{B}[m]$  is free  $F_K$ -module of finite rank. For  $k \notin K$ , we have a vertex algebra

$$\mathcal{B}_k = \mathcal{B} / (\kappa - k),$$

where  $(\kappa - k)$  is the ideal generated by  $\kappa - k$ . Clearly  $\dim_{\mathbb{C}}(\mathcal{B}_k[m]) = \text{rank}_{F_K}(\mathcal{B}[m])$  for all  $k \notin K$  and  $m \geq 0$ . We have a vertex algebra  $\mathcal{B}_{\infty} = \lim_{\kappa \rightarrow \infty} \mathcal{B}$  with basis  $\{\alpha_i \mid i \in I\}$ , where  $\{a_i \mid i \in I\}$  is any basis of  $\mathcal{B}$  over  $F_K$ , and  $\alpha_i = \lim_{\kappa \rightarrow \infty} a_i$ . By construction,  $\dim_{\mathbb{C}}(\mathcal{B}_{\infty}[m]) = \text{rank}_{F_K}(\mathcal{B}[m])$  for all  $m \geq 0$ . The vertex algebra structure on  $\mathcal{B}_{\infty}$  is defined by

$$(3.1) \quad \alpha_i \circ_n \alpha_j = \lim_{\kappa \rightarrow \infty} a_i \circ_n a_j, \quad i, j \in I, \quad n \in \mathbb{Z}.$$

The  $F_K$ -linear map  $\varphi : \mathcal{B} \rightarrow \mathcal{B}_{\infty}$  sending  $a_i \mapsto \alpha_i$  satisfies

$$(3.2) \quad \varphi(\omega \circ_n \nu) = \varphi(\omega) \circ_n \varphi(\nu), \quad \omega, \nu \in \mathcal{B}, \quad n \in \mathbb{Z}.$$

**Lemma 3.1** ([CLI], Lemma 8.1). *Let  $K \subset \mathbb{C}$  be at most countable, and let  $\mathcal{B}$  be a vertex algebra over  $F_K$  as above. Let  $U = \{\alpha_i \mid i \in I\}$  be a strong generating set for  $\mathcal{B}_\infty$ , and let  $T = \{a_i \mid i \in I\}$  be the corresponding subset of  $\mathcal{B}$ , so that  $\varphi(a_i) = \alpha_i$ . There exists a subset  $S \subset \mathbb{C}$  containing  $K$  which is at most countable, such that  $F_S \otimes_{F_K} \mathcal{B}$  is strongly generated by  $T$ . Here we have identified  $T$  with the set  $\{1 \otimes a_i \mid i \in I\} \subset F_S \otimes_{F_K} \mathcal{B}$ .*

**Corollary 3.2.** *For  $k \notin S$ , the vertex algebra  $\mathcal{B}_k = \mathcal{B}/(\kappa - k)$  is strongly generated by the image of  $T$  under the map  $\mathcal{B} \rightarrow \mathcal{B}_k$ .*

If  $U$  is a *minimal* strong generating set for  $\mathcal{B}_\infty$  it is not clear in general that  $T$  is a minimal strong generating set for  $\mathcal{B}$ , since there may exist relations of the form  $\lambda(k)\alpha_j = P$ , where  $P$  is a normally ordered polynomial in  $\{\alpha_i \mid i \in I, i \neq k\}$  and  $\lim_{k \rightarrow \infty} \lambda(k) = 0$ , although  $\lim_{k \rightarrow \infty} P$  is nontrivial. However, there is one condition which holds in many examples, under which  $T$  is a minimal strong generating set for  $\mathcal{B}$ .

**Proposition 3.3** ([CLII], Proposition 3.4). *Suppose that  $U = \{\alpha_i \mid i \in I\}$  is a minimal strong generating set for  $\mathcal{B}_\infty$  such that  $wt(\alpha_i) < N$  for all  $i \in I$ . If there are no normally ordered polynomial relations among  $\{\alpha_i \mid i \in I\}$  and their derivatives of weight less than  $N$ , the corresponding set  $T = \{a_i \mid i \in I\}$  is a minimal strong generating set for  $\mathcal{B}$ .*

In our main example  $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$ , suppose that

$$\dim(\mathfrak{g}_0^{\natural}) = n, \quad \dim(\mathfrak{g}_1^{\natural}) = 2m, \quad \dim(\mathfrak{g}_{-1/2}^{\text{ev}}) = 2r, \quad \dim(\mathfrak{g}_{-1/2}^{\text{odd}}) = s.$$

If we replace the generators  $J^a$ ,  $L$ , and  $G^u$  with  $\tilde{J}^a = \frac{1}{\sqrt{k}}J^a$ ,  $\tilde{L} = \frac{1}{\sqrt{k}}L$ , and  $\tilde{G}^u = \frac{1}{k}G^u$ , respectively, then it follows from Theorem 5.1 of [KWI] that all structure constants are rational functions of  $\sqrt{k}$  of degree at most zero. Moreover, the only structure constants which have degree zero as rational functions of  $\sqrt{k}$  are the following.

- (1) The second-order pole of  $\tilde{J}^a(z)\tilde{J}^b(w)$  when  $(a|b) \neq 0$ .
- (2) The third-order pole of  $\tilde{G}^u(z)\tilde{G}^v(w)$  when  $\langle u, v \rangle_{\text{ne}} \neq 0$ .
- (3) The fourth-order pole of  $\tilde{L}(z)\tilde{L}(w)$ .

Let  $\kappa$  be a formal variable satisfying  $\kappa^2 = k$ , and let  $F = F_K$  for  $K = \{0\}$ . Let  $\mathcal{W}$  be the vertex algebra with coefficients in  $F$  which is freely generated by  $\bar{J}^a, \bar{L}, \bar{G}^u$  satisfying the rescaled OPE relations. It follows that for  $k \neq 0$ , we have a surjective vertex algebra homomorphism

$$\mathcal{W} \rightarrow \mathcal{W}^k(\mathfrak{g}, e_{-\theta}), \quad \bar{J}^a \mapsto \frac{1}{\sqrt{k}}J^a, \quad \bar{L} \mapsto \frac{1}{\sqrt{k}}L, \quad \bar{G}^u \mapsto \frac{1}{k}G^u,$$

whose kernel is the ideal  $(\kappa - \sqrt{k})$ . Then  $\mathcal{W}^k(\mathfrak{g}, e_{-\theta}) \cong \mathcal{W}/(\kappa - \sqrt{k})$ . The limit

$$\mathcal{W}^\infty = \lim_{\kappa \rightarrow \infty} \mathcal{W} \cong \mathcal{H}(n) \otimes \mathcal{A}(m) \otimes \mathcal{T} \otimes \mathcal{G}_{\text{ev}}(r) \otimes \mathcal{G}_{\text{odd}}(s).$$

In this notation,

- (1)  $\mathcal{H}(n)$  is the rank  $n$  Heisenberg algebra, which has even generators  $\alpha^i, i = 1, \dots, n$ , and OPE relations

$$\alpha^i(z)\alpha^j(w) \sim \delta_{i,j}(z-w)^{-2}.$$

- (2)  $\mathcal{A}(m)$  is the rank  $m$  symplectic fermion algebra, which has odd generators  $e^i, f^i, i = 1, \dots, m$ , and operator products

$$e^i(z)f^j(w) \sim \delta_{i,j}(z-w)^{-2}.$$

(3)  $\mathcal{T}$  is a free field algebra with even generator  $t(z)$  in weight two satisfying

$$t(z)t(w) \sim (z - w)^{-4}.$$

(4)  $\mathcal{G}_{\text{ev}}(r)$  is a free field algebra with even generators  $a^i, b^i, i = 1, \dots, r$ , and operator products

$$a^i(z)b^j(w) \sim \delta_{i,j}(z - w)^{-3}.$$

(5)  $\mathcal{G}_{\text{odd}}(s)$  is a free field algebra with odd generators  $\varphi^i, i = 1, \dots, s$ , and operator products

$$\varphi^i(z)\varphi^j(w) \sim \delta_{i,j}(z - w)^{-3}.$$

Note that  $\mathcal{T}$ ,  $\mathcal{G}_{\text{ev}}(r)$ , and  $\mathcal{G}_{\text{odd}}(s)$  do not possess conformal vectors. However, they have quasi-conformal structures [FBZ] such that  $t$  has weight 2, and  $a^i, b^i, \varphi^i$  have weight  $\frac{3}{2}$ . To see this, note that  $\mathcal{T}$  can be regarded as the subalgebra of the rank one Heisenberg algebra generated by the derivative of the generator. Similarly,  $\mathcal{G}_{\text{ev}}(r)$  can be regarded as the subalgebra of the rank  $r$   $\beta\gamma$ -system generated by the derivatives of the generators. Finally,  $\mathcal{G}_{\text{odd}}(s)$  can be regarded as the subalgebra of the rank  $s$  free fermion algebra generated by the derivatives of the generators. The quasi-conformal structures on  $\mathcal{T}$ ,  $\mathcal{G}_{\text{ev}}(r)$ , and  $\mathcal{G}_{\text{odd}}(s)$  are then inherited from the conformal structures on the Heisenberg algebra,  $\beta\gamma$ -system, and free fermion algebra, respectively.

The full automorphism groups of  $\mathcal{G}_{\text{ev}}(r)$  and  $\mathcal{G}_{\text{odd}}(s)$  which preserves the weight gradation are the symplectic group  $\text{Sp}(2r)$  and the orthogonal group  $\text{O}(s)$ , respectively. It is necessary to understand the structure of orbifolds of  $\mathcal{G}_{\text{ev}}(r)$  and  $\mathcal{G}_{\text{odd}}(s)$  under arbitrary reductive groups. As in our previous studies of orbifolds of free field algebras, we first need to describe  $\mathcal{G}_{\text{ev}}(r)^{\text{Sp}(2r)}$  and  $\mathcal{G}_{\text{odd}}(s)^{\text{O}(s)}$ .

**Theorem 3.4.**  $\mathcal{G}_{\text{ev}}(r)^{\text{Sp}(2r)}$  has a minimal strong generating set

$$\omega^{2j+1} = \frac{1}{2} \sum_{i=1}^r ( : a^i \partial^{2j+1} b^i : - : (\partial^{2j+1} a^i) b^i : ), \quad 0 \leq j \leq r^2 + 3r - 1,$$

and is therefore of type  $\mathcal{W}(4, 6, \dots, 2r^2 + 6r + 2)$ . Moreover,  $\mathcal{G}_{\text{ev}}(r)$  is completely reducible as a  $\mathcal{G}_{\text{ev}}(r)^{\text{Sp}(2r)}$ -module, and all irreducible modules in this decomposition are highest-weight and  $C_1$ -cofinite according to Miyamoto's definition [Mi].

*Proof.* The first statement can be reduced to showing that, in the notation of Equation 9.1 of [LV],  $R_r(I) \neq 0$  for  $I = (1, 2, \dots, 2r + 2)$ . The explicit formula for  $R_r(I)$  is given by Theorem 4 of [LV], and it is clear that it is nonzero. The Zhu algebra of  $\mathcal{G}_{\text{ev}}(r)^{\text{Sp}(2r)}$  is abelian, which implies that the admissible irreducible modules are highest-weight. The proof of  $C_1$ -cofiniteness is the same as the proof of Lemma 8 of [LII].  $\square$

**Corollary 3.5.** For any reductive group  $G \subset \text{Sp}(2r)$ ,  $\mathcal{G}_{\text{ev}}(r)^G$  is strongly finitely generated.

*Proof.* This is the same as the proof of Theorem 15 of [LV]. First,  $\mathcal{G}_{\text{ev}}(r)^G$  is completely reducible as a  $\mathcal{G}_{\text{ev}}(r)^{\text{Sp}(2r)}$ -module. By a classical theorem of Weyl (Theorem 2.5A of [W]), it has an (infinite) strong generating set that lies in the sum of finitely many irreducible  $\mathcal{G}_{\text{ev}}(r)^{\text{Sp}(2r)}$ -modules. The result then follows from the strong finite generation of  $\mathcal{G}_{\text{ev}}(r)^{\text{Sp}(2r)}$  and the  $C_1$ -cofiniteness of these modules.  $\square$

There is one case where an explicit description of  $\mathcal{G}_{ev}(r)^G$  will be important to us, namely,  $G = \mathrm{GL}(r) \subset \mathrm{Sp}(2r)$ , where  $\{a^i\}$  spans a copy of the standard module  $\mathbb{C}^r$  and  $\{b^i\}$  spans a copy of  $(\mathbb{C}^r)^*$ .

**Theorem 3.6.**  $\mathcal{G}_{ev}(r)^{\mathrm{GL}(r)}$  has a minimal strong generating set

$$\nu^j = \sum_{i=1}^r : a^i \partial^j b^i :, \quad 0 \leq j \leq r^2 + 4r - 1,$$

and is therefore of type  $\mathcal{W}(3, 4, \dots, r^2 + 4r + 2)$ . Moreover,  $\mathcal{G}_{ev}(r)$  is completely reducible as a  $\mathcal{G}_{ev}(r)^{\mathrm{GL}(r)}$ -module, and all irreducible modules in this decomposition are highest-weight and  $C_1$ -cofinite according to Miyamoto's definition.

*Proof.* The method of [LI] for studying the  $\mathcal{W}_{1+\infty, -r}$ -algebra via its realization as the  $\mathrm{GL}(r)$ -invariants in the rank  $r$   $\beta\gamma$ -system can be applied in this case.  $\square$

**Theorem 3.7.**  $\mathcal{G}_{odd}(s)^{\mathrm{O}(s)}$  has a minimal strong generating set

$$\omega^{2j+1} = \frac{1}{2} \sum_{i=1}^s : \varphi^i \partial^{2j+1} \varphi^i :, \quad 0 \leq j \leq 2s - 1,$$

and is therefore of type  $\mathcal{W}(4, 6, \dots, 4s + 2)$ . Moreover,  $\mathcal{G}_{odd}(s)$  is completely reducible as a  $\mathcal{G}_{odd}(s)^{\mathrm{O}(s)}$ -module, and all irreducible modules in this decomposition are highest-weight and  $C_1$ -cofinite according to Miyamoto's definition.

*Proof.* This can be reduced to showing that, in the notation of Equation 11.1 of [LV],  $R_s(I, J) \neq 0$  for  $I = (1, 1, \dots, 1)$  and  $J = (2, 2, \dots, 2)$ , where both lists have length  $s + 1$ . This follows easily from the recursive formula given by Equation 11.5 of [LV].  $\square$

**Corollary 3.8.** For any reductive group  $G \subset \mathrm{O}(s)$ ,  $\mathcal{G}_{odd}(s)^G$  is strongly finitely generated.

Recall that the full automorphism groups of  $\mathcal{H}(n)$  and  $\mathcal{A}(m)$  are  $\mathrm{O}(n)$  and  $\mathrm{Sp}(2m)$ , respectively. Suppose that  $G$  is a reductive group of automorphisms of

$$\mathcal{H}(n) \otimes \mathcal{A}(m) \otimes \mathcal{G}_{ev}(r) \otimes \mathcal{G}_{odd}(s)$$

which preserves the tensor factors, so that

$$G \subset \mathrm{O}(n) \times \mathrm{Sp}(2m) \times \mathrm{Sp}(2r) \times \mathrm{O}(s).$$

By the same argument as the proof of Theorem 4.2 of [CLII], we obtain

**Corollary 3.9.**  $(\mathcal{H}(n) \otimes \mathcal{A}(m) \otimes \mathcal{G}_{ev}(r) \otimes \mathcal{G}_{odd}(s))^G$  is strongly finitely generated.

If  $G \subset \mathcal{G}_0^{\natural}$  is any reductive group of automorphisms of  $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$ , the orbifold  $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})^G$  is also a deformable family and

$$\lim_{k \rightarrow \infty} \mathcal{W}^k(\mathfrak{g}, e_{-\theta})^G \cong \mathcal{T} \otimes (\mathcal{H}(n) \otimes \mathcal{A}(m) \otimes \mathcal{G}_{ev}(r) \otimes \mathcal{G}_{odd}(s))^G.$$

The argument is the same as the proof of Lemma 5.1 and Corollary 5.2 of [CLII]. Since the right hand side is strongly finitely generated, it follows from Theorem 3.1 that  $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})^G$  is strongly finitely generated for generic values of  $k$ .

Finally, suppose that  $\mathfrak{g}' \subset \mathfrak{g}_0^\natural$  is a reductive Lie subalgebra of dimension  $d$ , i.e., a sum of abelian and simple summands. Let  $\mathcal{A}(\mathfrak{g}') \subset \mathcal{A}(\mathfrak{g}^\natural)$  be the corresponding affine vertex subalgebra, and let

$$\mathcal{C}^k = \text{Com}(\mathcal{A}(\mathfrak{g}'), \mathcal{W}^k(\mathfrak{g}, e_{-\theta})).$$

Note that  $\lim_{k \rightarrow \infty} \mathcal{A}(\mathfrak{g}') \cong \mathcal{H}(d)$  and

$$\lim_{k \rightarrow \infty} \mathcal{W}^k(\mathfrak{g}, e_{-\theta}) \cong \mathcal{H}(d) \otimes \mathcal{H}(n-d) \otimes \mathcal{A}(m) \otimes \mathcal{G}_{\text{ev}}(r) \otimes \mathcal{G}_{\text{odd}}(s).$$

Moreover,  $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$  is *good* in the sense of [CLII], so by Lemma 6.2 of [CLII],  $G$  acts on  $\mathcal{H}(n-d) \otimes \mathcal{A}(m) \otimes \mathcal{G}_{\text{ev}}(r) \otimes \mathcal{G}_{\text{odd}}(s)$ , and we have

$$\lim_{k \rightarrow \infty} \mathcal{C}^k \cong (\mathcal{H}(n-d) \otimes \mathcal{A}(m) \otimes \mathcal{G}_{\text{ev}}(r) \otimes \mathcal{G}_{\text{odd}}(s))^G.$$

Since  $G$  preserves the tensor factors, this algebra is strongly finitely generated. By Theorem 3.1, we obtain

**Theorem 3.10.** *For any  $\mathfrak{g}'$  as above,  $\mathcal{C}^k = \text{Com}(\mathcal{A}(\mathfrak{g}'), \mathcal{W}^k(\mathfrak{g}, e_{-\theta}))$  is strongly finitely generated for generic values of  $k$ .*

#### 4. GENERIC STRUCTURE OF $\text{COM}(\mathcal{A}(\mathfrak{g}^\natural), \mathcal{W}^k(\mathfrak{g}, e_{-\theta}))$ : SOME EXPLICIT RESULTS

In this section, we find minimal strong generating sets for  $\text{Com}(\mathcal{A}(\mathfrak{g}^\natural), \mathcal{W}^k(\mathfrak{g}, e_{-\theta}))$  for generic values of  $k$  in a few cases.

**The case  $\mathfrak{g} = \mathfrak{sl}_n$ .** Recall from [KWI] that  $(\mathfrak{sl}_n)^\natural = \mathfrak{gl}_{n-2}$ , and

$$(\mathfrak{sl}_n)_{-1/2} \cong \mathbb{C}^{n-2} \oplus (\mathbb{C}^{n-2})^*$$

as  $\mathfrak{gl}_{n-2}$ -modules. The affine vertex algebra  $\mathcal{A}(\mathfrak{gl}_{n-2})$  is isomorphic to  $V^{k+1}(\mathfrak{gl}_{n-2}) = \mathcal{H} \otimes V^{k+1}(\mathfrak{sl}_{n-2})$ , where  $\mathcal{H}$  is a rank one Heisenberg algebra. It follows that

$$\lim_{k \rightarrow \infty} \mathcal{W}^k(\mathfrak{sl}_n, e_{-\theta}) \cong \mathcal{H}((n-2)^2) \otimes \mathcal{T} \otimes \mathcal{G}_{\text{ev}}(n-2),$$

and

$$\lim_{k \rightarrow \infty} \text{Com}(V^{k+1}(\mathfrak{gl}_{n-2}), \mathcal{W}^k(\mathfrak{sl}_n, e_{-\theta})) \cong \mathcal{T} \otimes \mathcal{G}_{\text{ev}}(n-2)^{\text{GL}(n-2)}.$$

By Theorem 3.6,  $\mathcal{G}_{\text{ev}}(n-2)^{\text{GL}(n-2)}$  is of type  $\mathcal{W}(3, 4, \dots, n^2 - 2)$ , so we obtain

**Theorem 4.1.**  *$\text{Com}(V^{k+1}(\mathfrak{gl}_{n-2}), \mathcal{W}^k(\mathfrak{sl}_n, e_{-\theta}))$  is of type  $\mathcal{W}(2, 3, \dots, n^2 - 2)$  for generic values of  $k$ .*

In the case  $n = 3$ , this generalizes the result of [ACL] that  $\text{Com}(\mathcal{H}, \mathcal{W}(\mathfrak{sl}_3, e_{-\theta}))$  is generically of type  $\mathcal{W}(2, 3, 4, 5, 6, 7)$ .

**The case  $\mathfrak{g} = \mathfrak{sp}_{2n}$ .** Recall from [KWI] that for  $n \geq 2$ ,  $(\mathfrak{sp}_{2n})^\natural \cong \mathfrak{sp}_{2n-2}$  and that

$$(\mathfrak{sp}_{2n})_{-1/2} \cong \mathbb{C}^{2n-2}$$

as  $\mathfrak{sp}_{2n-2}$ -modules. Then  $\mathcal{A}(\mathfrak{sp}_{2n-2}) \cong V^{k+1/2}(\mathfrak{sp}_{2n-2})$ . It follows that

$$\lim_{k \rightarrow \infty} \mathcal{W}^k(\mathfrak{sp}_{2n}, e_{-\theta}) \cong \mathcal{H}(d) \otimes \mathcal{T} \otimes \mathcal{G}_{\text{ev}}(n-2), \quad d = \dim(\mathfrak{sl}_{2n-2}),$$

and that

$$\lim_{k \rightarrow \infty} \text{Com}(V^{k+1/2}(\mathfrak{sp}_{2n-2}), \mathcal{W}^k(\mathfrak{sp}_{2n}, e_{-\theta})) \cong \mathcal{T} \otimes \mathcal{G}_{\text{ev}}(n-2)^{\text{Sp}(2n-2)}.$$

By Theorem 3.4,  $\mathcal{G}_{\text{ev}}(n-2)^{\text{Sp}(2n-2)}$  is of type  $\mathcal{W}(4, 6, \dots, 2n^2 + 2n - 2)$ . We obtain

**Theorem 4.2.**  $\text{Com}(V^{k+1/2}(\mathfrak{sp}_{2n-2}, \mathcal{W}^k(\mathfrak{sp}_{2n}, e_{-\theta})))$  is of type  $\mathcal{W}(2, 4, \dots, 2n^2 + 2n - 2)$  for generic values of  $k$ .

**The case  $\mathfrak{g} = \mathfrak{sl}(2|m)$ ,  $m \neq 2$ .** In this case,  $\mathfrak{g}^\natural \cong \mathfrak{gl}_m$  and

$$\mathfrak{g}_{-1/2} \cong \mathbb{C}^m \oplus (\mathbb{C}^m)^*,$$

regarded as an odd vector space. Then

$$\lim_{k \rightarrow \infty} \mathcal{W}^k(\mathfrak{g}, e_{-\theta}) \cong \mathcal{H}(m^2) \otimes \mathcal{T} \otimes \mathcal{G}_{\text{odd}}(2m).$$

Letting  $\mathcal{C}^k = \text{Com}(\mathcal{A}(\mathfrak{gl}_m), \mathcal{W}^k(\mathfrak{g}, e_{-\theta}))$ , we have

$$\lim_{k \rightarrow \infty} \mathcal{C}^k \cong \mathcal{T} \otimes \mathcal{G}_{\text{odd}}(2m)^{\text{GL}(m)}.$$

It can be checked that  $\mathcal{G}_{\text{odd}}(2m)^{\text{GL}(m)}$  is purely even and of type  $\mathcal{W}(3, 4, \dots, 3m + 2)$ , so we obtain

**Theorem 4.3.** For  $\mathfrak{g} = \mathfrak{sl}(2|m)$  and  $m \neq 2$ ,  $\text{Com}(\mathcal{A}(\mathfrak{g}^\natural), \mathcal{W}^k(\mathfrak{g}, e_{-\theta}))$  is even and is generically of type  $\mathcal{W}(2, 3, \dots, 3m + 2)$ .

This generalizes the case  $m = 1$ , where  $\mathcal{W}^k(\mathfrak{sl}(2|1), e_{-\theta})$  is just the  $N = 2$  superconformal vertex algebra. In this case, the above commutant is known to be of type  $\mathcal{W}(2, 3, 4, 5)$  and is isomorphic to the universal parafermion algebra of  $\mathfrak{sl}_2$ .

**The case  $\mathfrak{g} = \mathfrak{sl}(2|2)/\mathbb{C}$ .** Recall that in this case,  $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$  is isomorphic to the small  $N = 4$  superconformal vertex algebra. We have  $\mathfrak{g}^\natural = \mathfrak{sl}_2$  and

$$\mathfrak{g}_{-1/2} \cong \mathbb{C}^2 \oplus (\mathbb{C}^2)^*,$$

regarded as an odd vector space. Letting  $\mathcal{C}^k = \text{Com}(\mathcal{A}(\mathfrak{sl}_2), \mathcal{W}^k(\mathfrak{g}, e_{-\theta}))$ , we have

$$\lim_{k \rightarrow \infty} \mathcal{C}^k \cong \mathcal{T} \otimes \mathcal{G}_{\text{odd}}(4)^{\text{SL}(2)}.$$

It is not difficult to show that  $\mathcal{G}_{\text{odd}}(4)^{\text{SL}(2)}$  is even and of type  $\mathcal{W}(3^3, 4, 5^3, 6, 7^3, 8)$ . In other words, a minimal strong generating set consists of three even fields in each weight 3, 5, 7, and one even field in weights 4, 6, 8. We obtain

**Theorem 4.4.** For  $\mathfrak{g} = \mathfrak{sl}(2|2)/\mathbb{C}$ ,  $\text{Com}(\mathcal{A}(\mathfrak{g}^\natural), \mathcal{W}^k(\mathfrak{g}, e_{-\theta}))$  is even and is generically of type  $\mathcal{W}(2, 3^3, 4, 5^3, 6, 7^3, 8)$ .

In this case,  $\text{Com}(\mathcal{A}(\mathfrak{g}^\natural), \mathcal{W}^k(\mathfrak{g}, e_{-\theta}))$  contains a subalgebra of type  $\mathcal{W}(2, 3, 4, 5, 6, 7, 8)$ .

## 5. THE STRUCTURE OF THE NONGENERIC SET

Given a coset of the form  $\mathcal{C}^k = \text{Com}(\mathcal{A}(\mathfrak{g}^\natural), \mathcal{W}^k(\mathfrak{g}, e_{-\theta}))$ , suppose that  $S$  is minimal strong generating set for  $\mathcal{C}^k$  that works for generic values of  $k$ , which corresponds to a minimal strong generating set for  $\mathcal{C}^\infty = \lim_{k \rightarrow \infty} \mathcal{C}^k$ . We call a value of  $k$  *nongeneric* if  $S$  does not strongly generate  $\mathcal{C}^k$ . In this section, we make some general remarks about the structure of the nongeneric set and analyze a few examples in greater detail.

In [ACL], it was shown that for generic values of  $k$ ,  $\mathcal{C}^k = \text{Com}(\mathcal{H}, \mathcal{W}^k(\mathfrak{sl}_3, e_{-\theta}))$  is of type  $\mathcal{W}(2, 3, 4, 5, 6, 7)$ . Moreover, we gave an explicit description of the nongeneric set; it consists only of  $\{-1, -\frac{3}{2}\}$ . This was much easier than general situation because cosets of Heisenberg algebras are better behaved than cosets of general affine VAs.

First issue is for which values of  $k$ , the natural infinite strong generating set coming from classical invariant theory will work. In general, this will work whenever  $k$  is a positive real number.

It is our belief that in a very general setting, the nongeneric set will consist of a discrete set of negative real numbers with compact closure, but a general approach to this problem seems out of reach at the moment. For the rest of this section, we will consider two cases where we can prove this statement.

First issue: let  $\mathcal{C}$  be the family. For which values of  $k$  is  $\mathcal{C}^k = \mathcal{C}/(\kappa - \sqrt{k})$ ? Not always true. At least if  $\mathfrak{g}'$  is simple and is positive real. Of course it can be true for more values too. Next: if we have a set of generators for limit that closes linearly under OPE, this is a strong generating set as long as above condition holds. If the set does not close linearly, it is less clear.

**The case  $\mathfrak{g} = \mathfrak{sp}_4$  and  $\mathfrak{g}^\dagger = \mathfrak{sp}_2$ .** The generators of  $\mathcal{W}^k(\mathfrak{sp}_4, e_{-\theta})$  are  $T, H, X, Y, G^\pm$ , where  $X, Y, H$  generate a copy of  $V^{k+1/2}(\mathfrak{sl}_2)$ ,  $T$  is a Virasoro of central charge  $c = -\frac{3(1+k)(1+2k)}{(3+k)}$ , and  $G^\pm$  are primary of weight  $3/2$  and satisfy

$$\begin{aligned} H(x)G^\pm(w) &\sim G^\pm(w)(z-w)^{-1}, & X(z)G^-(w) &\sim G^+(w)(z-w)^{-1}, & Y(z)G^+(w) &\sim G^-(w)(z-w)^{-1}, \\ G^+(z)G^+(w) &\sim (8+4k)X(w)(z-w)^{-2} + (4+2k)\partial X(w)(z-w)^{-1}, \\ G^-(z)G^-(w) &\sim -(8+4k)Y(w)(z-w)^{-2} - (4+2k)Y(w)(z-w)^{-1}, \\ G^+(z)G^-(w) &\sim -(4+10k+4k^2)(z-w)^{-3} - (4+2k)H(w)(z-w)^{-2} \\ &\quad + ((6+2k)T - 4 : XY : - : HH : -k\partial H)(w)(z-w)^{-1}. \end{aligned}$$

We use the notation  $a = \lim_{k \rightarrow \infty} \frac{1}{k}G^+$  and  $b = \lim_{k \rightarrow \infty} \frac{1}{k}G^-$  for the generators of  $\mathcal{G}_{\text{ev}}(1)$ , which satisfy

$$a(z)b(w) \sim -4(z-w)^{-3}.$$

Note that this normalization is different from the one used earlier. Recall that by classical invariant theory, the orbifold  $\mathcal{G}_{\text{ev}}(1)^{Sp(2)}$  has generators

$$\omega_{i,j} = : \partial^i a \partial^j b : - : \partial^j a \partial^i b :, \quad 0 \leq i < j,$$

and that the smaller set  $\{\omega_{0,n} \mid n = 1, 3, 5, \dots\}$  suffices.

Recall that the coset  $\mathcal{C}^k(2) = \text{Com}(V^{k+1/2}(\mathfrak{sp}_2), \mathcal{W}^k(\mathfrak{sp}_4))$  is generically of type  $\mathcal{W}(2, 4, 6, 8, 10)$  and has Virasoro element

$$L = T - \frac{1}{10+4k} (: HH : +4 : XY : -2\partial H)$$

with central charge

$$c = -\frac{6(2+k)^2(1+2k)}{(3+k)(5+2k)}.$$

In order to write down additional elements of  $\mathcal{C}^k$ , we first consider

$$u_{0,n} = : G^+ \partial^n G^- : - : (\partial^j G^+) G^- : \in \mathcal{W}^k(\mathfrak{sp}_4, e_{-\theta}), \quad n = 1, 3, 5, \dots$$

Clearly  $u_{0,n}$  has weight  $n+3$  and  $\lim_{k \rightarrow \infty} u_{0,n} = \omega_{0,n}$ , but  $u_{0,n} \notin \mathcal{C}^k$ . We shall construct suitable corrections

$$(5.1) \quad U_{0,n} = u_{0,n} + P_n \in \mathcal{C}^k,$$

such that  $P_n$  is a normally ordered polynomial in  $T, H, X, Y, u_{0,1}, \dots, u_{0,n-2}$  and their derivatives, and  $\lim_{k \rightarrow \infty} P_n = 0$ . First, for  $k \neq -\frac{9}{2}$ , we define

$$\begin{aligned} U_{0,1} = & u_{0,1} + \frac{2(2+k)(3+k)}{9+2k} \partial^2 T + \frac{13+8k+k^2}{3(9+2k)} \partial^3 H \\ & - \frac{2}{9+2k} : (\partial^2 H)H : - \frac{2+k}{2(9+2k)} : (\partial H)\partial H : - \frac{2(2+k)}{9+2k} : (\partial X)\partial Y : \\ & + \frac{2}{9+2k} : THH : + \frac{8}{9+2k} : TXY : - \frac{4}{9+2k} : T\partial H : + \frac{4}{9+2k} : H(\partial X)Y : \\ & + \frac{4}{9+2k} : HX\partial Y : - \frac{2(8+3k)}{9+2k} : TT : + \frac{2}{9+2k} : (\partial H)HH : \\ & - \frac{2(3+k)}{9+2k} : (\partial T)H : + \frac{2}{9+2k} : HG^+G^- : + \frac{2}{9+2k} : XG^-G^- : - \frac{2}{9+2k} : YG^+G^+ : \end{aligned}$$

This lies in  $\mathcal{C}^k$  but is not primary with respect to  $L$ . Instead, it satisfies

(5.2)

$$\begin{aligned} L(z)U_{0,1}(w) \sim & -\frac{12(2+k)^2(1+2k)(18+7k)}{(3+k)(9+2k)}(z-w)^{-6} + \frac{4(5+2k)(3+17k+7k^2)}{(3+k)(9+2k)}L(w)(z-w)^{-4} \\ & + \frac{2(5+2k)(18+7k)}{9+2k}L(w)(z-w)^{-3} + 4U_{0,1}(w)(z-w)^{-2} + \partial U_{0,1}(w)(z-w)^{-1}. \end{aligned}$$

Whenever  $k$  is not a root of  $p(x) = 30x^3 + 113x^2 + 59x - 105$ ,  $U_{0,1}$  can be corrected uniquely to a primary element by adding

$$\frac{2(5+2k)^2(-147-32k+14k^2)}{(9+2k)(-105+59k+113k^2+30k^3)} : LL : - \frac{(3+k)(5+2k)(-273+22k+157k^2+42k^3)}{(9+2k)(-105+59k+113k^2+30k^3)} \partial^2 L,$$

but it is easier for our purposes to work with  $U_{0,1}$ .

Next, we need the following calculation:

$$u_{0,1} \circ_1 u_{0,n} = f(k, n)u_{0,n+2} + P_n, \quad n = 1, 3, 5, \dots,$$

where

$$f(k, n) = \frac{2(3+k)(5+n)(7+2k+5n+2kn)(14+4k+5n+2kn)}{(9+2k)(1+n)(2+n)},$$

and  $P_n$  is a normally ordered polynomial in generators  $T, X, Y, H, u_{0,1}, u_{0,3}, \dots, u_{0,n}$ , and their derivatives. The proof of this formula is omitted, and is similar to a statement in (). One first verifies that  $f(k, n)$  is a rational function of  $k$  and  $n$  and then the specific formula can be obtained by calculating enough values with a computer. Let

$$S = \left\{-\frac{9}{2}, -3\right\} \cup \left\{-\frac{7+5n}{2(1+n)} \mid n \geq 1\right\} \cup \left\{-\frac{14+5n}{2(2+n)} \mid n \geq 1\right\},$$

which is the set of values of  $k$  for which  $f(n, k) = 0$  or is undefined.

We would like the corrections  $U_{0,n}$  in (5.1) to satisfy

$$U_{0,1} \circ_1 U_{0,n} = f(k, n)U_{0,n+2} + Q_n,$$

where  $Q_n$  is a normally ordered polynomial in  $L, U_{0,1}, U_{0,3}, \dots, U_{0,n-2}$ , and this can be achieved by a bootstrap procedure as we now demonstrate.

One can check by computer that

$$U_{0,1} \circ_1 U_{0,1} = f(k, 1)u_{0,3} + P_1,$$

where  $P_1$  depends on  $T, H, X, Y, u_{0,1}$  and their derivatives. As a first attempt to correct  $u_{0,3}$  to an element of  $\mathcal{C}^k$ , define

$$\tilde{U}_{0,3} = u_{0,3} + \frac{1}{f(k, 1)}P_1,$$

which certainly lies in  $\mathcal{C}^k$  and satisfies  $\lim_{k \rightarrow \infty} \tilde{U}_{0,3} = \omega_{0,3}$ . However,  $\tilde{U}_{0,3}$  is not the desired correction since

$$U_{0,1} \circ_1 \tilde{U}_{0,3} \neq f(k, 3)u_{0,5} + P_3$$

for any normally ordered polynomial  $P_3$  in  $T, H, X, Y, u_{0,1}, u_{0,3}$  and their derivatives. The reason is that the terms  $: T(\partial G^+)G^- :$  and  $: TG^+(\partial G^-) :$  appear in  $P_1$  with nonzero coefficient, and if we express  $U_{0,1} \circ_1 (: T(\partial G^+)G^- :)$  and  $U_{0,1} \circ_1 (: TG^+(\partial G^-) :)$  as normally ordered polynomials in  $T, H, X, Y, u_{0,1}, u_{0,3}, u_{0,5}$ , the coefficient of  $u_{0,5}$  is nonzero. It is easy to check the only normally ordered monomials  $\nu$  in  $T, H, X, Y, u_{0,1}$  which have the property that  $U_{0,1} \circ_1 \nu$  has a nontrivial coefficient of  $u_{0,5}$  are  $: T(\partial G^+)G^- :$  and  $: TG^+(\partial G^-) :$ . Moreover, these terms can be eliminated as follows: we define

$$U_{0,3} = \tilde{U}_{0,3} - \frac{6}{19 + 6k} : LU_{0,1} : .$$

Then the terms  $: T(\partial G^+)G^- :$  and  $: TG^+(\partial G^-) :$  do not appear in  $U_{0,3}$ , and

$$(5.3) \quad U_{0,1} \circ_1 U_{0,3} = f(k, 3)u_{0,5} + P_3,$$

where  $P_3$  is a normally ordered polynomial in  $T, H, X, Y, u_{0,1}, u_{0,3}$  and their derivatives. Also, note that

$$U_{0,1} \circ_1 U_{0,1} = f(k, 1)U_{0,3} + \frac{6f(k, 1)}{19 + 6k} : LU_{0,1} : .$$

As above, we attempt to correct  $u_{0,5}$  by defining

$$\tilde{U}_{0,5} = u_{0,5} + \frac{1}{f(k, 3)}P_3,$$

which lies in  $\mathcal{C}^k$  and satisfies  $\lim_{k \rightarrow \infty} \tilde{U}_{0,5} = \omega_{0,5}$ . However,

$$U_{0,1} \circ_1 \tilde{U}_{0,5} \neq f(k, 5)u_{0,7} + P_5$$

for any  $P_5$  depending only on  $T, H, X, Y, u_{0,1}, u_{0,3}, u_{0,5}$  and their derivatives. As before, the reason is that  $P_3$  contains terms  $\nu$  of the form

$$(5.4) \quad : TT(\partial G^+)G^- :, \quad : TTG^+(\partial G^1) :, \quad : \partial^i T \partial^j G^+ \partial^k G^- :, \quad i + j + k = 3,$$

which all have the property that the coefficient of  $u_{0,7}$  in  $U_{0,1} \circ_k \nu$  is nonzero. However, we can correct  $\tilde{U}_{0,5}$  as follows:

$$\begin{aligned} U_{0,5} = & \tilde{U}_{0,5} - \frac{30(-1 + 5k + 2k^2)}{(3 + k)(11 + 4k)(19 + 6k)(29 + 10k)} : LLU_{0,1} : \\ & - \frac{5(42 + 21k + 2k^2)}{(3 + k)(11 + 4k)(29 + 10k)} : LU_{0,3} : \\ & - \frac{5(12234 + 13450k + 5169k^2 + 788k^3 + 36k^4)}{2(3 + k)^2(11 + 4k)(19 + 6k)(29 + 10k)} : (\partial^2 L)U_{0,1} : \end{aligned}$$

$$-\frac{5(9523 + 8130k + 2186k^2 + 180k^3)}{4(3+k)(11+4k)(19+6k)(29+10k)} : (\partial L)\partial U_{0,1} :$$

$$-\frac{5(5+2k)(491+256k+30k^2)}{2(3+k)(11+4k)(19+6k)(29+10k)} : L\partial^2 U_{0,1} : .$$

Then  $U_{0,5}$  has no terms of the form (5.4), and

$$U_{0,1} \circ_1 U_{0,5} = f(k, 5)u_{0,7} + P_5,$$

where  $P_5$  is a normally ordered polynomial in  $T, X, Y, u_{0,1}, u_{0,3}, u_{0,5}$  and their derivatives. Also, it is immediate from the above formula for  $U_{0,5}$  that

$$U_{0,1} \circ_1 U_{0,3} = f(k, 3)U_{0,5} + Q_3,$$

where  $Q_3$  is a linear combinations of  $: LLU_{0,1} :, : LU_{0,3} :, : (\partial^2 L)U_{0,1} :, : (\partial L)(\partial U_{0,1}) :,$  and  $: L(\partial^2 U_{0,1}) : .$  Finally, note that the only values of  $k$  where the denominators of any terms appearing in  $U_{0,3}$  and  $U_{0,5}$  vanishes are elements of  $S$ . This is a consequence of (5.2) together with the fact that  $U_{0,3}$  and  $U_{0,5}$  lie in the algebra generated by  $L$  and  $U_{0,1}$ .

More generally, we can continue this process and construct elements  $U_{0,n}$  for all  $n = 1, 3, 5, \dots$ , with the property that

$$U_{0,1} \circ_1 U_{0,n} = f(k, n)U_{0,n+2} + Q_n$$

where  $Q_n$  is a linear combination of elements of the form  $\partial^{i_1} L \dots \partial^{i_r} L \partial^k U_{0,m}$  for  $m = 1, 3, \dots, n$  and  $i_1, \dots, i_r, k \geq 0$ . Moreover, since  $U_{0,n}$  all lie in the algebra generated by  $L$  and  $U_{0,1}$ , all points where the denominator of any term appearing in  $U_{0,n}$  vanishes lie in  $S$ . Moreover, it is immediate that  $\{L, U_{0,n} \mid n = 1, 3, 5, \dots\}$  close under OPE and therefore strongly generated a vertex algebra inside  $\mathcal{C}^k$ .

**Corollary 5.1.** *For  $k \notin S$ ,  $\{L, U_{0,2j+1} \mid j \geq 0\}$  strongly generates the commutant  $\mathcal{C}^k$  whenever  $\mathcal{C}^k = \mathcal{C}/(\kappa - \sqrt{k})$ . In particular, this set generates  $\mathcal{C}_k$  for all real numbers  $k > -2$ .*

*Proof.* For generic values of  $k$ ,  $\mathcal{C}^k$  is generated by  $U_{2j+1}$ , and for  $k \notin S$ , we have the linear span of all these elements, and in particular all normally ordered monomials in these generators. Need to know that for positive integer values, the specialization gives everything. □

Next, we consider normally ordered relations among the generators. Recall that the first normally ordered relation among generators  $\omega_{i,j}$  of  $(\mathcal{G}(1)_{\text{ev}})^{Sp(2)}$  occurs at weight 12, and is a quantum correction of a classical Pfaffian relation. The leading term is

$$: \omega_{0,1}\omega_{2,3} : - : \omega_{0,2}\omega_{1,3} : + : \omega_{0,3}\omega_{1,2} :,$$

and the subleading terms can all be expressed as normally ordered monomials  $\omega_{0,n}$  for  $n = 1, 3, 5, 7, 9$ . In fact, the relation can be expressed as

$$(5.5) \quad \frac{1}{6}\omega_{0,9} = : \omega_{0,1}\omega_{0,5} : - 3 : \omega_{0,1}\partial^2\omega_{0,3} : + 2 : \omega_{0,1}\partial^4\omega_{0,1} : - : (\partial\omega_{0,1})\partial^3\omega_{0,1} : + : (\partial\omega_{0,1})\partial\omega_{0,3} :$$

$$+ : \omega_{0,3}\partial^2\omega_{0,1} : - : \omega_{0,3}\omega_{0,3} : - \frac{5}{3}\partial^2\omega_{0,7} + \frac{19}{2}\partial^4\omega_{0,5} - \frac{587}{30}\partial^6\omega_{0,3} + \frac{119}{10}\partial^8\omega_{0,1}.$$

This shows that  $\omega_{0,9}$  can be expressed as a normally ordered polynomial in  $\omega_{0,1}, \dots, \omega_{0,7}$  and their derivatives. Starting with (5.5), and using the fact that

$$\omega_{0,1} \circ_1 \omega_{0,n} = -4(n+5)\omega_{0,n+2} + \nu_n, \quad n = 1, 3, 5, \dots,$$

where  $\nu_n$  is a linear combination of  $\partial^2\omega_{0,n}, \partial^4\omega_{n-2}, \dots, \partial^{n+1}\omega_{0,1}$ , one can construct decoupling relations

$$\omega_{0,n} = Q_n(\omega_{0,1}, \omega_{0,3}, \omega_{0,5}, \omega_{0,7}), \quad n = 1, 3, 5, \dots$$

This shows that  $\mathcal{G}_{\text{ev}}(1)^{\text{Sp}(2)}$  is of type  $\mathcal{W}(2, 4, 6, 8, 10)$  with minimal strong generating set  $\{\omega_{0,1}, \dots, \omega_{0,7}\}$ , and is a special case of Theorem 4.2.

Recall the elements  $U_{0,n}$  which satisfy  $\lim_{k \rightarrow \infty} U_{0,n} = \omega_{0,n}$ . The above relation 5.5 can be deformed to a relation

$$\lambda(k)U_{0,9} = R_9(L, U_{0,1}, U_{0,3}, U_{0,5}, U_{0,7}).$$

It can be verified by a lengthy computer calculation that

$$\lambda(k) = \frac{(2+k)(39+14k)(49+18k)}{756(9+2k)}.$$

Note that the set of values of  $k$  where the numerator or denominator of  $\lambda(k)$  vanishes lies in  $S$ . Using the fact that  $U_{0,1} \circ_1 U_{0,n} = f(k, n)U_{0,n+2} + Q_n$ , we can construct similar decoupling relations

$$U_{0,n} = R_n(L, U_{0,1}, U_{0,3}, U_{0,5}, U_{0,7}),$$

for all  $n = 1, 3, 5, \dots$  and  $k \notin S$ . This implies

**Theorem 5.2.** *For all real numbers  $k > -\frac{5}{2}$ ,  $\mathcal{C}^k$  is of type  $\mathcal{W}(2, 4, 6, 8, 10)$  with minimal strong generating set  $\{L, U_{0,1}, U_{0,3}, U_{0,5}, U_{0,7}\}$ .*

**The case  $\mathfrak{g} = \mathfrak{sl}_4$  and  $\mathfrak{g}^\natural = \mathfrak{gl}_2$ .** The generators of  $\mathcal{W}^k(\mathfrak{sl}_4, e_{-\theta})$  are  $J, X, Y, H, T, G^{1,\pm}, G^{2,\pm}$ , where  $T$  is a Virasoro element of central charge  $c = -\frac{3k(3+2k)}{4+k}$ ,  $J, H, X, Y$  are primary of weight one, and  $G^{1,\pm}, G^{2,\pm}$  are primary of weight  $\frac{3}{2}$ . Moreover,  $H, X, Y$  generate a copy of  $V^{k+1}(\mathfrak{sl}_2)$ ,  $J$  commutes with  $H, X, Y$  and generates a Heisenberg algebra, and we have the following OPE relations:

$$\begin{aligned} J(z)J(w) &\sim 4(2+k)(z-w)^{-2}, & J(z)X^+(w) &\sim 2X^+(w), \\ G^{1,-}(z)G^{1,+}(w) &\sim -2(k+2)X(w)(z-w)^{-2} + (: JX : -(k+2)\partial X)(w)(z-w)^{-1}, \\ G^{2,-}(z)G^{2,+}(w) &\sim -2(k+2)Y(w)(z-w)^{-2} + (: JY : -(k+2)\partial Y)(w)(z-w)^{-1}, \\ G^{1,-}(z)G^{2,+}(w) &\sim -2(k+1)(k+2)(z-w)^{-3} + ((k+1)J - (k+2)H)(w)(z-w)^{-2} \\ &+ \left( (4+k)T - \frac{3}{8} : JJ : + \frac{1}{2} : HJ : - \frac{1}{2} : HH : - 2 : XY : + \frac{1+k}{2} \partial J - \frac{k}{2} \partial H \right) (w)(z-w)^{-1}, \\ G^{1,+}(z)G^{2,-}(w) &\sim 2(k+1)(k+2)(z-w)^{-3} + ((k+1)J + (k+2)H)(w)(z-w)^{-2} \\ &+ \left( -(4+k)T + \frac{3}{8} : JJ : + \frac{1}{2} : HJ : + \frac{1}{2} : HH : + 2 : XY : + \frac{1+k}{2} \partial J + \frac{k}{2} \partial H \right) (w)(z-w)^{-1}. \end{aligned}$$

We use the notation

$$a^i = \lim_{k \rightarrow \infty} \frac{1}{k} G^{i,+}, \quad b^i = \lim_{k \rightarrow \infty} \frac{1}{k} G^{i,-}, \quad i = 1, 2,$$

for the generators of  $\mathcal{G}_{\text{ev}}(2)$ , which satisfy

$$a^i(z)b^j(w) \sim 4\delta_{i,j}(z-w)^{-3}.$$

Note that this normalization is different from the one used earlier. By classical invariant theory, the orbifold  $(\mathcal{G}_{\text{ev}}(2))^{GL(2)}$  has strong generators

$$\omega_{i,j} = : \partial^i a^1 \partial^j b^2 : + : \partial^j a^2 \partial^i b^1 :, \quad i, j \geq 0,$$

and the smaller set  $\{u_{0,n} \mid n \geq 0\}$  suffices. Recall that the coset

$$\mathcal{C}^k(4) = \text{Com}(V^{k+1}(\mathfrak{gl}_2), \mathcal{W}^k(\mathfrak{sl}_4, e_{-\theta}))$$

is generically of type  $\mathcal{W}(2, 3, \dots, 14)$  and has Virasoro element

$$L = T - \frac{1}{8(2+k)} : JJ : - \frac{1}{4(3+k)} : HH : - \frac{1}{3+k} : XY : + \frac{1}{2(3+k)} \partial H.$$

Next, consider

$$u_{0,n} = : G^{1,-} \partial^n G^{2,+} : + : (\partial^n G^{1,+}) G^{2,-} :, \quad n \geq 0.$$

These satisfy  $\lim_{k \rightarrow \infty} u_{0,n} = \omega_{i,j}$ , but do not lie in  $\mathcal{C}^k$  and we would like to find suitable corrections. First, define

$$\begin{aligned} U_{0,0} = u_{0,0} - \frac{14+5k}{24(2+k)^2} : JJJ : - \frac{1}{2(2+k)} : JHH : - \frac{2}{2+k} : JXY : - \frac{1}{2} : (\partial J)H : \\ - \frac{k}{2(2+k)} : J(\partial H) : + \frac{4+k}{2+k} : TJ : - \frac{16+9k+2k^2}{6(2+k)} \partial^2 J. \end{aligned}$$

It can be verified that  $U_{0,0}$  lies in  $\mathcal{C}^k$  and is primary with respect to  $L$ . In order to find the remaining correction we need the following calculation. For all  $k \geq 0$ , we have

$$u_{0,0} \circ_1 u_{0,n} = f(n, k) u_{0,n+1} + P_n, \quad f(n, k) = - \frac{(k+4)(n+4)(5+k+3n+kn)}{(n+1)},$$

where  $P_n$  can be expressed as a normally ordered polynomial in  $T, J, H, X, Y, u_{0,0}, \dots, u_{0,n}$  and their derivatives. Let

$$S = \{-4\} \cup \left\{ -\frac{5+3n}{1+n} \mid n \geq 0 \right\},$$

which is the set of values of  $k$  where  $f(n, k)$  is either zero or undefined.

We have  $U_{0,0} \circ_1 U_{0,0} = f(0, k) u_{0,1} + P_0$ . Clearly  $\tilde{U}_{0,1} = u_{0,1} + \frac{1}{f(0,k)} P_0$  lies in  $\mathcal{C}^k$  and satisfies  $\lim_{k \rightarrow \infty} \tilde{U}_{0,1} = \omega_{0,1}$ , but  $\tilde{U}_{0,1}$  is not the desired correction of  $u_{0,1}$  since

$$U_{0,0} \circ_1 \tilde{U}_{0,1} \neq f(1, k) u_{0,2} + P_1,$$

for any  $P_1$  depending only on  $J, H, X, Y, T, u_{0,0}$ . The problem is that  $P_0$  contains the terms  $: TG^{1,-} G^{2,+} :$  and  $T : G^{1,+} G^{2,-} :$  which have the property that  $U_{0,0} \circ_1 (: TG^{1,-} G^{2,+} :)$  and  $U_{0,0} \circ_1 (: TG^{1,+} G^{2,-} :)$  have a nonzero coefficient of  $u_{0,1}$  when expressed as a normally ordered polynomial in  $J, H, X, Y, T, u_{0,0}, u_{0,1}$ . This can be corrected as follows:

$$U_{0,1} = \tilde{U}_{0,1} + : LU_{0,0}.$$

This has the property that  $U_{0,0} \circ_1 U_{0,1} = f(1, k) u_{0,2} + P_1$  where  $P_1$  depends only on  $J, H, X, Y, T, u_{0,0}$ . Moreover,

$$U_{0,0} \circ_1 U_{0,1} = U_{0,1} + : LU_{0,0}.$$

As above, we can construct elements  $U_{0,n}$  for all  $n \geq 0$  with the property that  $U_{0,0} \circ_1 U_{0,n} = f(n, k) U_{0,n+1} + Q_n$  where  $Q_n$  is a linear combination of the fields

$$: (\partial^{i_1} L) \cdots (\partial^{i_r} L) (\partial^k U_{0,m}) :, \quad m = 0, 1, \dots, n, \quad i_1, \dots, i_r, k \geq 0.$$

Moreover, since  $U_{0,n}$  all lie in the algebra generated by  $L$  and  $U_{0,0}$ , all points where the denominator of any term appearing in  $U_{0,n}$  vanishes lie in  $S$ . Moreover, it is immediate that  $\{L, U_{0,n} \mid n \geq 0\}$  close under OPE and therefore strongly generate a vertex algebra inside  $\mathcal{C}^k$ .

**Theorem 5.3.** *For all real numbers  $k > -3$ ,  $\mathcal{C}^k(4)$  is strongly generated by  $\{L, U_{0,n} \mid n \geq 0\}$ .*

Now we consider relations normally ordered relations among these generators. Recall that in  $(\mathcal{G}_{\text{ev}}(2))^{GL(2)}$ , the first normally ordered relation among the generators  $\omega_{i,j}$  occurs at weight 15 and has leading term

$$\begin{aligned} & : \omega_{0,0} \omega_{1,1} \omega_{2,2} : - : \omega_{0,0} \omega_{1,2} \omega_{2,1} : - : \omega_{0,1} \omega_{1,0} \omega_{2,2} : \\ & + : \omega_{0,1} \omega_{1,2} \omega_{2,0} : - : \omega_{0,2} \omega_{1,1} \omega_{2,0} : + : \omega_{0,2} \omega_{1,0} \omega_{2,1} :, \end{aligned}$$

which is a normally ordering of a classical determinantal relation in  $\text{gr}((\mathcal{G}_{\text{ev}}(2)))^{GL(2)}$ . The subleading terms can all be expressed as normally ordered monomials in  $L$  and  $\omega_{0,n}$  for  $j = n = 0, 1, \dots, 12$ . In fact, the relation can be rewritten in the form

$$\lambda \omega_{0,12} = P(L, \omega_{0,0}, \omega_{0,1}, \dots, \omega_{0,11}),$$

where  $P$  is a normally ordered polynomial in  $L, \omega_{0,0}, \omega_{0,1}, \dots, \omega_{0,11}$  and their derivatives, and  $\lambda \neq 0$ . It follows that among the generators  $L, U_{0,0}, U_{0,1}, \dots, U_{0,11}$  for  $\mathcal{C}^k(4)$ , there is a normally ordered relation

$$\lambda(k) U_{0,12} = Q(L, U_{0,0}, U_{0,1}, \dots, U_{0,11}),$$

where  $\lim_{k \rightarrow \infty} \lambda(k) = \lambda$  and  $\lim_{k \rightarrow \infty} Q = P$ . Moreover, all roots of denominators appear in this relation belong to the set  $S$  above. Unfortunately, it is too difficult at the moment to compute the numerator of  $\lambda(k)$ , although it has finitely many roots. By analogy with the case of  $\mathfrak{sp}_4$  in the previous section, we conjecture that the set  $T$  of distinct roots of the numerator of  $\lambda(k)$  lies in  $S$ . An immediate consequence is

**Theorem 5.4.** *For all but finitely many real numbers  $k > -3$ ,  $\mathcal{C}^k(4)$  is of type  $\mathcal{W}(2, 3, \dots, 14)$  and has a minimal strong generating set  $\{L, U_{0,n} \mid n = 0, 1, \dots, 11\}$ .*

## 6. COSETS OF $\mathcal{W}_k(\mathfrak{g}, e_{-\theta})$ AT NONGENERIC LEVELS

Of much more interest than cosets  $\mathcal{C}^k$  of  $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$  for generic values of  $k$ , are cosets  $\mathcal{C}_k$  of the simple quotient  $\mathcal{W}_k(\mathfrak{g}, e_{-\theta})$  when  $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$  is not simple. In [ACL], one such family was considered, namely,

$$\mathcal{C}_{p/2-3} = \text{Com}(\mathcal{H}, \mathcal{W}_{p/2-3}(\mathfrak{sl}_3, e_{-\theta})).$$

It was shown in [ArI] that for  $p = 5, 7, 9, \dots$ ,  $\mathcal{W}_{p/2-3}(\mathfrak{sl}_3, e_{-\theta})$  is  $C_2$ -cofinite and rational, and the main result of [ACL] is that  $\mathcal{C}_{p/2-3}$  is isomorphic to the principal, rational  $\mathcal{W}(\mathfrak{sl}_{p-3})$ -algebra with central charge  $c = -\frac{3}{p}(p-4)^2$ . Recall that

$$\mathcal{C}^{p/2-3} = \text{Com}(\mathcal{H}, \mathcal{W}^{p/2-3}(\mathfrak{sl}_3, e_{-\theta}))$$

is of type  $\mathcal{W}(2, 3, 4, 5, 6, 7)$  for all  $p$  as above. Since the natural map  $\mathcal{C}^{p/2-3} \rightarrow \mathcal{C}_{p/2-3}$  is surjective, this family of  $\mathcal{W}(\mathfrak{sl}_{p-3})$ -algebras has the following *uniform truncation property*; it is of type  $\mathcal{W}(2, 3, 4, 5, 6, 7)$  for all  $p > 9$ , even though the universal  $\mathcal{W}(\mathfrak{sl}_{p-3})$ -algebra is of type  $\mathcal{W}(2, 3, \dots, p-3)$ .

In fact,  $\mathcal{W}_{p/2-3}(\mathfrak{sl}_3, e_{-\theta})$  is a simple current extension of  $V_L \otimes \mathcal{W}(\mathfrak{sl}_{p-3})$ , where  $V_L$  is the lattice vertex algebra with  $L = \sqrt{3p-9}\mathbb{Z}$ . This is a surprising coincidence, and it indicates that principal, rational  $\mathcal{W}$ -algebras may be important building blocks for more general rational  $\mathcal{W}$ -algebras. One of the goals of this paper is to give evidence that this phenomenon is not restricted to the rational levels.

More generally, suppose that  $k$  is a value for which  $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$  is not simple. Let  $\mathcal{I}$  be the maximal proper ideal of  $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$  graded by conformal weight, so that

$$\mathcal{W}_k(\mathfrak{g}, e_{-\theta}) = \mathcal{W}^k(\mathfrak{g}, e_{-\theta})/\mathcal{I}$$

is simple. Let  $\mathfrak{g}' \subset \mathfrak{g}_0^{\natural}$  be a simple Lie subalgebra, so that  $\mathcal{A}(\mathfrak{g}') \cong V^\ell(\mathfrak{g}')$  for some  $\ell$ . Let  $\mathcal{J}$  denote the kernel of the map  $V^\ell(\mathfrak{g}') \rightarrow \mathcal{W}_k(\mathfrak{g}, e_{-\theta})$ , and suppose that  $\mathcal{J}$  is maximal so that  $V^\ell(\mathfrak{g}')/\mathcal{J} \cong V_\ell(\mathfrak{g}')$ . Finally, let

$$\mathcal{C}^k = \text{Com}(V^\ell(\mathfrak{g}'), \mathcal{W}^k(\mathfrak{g}, e_{-\theta})), \quad \mathcal{C}_k = \text{Com}(V_\ell(\mathfrak{g}'), \mathcal{W}_k(\mathfrak{g}, e_{-\theta})).$$

There is always a vertex algebra homomorphism

$$\pi_k : \mathcal{C}^k \rightarrow \mathcal{C}_k,$$

but in general this map need not be surjective. In order to apply our results on the generic behavior of  $\mathcal{C}^k$  to the structure of  $\mathcal{C}_k$ , the following problems must be solved.

- (1) Find conditions for which  $\pi_k$  is surjective. In this case, a strong generating set for  $\mathcal{C}^k$  descends to a strong generating set for  $\mathcal{C}_k$ .
- (2) Let  $S \subset \mathcal{C}^k$  be a strong generating set for  $\mathcal{C}^k$  for generic values of  $k$ . We call  $k \in \mathbb{C}$  *nongeneric* if  $\mathcal{C}^k$  is not strongly generated by  $S$ . Determine which values of  $k$  are generic.

By Theorem 8.1 of [CLII], if  $\ell + h^\vee$  is a positive real number, where  $h^\vee$  is the dual Coxeter number of  $\mathfrak{g}'$ , then  $\pi_k : \mathcal{C}^k \rightarrow \mathcal{C}_k$  is surjective. A similar result holds if  $\mathfrak{g}'$  is any reductive Lie subalgebra such that this condition holds for each simple summand.

**Type  $C$  series at positive half-integer levels.** We consider the case  $\mathfrak{g} = \mathfrak{sp}_{2n}$  for  $n \geq 2$ ,  $\mathfrak{g}' = \mathfrak{g}_0^{\natural} = \mathfrak{sp}_{2n-2}$ , and  $k$  a half-integer such that  $k + 1/2$  is a positive integer. It is known that the maximal proper graded ideal  $\mathcal{I}_k \subset \mathcal{W}^k(\mathfrak{sp}_{2n}, e_{-\theta})$  is generated by  $(J^e)^{k+3/2}$ , where  $e \in \mathfrak{sp}_{2n-2}$  denotes the highest root. Therefore we have an embedding

$$V_{k+1/2}(\mathfrak{sp}_{2n-2}) \hookrightarrow \mathcal{W}_k(\mathfrak{sp}_{2n}, e_{-\theta}).$$

It is known that  $\mathcal{W}^k(\mathfrak{sp}_{2n}, e_{-\theta})$  is  $C_2$ -cofinite and rational [ArIII]. One therefore expects that the coset

$$\mathcal{C}_k(n) = \text{Com}(V_{k+1/2}(\mathfrak{sp}_{2n-2}), \mathcal{W}_k(\mathfrak{sp}_{2n}, e_{-\theta}))$$

is  $C_2$ -cofinite and rational as well, but since  $\mathcal{W}_k(\mathfrak{sp}_{2n}, e_{-\theta})$  is not a simple current extension of  $V_{k+1/2}(\mathfrak{sp}_{2n-2}) \otimes \mathcal{C}^k(n)$ , this is out of reach at the moment.

**Proposition 6.1.** *For all  $n \geq 2$  and  $k$  such that  $k + 1/2$  is a positive integer,  $\mathcal{C}_k(n)$  is simple.*

We now specialize to the case  $n = 2$ . Recall that for all real numbers  $k > -\frac{5}{2}$ ,  $\mathcal{C}^k(2)$  is of type  $\mathcal{W}(2, 4, 6, 8, 10)$  and the map  $\pi_k : \mathcal{C}^k(2) \rightarrow \mathcal{C}_k(2)$  is surjective. We use the same notation  $\{L, U_{0,1}, U_{0,3}, U_{0,5}, U_{0,7}\}$  for the images of the generators in  $\mathcal{C}_k(2)$ . Since  $\pi_k$  is surjective, this set strongly generates  $\mathcal{C}_k$ , but it need not be minimal.

It was shown by Kawasetsu [Ka] that  $\mathcal{C}_{1/2}(2)$  is isomorphic to the rational Virasoro algebra with  $c = -\frac{25}{7}$ . Here we give an alternative proof of this result. It is easy to verify that  $U_{0,1} + \frac{132}{5} : LL : -\frac{21}{2} \partial^2 L$  lies in the ideal  $\mathcal{I}_{1/2} \subset \mathcal{C}^{1/2}(2)$ , so we have the relation

$$U_{0,1} = -\frac{132}{5} : LL : + \frac{21}{2} \partial^2 L$$

in  $\mathcal{C}_{1/2}(2)$ . By applying the operator  $U_{0,1} \circ_1$  successively to this relation, we obtain relations

$$U_{0,n} = P_n(L), \quad n = 3, 5, 7$$

in  $\mathcal{C}_{1/2}(2)$ , where  $P_n(L)$  is a normally ordered polynomial in  $L$  and its derivatives. We conclude that  $\mathcal{C}_{1/2}(2)$  is strongly generated by  $L$ . Since  $\mathcal{C}_{1/2}(2)$  is simple we recover Kawasetsu's result.

Next, in the case  $k = \frac{3}{2}$  we have the following result.

**Theorem 6.2.**  $\mathcal{C}_{3/2}(2)$  is isomorphic to the principal, rational  $\mathcal{W}(\mathfrak{sp}_4)$ -algebra with central charge  $c = -\frac{49}{6}$ .

*Proof.* In weight 6, it can be checked by computer that the element

$$\begin{aligned} U_{0,3} - \frac{39664}{1701} : LLL : + \frac{117533}{5103} : (\partial^2 L)L : + \frac{1679}{972} : (\partial L)(\partial L) : \\ - \frac{3}{2} : LU_{0,1} : + \frac{34801}{20412} \partial^4 L + \frac{37}{4536} : \partial^2 U_{0,1} \end{aligned}$$

in  $\mathcal{C}^{3/2}(2)$  lies in the ideal  $\mathcal{I}_{3/2}$ . Therefore the corresponding element in  $\mathcal{C}_{3/2}(2)$  is a relation expressing  $U_{0,3}$  as a normally ordered polynomial  $P_3(L, U_{0,1})$  and  $L, U_{0,1}$  and their derivatives. By applying  $U_{0,1} \circ_1$  successively to this relation, and using the fact that  $U_{0,1} \circ_1 U_{0,n} = f(3/2, n)U_{0,n+2} + Q_n$  for  $n = 1, 3, 5, \dots$ , we obtain relations  $U_{0,5} = R_5(L, U_{0,1})$  and  $U_{0,7} = R_7(L, U_{0,1})$ , where  $R_5$  and  $R_7$  are normally ordered polynomials in  $L, U_{0,1}$  and their derivatives. Therefore  $\mathcal{C}_{3/2}(2)$  is of type  $\mathcal{W}(2, 4)$  with strong generators  $\{L, U_{0,1}\}$ . Checking that it is actually a  $\mathcal{W}(\mathfrak{sp}_4)$  is straightforward by computer using the explicit formulas for  $\mathcal{W}(\mathfrak{sp}_4)$  appearing in [ZhII].  $\square$

The central charge of  $\mathcal{C}_k(n)$  is

$$-\frac{(1+2k)(3+3k+2n-n^2)}{1+k+n}.$$

This is the same as the central charge of the principal, rational  $\mathcal{W}$ -algebra  $\mathcal{W}_s(\mathfrak{sp}_{2m}, f_{prin})$ , where

$$m = k + 1/2, \quad s + (k + 3/2) = p/q, \quad (p, q) = (n + k + 1/2, 2n + 2k + 2).$$

More generally, based on the equality of central charges we make the following conjecture.

**Conjecture 6.3.** For all  $n \geq 2$  and  $k$  such that  $k + 1/2$  is a positive integer,  $\mathcal{C}_k(n)$  is isomorphic to the  $C_2$ -cofinite, rational principal  $\mathcal{W}$ -algebra  $\mathcal{W}_s(\mathfrak{sp}_{2m}, f_{prin})$ , where

$$m = k + 1/2, \quad s + (k + 3/2) = p/q, \quad (p, q) = (n + k + 1/2, 2n + 2k + 2).$$

**Type A series at positive integer levels.** We next consider the case  $\mathfrak{g} = \mathfrak{sl}_n$  for  $n \geq 4$ ,  $\mathfrak{g}' = \mathfrak{g}^{\natural} = \mathfrak{gl}_{n-2}$ , and  $k$  a non-negative integer. The maximal proper graded ideal  $\mathcal{I}_k \subset \mathcal{W}^k(\mathfrak{sl}_n, e_{-\theta})$  is generated by  $(J^e)^{k+2}$ , where  $e \in \mathfrak{sl}_{n-2}$  denotes the highest root. Therefore we have an embedding

$$V_{k+1}(\mathfrak{gl}_{n-2}) = \mathcal{H} \otimes V_{k+1}(\mathfrak{sl}_{n-2}) \hookrightarrow \mathcal{W}_k(\mathfrak{sl}_n, e_{-\theta}),$$

and the above result shows that  $\pi_k : \mathcal{C}^k(n) \rightarrow \mathcal{C}_k(n)$  is surjective.

We specialize to the case  $n = 4$ . Recall that for all real numbers  $k > -3$ ,  $\mathcal{C}_k(4)$  is strongly generated by the fields  $\{L, U_{0,n} \mid n \geq 0\}$  and  $\pi_k : \mathcal{C}^k(4) \rightarrow \mathcal{C}_k(4)$  is surjective.

**Theorem 6.4.**  $\mathcal{C}_0(4)$  is isomorphic to the simple Zamolodchikov  $\mathcal{W}_3$ -algebra with  $c = -2$ .

*Proof.* It is not difficult to verify by computer that the element

$$U_{0,1} + \frac{16}{5} : LL : -\frac{3}{5} \partial^2 L$$

in  $\mathcal{C}^0(4)$  lies in the ideal  $\mathcal{I}_0$ , and hence gives rise to the relation  $U_{0,1} = -\frac{16}{5} : LL : +\frac{3}{5} \partial^2 L$  in  $\mathcal{C}_0(4)$ . By applying  $U_{0,0} \circ_1$  successively to this relation, and using the fact that  $U_{0,0} \circ_1 U_{0,n} = f(k, n)U_{0,n+1} + Q_n$ , we obtain relations

$$U_{0,n} = P_n(L, U_{0,0}), \quad n \geq 0, \quad n \geq 0.$$

It follows that  $\mathcal{C}_0(4)$  is strongly generated by  $\{L, U_{0,0}\}$ . Finally, it is not difficult to check by computer that they satisfy the OPE relations of  $\mathcal{W}_3$  with  $c = -2$ .  $\square$

**Theorem 6.5.**  $\mathcal{C}_1(4)$  is isomorphic to the simple parafermion algebra  $K_{-6/5}(\mathfrak{sl}_2) = \text{Com}(\mathcal{H}, V_{-6/5}(\mathfrak{sl}_2))$  at level  $-6/5$ , which has central charge  $c = -11/2$ .

*Proof.* It can be checked by computer that there is a relation in  $\mathcal{C}_1(4)$  at weight 6 of the form

$$U_{0,3} = P_3(L, U_{0,0}, U_{0,1}, U_{0,2}),$$

where  $P_3$  is a normally ordered polynomial in  $L, U_{0,0}, U_{0,1}, U_{0,2}$ , and their derivatives. Applying  $U_{0,0} \circ_1$  successively to this relation, we obtain relations

$$U_{0,n} = P_n(L, U_{0,0}, U_{0,1}, U_{0,2}), \quad n \geq 0,$$

so  $\mathcal{C}_1(4)$  is of type  $\mathcal{W}(2, 3, 4, 5)$  with strong generators  $\{L, U_{0,0}, U_{0,1}, U_{0,2}\}$ . It is then a straightforward but lengthy computer calculation to verify that these generators satisfy the OPE relations of the above parafermion algebra.  $\square$

**Remark 6.6.** One might guess that  $K_{-6/5}(\mathfrak{sl}_2)$  is isomorphic to the simple, principal  $\mathcal{W}(\mathfrak{sl}_5)$  algebra with  $c = -\frac{11}{2}$ . However, this turns out to be false, which can be shown using the explicit OPE relations for  $\mathcal{W}(\mathfrak{sl}_5)$  given in [Zhi].

In [FS], Feigin and Semikhatov introduced a remarkable sequence of vertex algebra which they call  $\mathcal{W}_n^{(2)}$ . They depend on a complex parameter  $\ell$  called level and we use the notation  $\mathcal{W}_{n,\ell}^{(2)}$ . They are freely generated of type  $\mathcal{W}(1, 2, \dots, n-1, n/2, n/2)$  and have central charge

$$c = -\frac{((\ell+n)(n-1)-n)((\ell+n)(n-2)n-n^2+1)}{\ell+n}.$$

In a recent paper [G], Genra has shown that  $\mathcal{W}_n^{(2)}$  is isomorphic to the subregular  $\mathcal{W}$ -algebra of  $\mathfrak{sl}_n$ , a fact which was conjectured in [ACGHR] and known previously for  $n = 3$ .

Note that  $\mathcal{C}_k(n)$  has central charge

$$c = -\frac{(1+k)(-1+2k+n)(3k+2n)}{(-1+k+n)(k+n)},$$

which coincides with the central charge of  $\text{Com}(\mathcal{H}, \mathcal{W}_{k+1, \ell}^{(2)})$  for  $\ell = -\frac{1+k^2+kn}{k+n}$ . Based on this observation, we make the following

**Conjecture 6.7.** *For all integers  $n \geq 4$  and  $k \geq 0$ ,  $\mathcal{C}_k(n)$  is isomorphic to  $\text{Com}(\mathcal{H}, \mathcal{W}_{k+1, \ell}^{(2)})$ , where  $\mathcal{W}_{k+1, \ell}^{(2)}$  denotes the simple Feigin-Semikhatov algebra at level  $\ell = -\frac{1+k^2+kn}{k+n}$ .*

Since  $\mathcal{W}_{1, \ell}^{(2)}$  is isomorphic to the rank one  $\beta\gamma$ -system  $\mathcal{S}$  for all  $\ell$ , and  $\text{Com}(\mathcal{H}, \mathcal{S})$  is well known to be  $\mathcal{W}_{3, -2}$ , this conjecture holds for  $n = 4$  and  $k = 0$ . Similarly, for  $k = 1$ ,  $\mathcal{W}_{2, \ell}^{(2)} \cong V^\ell(\mathfrak{sl}_2)$  so the conjecture holds for  $n = 4$  and  $k = 2$  as well. We now prove a more general statement using Theorem 2.1 that implies our conjecture for  $k = 0$  and all  $n \geq 4$ .

Let  $\mathcal{E}(n)$  be the rank  $n$   $bc$ -system with generators  $b^i, c^i$ , and let  $\mathcal{S}(m)$  be the rank  $m$   $\beta\gamma$ -system with generators  $\beta^i, \gamma^i$ . Let  $\mathcal{A}(1)$  the rank one symplectic fermion algebra with generators  $X^\pm$ . Define a  $U(1)$  action on  $\mathcal{E}(n) \otimes \mathcal{S}(m) \otimes \mathcal{A}(1)$  by

$$X^\pm \mapsto \lambda^{\pm 1} X^\pm, \quad b^i \mapsto \lambda b^i, \quad c^i \mapsto \lambda^{-1} c^i, \quad \beta^j \mapsto \lambda \beta^j, \quad \gamma^j \mapsto \lambda^{-1} \gamma^j$$

Here the  $1 + n + m$  vectors  $\{X^+, b^i, \beta^j\}$  and  $\{X^-, c^i, \gamma^j\}$  should be viewed as carrying the standard and respectively conjugate representation of  $\mathfrak{gl}(n+1|m)$ . With this notation, we have the following result.

**Theorem 6.8.** *For all  $(n, m) \neq (0, 2)$ ,  $\mathcal{W}_0(\mathfrak{sl}(n+2|m)) \cong (\mathcal{E}(n) \otimes \mathcal{S}(m) \otimes \mathcal{F}(1))^{U(1)}$ .*

*Proof.* In [CKLR] it is proven that  $(\mathcal{E}(n) \otimes \mathcal{S}(m))^{U(1)} \cong L_1(\mathfrak{gl}(m|n))$  (see also [AP, KWII]). Further  $\mathcal{A}(1)^{U(1)}$  is well known to be Zamolodchikov's  $\mathcal{W}_3$  algebra. It is clear that the vectors  $:c^i X^+ :$  and  $:b^i X^- :$  give then a strong generating set and it is then easy to see that the dimension 3 field of  $\mathcal{W}_3$  (one can take  $X^+ \partial X^-$  for it) is a normal ordered product of the other strong generators and their derivatives. So all conditions for Theorem 2.1 are satisfied. The orbifold is simple [DLM].  $\square$

**Remark 6.9.** *If  $(n, m) = (0, 2)$  then the orbifold  $(\mathcal{S}(2))^{U(1)}$  is larger than just  $L_{-1}(\mathfrak{gl}_2)$ . In that case also  $\mathfrak{gl}(2|2)$  is not reductive and it is better to consider the simple quotient  $\mathfrak{psl}(2|2)$ . In that case, the minimal  $\mathcal{W}$ -algebra is the small  $N = 4$  super Virasoro algebra and constructions of it are given in [CKLR, ?].*

**Corollary 6.10.** *Let*

$$\mathcal{D}_0(n, m) = \text{Com}(L_1(\mathfrak{gl}(n|m)), \mathcal{W}_0(\mathfrak{sl}(n+2|m))).$$

*Then for all  $(n, m) \neq (0, 2)$ ,*

$$\mathcal{C}_0(n, m) \cong \mathcal{A}(1)^{U(1)} \cong \mathcal{W}_{3, -2}.$$

*In particular,  $\mathcal{C}_0(n) = \mathcal{D}_0(n, 0) \cong \mathcal{W}_{3, -2}$  for all  $n \geq 4$ , so Conjecture 6.7 holds for all  $n \geq 4$  and  $k = 0$ .*

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