# SYMMETRIC TIGHT FRAMELET FILTER BANKS WITH THREE HIGH-PASS FILTERS

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Abstract. In this paper we continue our investigation of symmetric tight framelet filter banks (STFFBs) with a minimum number of generators in [7]. In particular, we shall systematically study STFFBs with three high-pass filters which are derived from the oblique extension principle. To our best knowledge, except the papers [1, 11], there are no other papers in the literature so far systematically studying this problem. In this paper we show that there are two different types, types I and II, of STFFBs with three high-pass filters. Then we provide a detailed analysis and a complete algorithm to obtain all type I STFFBs with three high-pass filters. Our results not only significantly generalize the results in [1, 11], but also help us answer several unresolved problems on STFFBs. Based on [7], we also propose an algorithm to construct all type II STFFBs with three high-pass filters and with the shortest possible filter supports. Several examples are given to illustrate the results and algorithms in this paper.

## 1. Introduction and Motivations

Motivated by the interesting papers by Chui and He [1] and Han and Mo [11], continuing our lines developed in [7, 9] on symmetric tight framelet filter banks with a minimum number of generators, in this paper we are particularly interested in systematically studying and developing algorithms to construct all symmetric tight framelet filter banks with three high-pass filters and with the shortest possible filter supports.

To proceed further, let us recall some definitions and notation. By $l_0(\mathbb{Z})$ we denote the linear space of all sequences $u = \{u(k)\}_{k\in\mathbb{Z}} : \mathbb{Z} \to \mathbb{C}$ on $\mathbb{Z}$ such that $\{k \in \mathbb{Z} \ : \ u(k) \ne 0\}$ is a finite set. For $u = \{u(k)\}_{k\in\mathbb{Z}} \in l_0(\mathbb{Z})$, its $z$-transform is a Laurent polynomial defined to be $\mathsf{u}(z) := \sum_{k\in\mathbb{Z}} u(k)z^k$. For a matrix $\mathsf{P}(z) = \sum_{k\in\mathbb{Z}} P_k z^k$ of Laurent polynomials, we define $\mathsf{P}^\star(z) := \sum_{k\in\mathbb{Z}} \overline{P_k}^{\mathsf{T}} z^{-k}$, where $\overline{P_k}^{\mathsf{T}}$ denotes the complex conjugate of the transpose of the matrix $P_k$.

The oblique extension principle introduced in [2, 3] is a general procedure to construct tight wavelet frames through the design of tight framelet filter banks. Let $\Theta, a, b_1, \dots, b_s \in l_0(\mathbb{Z})$ with $\Theta^\star = \Theta$. We say that $\{a; b_1, \dots, b_s\}_\Theta$ is *a tight framelet filter bank* if

$$\begin{bmatrix} \mathsf{b}_1(z) & \cdots & \mathsf{b}_s(z) \\ \mathsf{b}_1(-z) & \cdots & \mathsf{b}_s(-z) \end{bmatrix} \begin{bmatrix} \mathsf{b}_1(z) & \cdots & \mathsf{b}_s(z) \\ \mathsf{b}_1(-z) & \cdots & \mathsf{b}_s(-z) \end{bmatrix}^\star = \mathcal{M}_{a,\Theta}(z), \tag{1.1}$$

where

$$\mathcal{M}_{a,\Theta}(z) := \begin{bmatrix} \Theta(z) - \Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(z) & -\Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(-z) \\ -\Theta(z^2)\mathsf{a}(-z)\mathsf{a}^\star(z) & \Theta(-z) - \Theta(z^2)\mathsf{a}(-z)\mathsf{a}^\star(-z) \end{bmatrix}. \tag{1.2}$$

In particular we write $\{a; b_1, \dots, b_s\}$ for $\{a; b_1, \dots, b_s\}_{\boldsymbol{\delta}}$, where $\boldsymbol{\delta}$ is the *Dirac sequence* such that $\boldsymbol{\delta}(0) = 1$ and $\boldsymbol{\delta}(k) = 0$ for all $k \in \mathbb{Z}\backslash\{0\}$. Recall that a sequence $u : \mathbb{Z} \to \mathbb{C}$ has *symmetry* if

$$u(k) = \epsilon u(c-k), \qquad \forall\, k \in \mathbb{Z} \quad \text{with} \quad \epsilon \in \{-1, 1\},\ c \in \mathbb{Z}. \tag{1.3}$$

The filter $u$ is symmetric if (1.3) holds with $\epsilon = 1$, and is antisymmetric if (1.3) holds with $\epsilon = -1$.

Note that (1.1) implies $\mathcal{M}^\star_{a,\Theta} = \mathcal{M}_{a,\Theta}$, from which we must have $\Theta^\star = \Theta$. Consequently, since $\Theta^\star = \Theta$, we see that $\Theta$ is symmetric if and only if $\Theta$ has real coefficients.

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Since filters that we consider in this paper are not necessarily real-valued, there is another closely related but different notion of symmetry. We say that $u$ has *complex symmetry* if

$$u(k)=\epsilon\overline{u(c-k)}, \qquad \forall\, k\in\mathbb{Z} \quad \text{with} \quad \epsilon\in\{-1,1\},\ c\in\mathbb{Z}. \tag{1.4}$$

Obviously, for a real-valued sequence $u$, there is no difference between symmetry and complex symmetry.

For a given low-pass filter $a$ and a moment correcting filter $\Theta$, to obtain high-pass filters $b_1,\ldots,b_s$ in a tight framelet filter bank, we have to factorize the given matrix $\mathcal{M}_{a,\Theta}$ in (1.2) so that (1.1) holds. To reduce computational complexity in the implementation of a tight framelet filter bank, we often prefer a small number $s$ of high-pass filters. If $s=1$, then we must have $\det(\mathcal{M}_{a,\Theta}(z))=0$ for all $z\in\mathbb{C}\backslash\{0\}$ which is too restrictive to be satisfied by many filters $a$ and $\Theta$. In fact, a tight framelet filter bank $\{a;b_1\}_\Theta$ with $s=1$ is essentially an orthogonal wavelet filter bank, see [8, Theorem 7]. When $s=2$, a necessary and sufficient condition has been given in [7, Theorem 4.2] (also see [9, 12] for special cases) in terms of the filters $a$ and $\Theta$ such that $\{a;b_1,b_2\}_\Theta$ is a tight framelet filter bank with [complex] symmetry. Moreover, several algorithms have been proposed in [7, 9] to construct tight framelet filter banks $\{a;b_1,b_2\}_\Theta$ with [complex] symmetry. However, for any given low-pass filter $a$ and a moment correcting filter $\Theta$, the necessary and sufficient condition in [7] is still too restrictive. As a matter of fact, there are only a handful examples of symmetric tight framelet filter banks $\{a;b_1,b_2\}_\Theta$ with two high-pass filters known in the literature ([2, 3, 7, 9, 12, 13, 14, 15] and references therein).

To have more flexibility in constructing tight framelet filter banks with [complex] symmetry from a given low-pass filter $a$ and a moment correcting filter $\Theta$, it is very natural to consider more than two high-pass filters. This naturally leads us to study in this paper symmetric tight framelet filter banks with three high-pass filters. For the particular case $s=3$, the perfect reconstruction condition in (1.1) can be rewritten as

$$\Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(z)+\mathsf{b}_1(z)\mathsf{b}_1^\star(z)+\mathsf{b}_2(z)\mathsf{b}_2^\star(z)+\mathsf{b}_3(z)\mathsf{b}_3^\star(z)=\Theta(z) \tag{1.5}$$

and

$$\Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(-z)+\mathsf{b}_1(z)\mathsf{b}_1^\star(-z)+\mathsf{b}_2(z)\mathsf{b}_2^\star(-z)+\mathsf{b}_3(z)\mathsf{b}_3^\star(-z)=0. \tag{1.6}$$

Currently, there are two particular constructions proposed in [1, 11] for designing symmetric tight framelet filter banks $\{a;b_1,b_2,b_3\}_\Theta$ with particular choices of moment correcting filters $\Theta$. For the special case $\Theta=\boldsymbol{\delta}$, Chui and He [1] found a simple solution for constructing a real-valued symmetric tight framelet filter bank $\{a;b_1,b_2,b_3\}$. More precisely, for any real-valued low-pass filter $a$ having symmetry and satisfying

$$\mathsf{a}(z)\mathsf{a}^\star(z)+\mathsf{a}(-z)\mathsf{a}^\star(-z)\leqslant 1, \qquad \forall\, z\in\mathbb{T}:=\{\zeta\in\mathbb{C}\ :\ |\zeta|=1\}, \tag{1.7}$$

define filters $b_1,b_2,b_3$ by (see [1, Proof of Theorem 3])

$$\mathsf{b}_1(z):=[\mathsf{u}(z^2)+z\mathsf{u}^\star(z^2)]/2, \qquad \mathsf{b}_2(z):=[\mathsf{u}(z^2)-z\mathsf{u}^\star(z^2)]/2, \qquad \mathsf{b}_3(z):=z\mathsf{a}^\star(-z), \tag{1.8}$$

where $\mathsf{u}$ is a Laurent polynomial with real coefficients obtained via the Fejér-Riesz lemma through

$$1-\mathsf{a}(z)\mathsf{a}^\star(z)-\mathsf{a}(-z)\mathsf{a}^\star(-z)=\mathsf{u}(z^2)\mathsf{u}^\star(z^2). \tag{1.9}$$

Then it is straightforward to directly check that $\{a;b_1,b_2,b_3\}$ is a real-valued tight framelet filter bank with symmetry. Conversely, if $\{a;b_1,b_2,b_3\}$ is a tight framelet filter bank, then the condition in (1.7) on the filter $a$ must hold ([1]). Indeed, from the perfect reconstruction condition in (1.1), we must have $\det(\mathcal{M}_{a,\boldsymbol{\delta}}(z))\geqslant 0$ for all $z\in\mathbb{T}$. Since $\det(\mathcal{M}_{a,\boldsymbol{\delta}}(z))=1-\mathsf{a}(z)\mathsf{a}^\star(z)-\mathsf{a}(-z)\mathsf{a}^\star(-z)$, we see that (1.7) must hold.

We now describe the method in [11]. Let $a$ be a real-valued low-pass filter with symmetry. Suppose that there exists a Laurent polynomial $\boldsymbol{\theta}$ with symmetry and real coefficients such that

$$\boldsymbol{\theta}^\star(-z)\boldsymbol{\theta}(z)=\boldsymbol{\theta}^\star(z)\boldsymbol{\theta}(-z), \qquad \boldsymbol{\theta}^\star(z)\boldsymbol{\theta}(-z)-\Theta(z^2)\geqslant 0, \qquad \forall\, z\in\mathbb{T}, \tag{1.10}$$

where

$$\Theta(z):=\boldsymbol{\theta}^\star(z)[\mathsf{a}(z)\mathsf{a}^\star(z)\boldsymbol{\theta}(-z)+\mathsf{a}(-z)\mathsf{a}^\star(-z)\boldsymbol{\theta}(z)]. \tag{1.11}$$

Define

$$\mathsf{b}_1(z) := \mathsf{a}(z)[\mathsf{v}(z^2)+\mathsf{v}^\star(z^2)]/2, \quad \mathsf{b}_2(z) := \mathsf{a}(z)[\mathsf{v}(z^2)-\mathsf{v}^\star(z^2)]/2, \quad \mathsf{b}_3(z) := z\mathsf{a}^\star(-z)\boldsymbol{\theta}^\star(z), \tag{1.12}$$

where $\mathsf{v}$ is a Laurent polynomial with real coefficients obtained via the Fejér-Riesz lemma such that $\mathsf{v}(z)\mathsf{v}^\star(z) = \boldsymbol{\theta}_0(z)$ with $\boldsymbol{\theta}_0(z^2) := \boldsymbol{\theta}^\star(z)\boldsymbol{\theta}(-z) - \Theta(z^2)$. By direct calculation, one can easily verify that $\{a; b_1, b_2, b_3\}_\Theta$ is a real-valued tight framelet filter bank with symmetry. Note that the condition in (1.10) becomes (1.7) when $\boldsymbol{\theta} = 1$. Under the condition that $\mathsf{a}(1) = 1$ and $\overline{\boldsymbol{\theta}(1)}\boldsymbol{\theta}(-1) = 1$, it has been proved in [11, Theorem 1.2] that such a desired Laurent polynomial $\boldsymbol{\theta}$ always exists provided that the standard refinable function associated with the low-pass filter $a$ has stable integer shifts. But generally the length of the moment correcting filter $\Theta$ in (1.11) is long.

The problem of symmetric tight framelet filter banks (STFFBs) with three high-pass filters seems to be completely and satisfactorily solved in [1, 11] at least for special cases of moment correcting filters. This probably partially explains that to our best knowledge there are no papers other than [1, 10, 11] available in the literature addressing STFFBs with three high-pass filters. On one hand, using three high-pass filters, we can increase the flexibility and freedom in the construction of STFFBs. On the other hand, such added flexibility and freedom by using three high-pass filters also make the task much more difficult in finding all possible STFFBs with three high-pass filters and with short supports. This paper is largely motivated by the interesting papers [1, 11] by re-examining the problem of STFFBs with three high-pass filters in a systematic way. To explain our motivations better, let us recall some definitions. For a filter $u = \{u(k)\}_{k\in\mathbb{Z}} \in l_0(\mathbb{Z})$, if $u(m)u(n) \neq 0$ and $u(k) = 0$ for all $k \in \mathbb{Z}\backslash[m, n]$, then we define the filter support and length of $u$ to be

$$\operatorname{fsupp}(\mathsf{u}) := \operatorname{fsupp}(u) := [m, n], \qquad \operatorname{len}(\mathsf{u}) := \operatorname{len}(u) := |\operatorname{fsupp}(u)| := n - m.$$

The filter support $\operatorname{fsupp}(u)$ is simply the shortest interval containing all the positions of the nonzero coefficients of $u$. From (1.5) and the fact $\Theta^\star = \Theta$ which is implied by (1.6), it is easy to see ([7, 8]) that

$$\max(\operatorname{len}(b_1), \operatorname{len}(b_2), \operatorname{len}(b_3)) \geqslant \operatorname{len}(a) + \operatorname{len}(\Theta).$$

From any given filters $a, \Theta$ with [complex] symmetry, it is natural and important to construct all tight framelet filter banks $\{a; b_1, b_2, b_3\}_\Theta$ having [complex] symmetry and the shortest possible filter supports:

$$\max(\operatorname{len}(b_1), \operatorname{len}(b_2), \operatorname{len}(b_3)) = \operatorname{len}(a) + \operatorname{len}(\Theta). \tag{1.13}$$

We now illustrate that it is not always possible to derive a symmetric tight framelet filter bank via (1.8) or (1.12) such that the condition in (1.13) holds for the shortest possible filter support. Let $a \in l_0(\mathbb{Z})$ be a nontrivial filter having symmetry and satisfying (1.7). Assume that

$$\operatorname{len}(a) \text{ is an even integer.} \tag{1.14}$$

Then we must have $\operatorname{len}(1 - \mathsf{a}(z)\mathsf{a}^\star(z) - \mathsf{a}(-z)\mathsf{a}^\star(-z)) = 2\operatorname{len}(a)$ and hence, any solution of a Laurent polynomial $\mathsf{u}$ to (1.9) must satisfy $\operatorname{len}(\mathsf{u}) = \operatorname{len}(a)/2$. Note that the two endpoints of the interval $\operatorname{fsupp}(\mathsf{u}(z^2))$ must be even integers while the two endpoints of the interval $\operatorname{fsupp}(z\mathsf{u}(z^{-2}))$ must be odd integers. Therefore, regardless of the choice of $\mathsf{u}$, for the high-pass filters $b_1$ and $b_2$ defined in (1.8), $\operatorname{len}(b_1)$ must be an odd integer, and $\operatorname{len}(b_1) = \operatorname{len}(b_2) \geqslant 2\operatorname{len}(\mathsf{u}) = \operatorname{len}(a)$. Since $\operatorname{len}(a)$ is an even integer, we conclude that $\operatorname{len}(b_1) = \operatorname{len}(b_2) > \operatorname{len}(a)$. Therefore, under the condition in (1.14), any tight framelet filter bank $\{a; b_1, b_2, b_3\}$, which is constructed via (1.8), cannot have the shortest possible filter support as described in (1.13). The same phenomenon can be said to (1.12) if $\frac{1}{2}\operatorname{len}(\Theta)$ is an odd integer. There are many filters having symmetry and satisfying both (1.7) and (1.14). For example, the $B$-spline filter $a_m^B$ of order $m$ is defined to be

$$\mathsf{a}_m^B(z) := 2^{-m}(1+z)^m, \qquad m \in \mathbb{N}. \tag{1.15}$$

Obviously, all $B$-spline filters satisfy the condition in (1.7) and $\operatorname{len}(a_m^B) = m$. For example, let $a := a_4^B$ be the $B$-spline filter of order 4. On one hand, the necessary and sufficient condition in [7, Theorem 4.2] fails for both symmetry and complex symmetry with $\Theta = \boldsymbol{\delta}$. Hence, it is impossible to derive a tight framelet filter bank $\{a; b_1, b_2\}$ with symmetry or complex symmetry from the low-pass filter $a$. On the other hand, since the condition in (1.7) is obviously true, a tight framelet filter bank $\{a; b_1, b_2, b_3\}$ with symmetry and real coefficients can be derived from the low-pass filter $a$ via (1.8). Since $\mathrm{len}(a) = 4$ is an even integer, any tight framelet filter bank obtained via (1.8) cannot have the shortest filter support as described in (1.13). This naturally motivates us to ask the following question:

Q1: *For every $B$-spline filter $a^B_m$ of order $m \in \mathbb{N}$, is it possible to construct a symmetric real-valued tight framelet filter bank $\{a^B_m; b_1, b_2, b_3\}$ such that* $\max(\mathrm{len}(b_1), \mathrm{len}(b_2), \mathrm{len}(b_3)) = \mathrm{len}(a^B_m)$*?*

Though the above problem for $B$-spline filters has been checked for the particular cases $m = 3, \dots, 6$ in [1], it remained unclear whether a positive answer to all $B$-spline filters is always possible. More generally, since in this paper we are interested in obtaining all symmetric tight framelet filter banks with three high-pass filters and with the shortest possible filter support, it is of interest to ask whether or not the methods proposed in [1, 11] essentially yield all symmetric tight framelet filter banks $\{a; b_1, b_2, b_3\}$ having the shortest possible filter support. Based on our results in this paper, we shall show that for every $B$-spline filter $a^B_m$, we can always construct a symmetric real-valued tight framelet filter bank $\{a^B_m; b_1, b_2, b_3\}$ such that $\max(\mathrm{len}(b_1), \mathrm{len}(b_2), \mathrm{len}(b_3)) = \mathrm{len}(a^B_m)$. We shall also show that many symmetric tight framelet filter banks with the shortest possible filter support cannot be obtained via the methods in [1, 11].

The main purpose of this paper is to find and construct all possible tight framelet filter banks $\{a; b_1, b_2, b_3\}_\Theta$ with [complex] symmetry and with the shortest possible filter supports, when filters $a$ and $\Theta$ are given in advance and satisfy the necessary condition $\det(\mathcal{M}_{a,\Theta}(z)) \geqslant 0$ for all $z \in \mathbb{T}$. On one hand, the method in [1] can only handle the special case $\Theta = \boldsymbol{\delta}$ and its generalization to a general moment correcting filter is not available in the literature. On the other hand, the method in [11] can only handle moment correcting filters $\Theta$ taking the particular form in (1.11). Therefore, it is of interest for us to find an algorithm to handle the general case for constructing tight framelet filter banks $\{a; b_1, b_2, b_3\}_\Theta$ with [complex] symmetry from any given filters $a$ and $\Theta$.

By studying the relations of symmetry centers of filters in a symmetric tight framelet filter bank with three high-pass filters, we classify all such tight framelet filter banks into type I and type II. A detailed study of type I symmetric tight framelet filter banks $\{a; b_1, b_2, b_3\}_\Theta$ leads to a complete algorithm (for type I), which enables us to obtain all type I symmetric tight framelet filter banks with three high-pass filters (and without any support constraint). This not only generalizes the method in [1] (see (1.8)) to a general moment correcting filter, but also widens the method in [1] even for the special case $\Theta = \boldsymbol{\delta}$ by finding new tight framelet filter banks $\{a; b_1, b_2, b_3\}$ which are impossible to obtain via (1.8). Based on these results, we are able to show that a symmetric real-valued tight framelet filter bank $\{a^B_m; b_1, b_2, b_3\}$ having the shortest possible filter support can always be constructed from every $B$-spline filter $a^B_m$. This affirmatively settles Q1. In fact, our algorithm for type I in this paper can construct not only all type I symmetric tight framelet filter banks with three high-pass filters but also some type II tight framelet filter banks with three high-pass filters. Built on results in [7] for symmetric tight framelet filter banks with two high-pass filters, in this paper we also propose another algorithm (for type II) to construct all symmetric tight framelet filter banks with three high-pass filters and with the shortest possible filter supports.

This paper is also motivated by the interpolatory filter $a^I_4 = \{-\frac{1}{32}, 0, \frac{9}{32}, \underline{\boldsymbol{\frac{1}{2}}}, \frac{9}{32}, 0, -\frac{1}{32}\}_{[-3,3]}$ which satisfies the condition in (1.7) with $\mathrm{len}(a^I_4) = 6$. Therefore, as we discussed above, a real-valued symmetric tight framelet filter bank $\{a^I_4; b_1, b_2, b_3\}$ satisfying $\max(\mathrm{len}(b_1), \mathrm{len}(b_2), \mathrm{len}(b_3)) = \mathrm{len}(a^I_4)$ cannot be constructed using the method in (1.8). Indeed, the example $\{a^I_4; b_1, b_2, b_3\}$ constructed in [1, Example 9] has $\max(\mathrm{len}(b_1), \mathrm{len}(b_2), \mathrm{len}(b_3)) = 7 > \mathrm{len}(a^I_4)$. More generally, we shall show in Example 4 of this paper that every tight framelet filter bank $\{a^I_4; b_1, b_2, b_3\}$ with symmetry, which is constructed by our algorithm for type I (see Algorithm 1), must satisfy $\max(\mathrm{len}(b_1), \mathrm{len}(b_2), \mathrm{len}(b_3)) \geqslant 7 > \mathrm{len}(a^I_4)$. On the other hand, we shall prove in Theorem 6 that for any filter $a$ having complex symmetry and satisfying (1.7), we can always construct a (complex-valued) tight framelet filter bank $\{a; b_1, b_2, b_3\}$ with complex symmetry such that $\max(\mathrm{len}(b_1), \mathrm{len}(b_2), \mathrm{len}(b_3)) = \mathrm{len}(a)$. As a consequence, we indeed can construct a (complex-valued) tight framelet filter bank $\{a^I_4; b_1, b_2, b_3\}$ with complex symmetry and with the shortest possible filter support. This naturally leads us to the following question:

Q2: *Does there exist a real-valued symmetric tight framelet filter bank* $\{a_4^I; b_1, b_2, b_3\}$ *such that* $\max(\mathrm{len}(b_1), \mathrm{len}(b_2), \mathrm{len}(b_3)) = \mathrm{len}(a_4^I)$*?*

Using our algorithm for type II (see Algorithm 2) developed in this paper, we shall provide a positive answer to the above question Q2.

The structure of the paper is as follows. In Section 2 we shall study the relations of symmetry centers of a symmetric tight framelet filter bank $\{a; b_1, b_2, b_3\}_\Theta$. Using the relations between symmetry centers, we naturally classify all symmetric tight framelet filter banks $\{a; b_1, b_2, b_3\}_\Theta$ into type I and type II. We then provide a detailed analysis and a complete algorithm for type I symmetric tight framelet filter banks $\{a; b_1, b_2, b_3\}_\Theta$ in Section 2. To illustrate our algorithm developed in Section 2 for type I, we present several examples of tight framelet filter banks $\{a; b_1, b_2, b_3\}_\Theta$ with [complex] symmetry in Section 3. In Section 4, we first investigate the symmetry patterns of a symmetric tight framelet filter bank $\{a; b_1, b_2, b_3\}_\Theta$ having the shortest possible filter supports in (1.13). Based on this result and [7] for symmetric tight framelet filter banks with two high-pass filters, we propose an algorithm for type II in Section 4 to construct all symmetric tight framelet filter banks with three high-pass filters and with the shortest possible filter supports. Using our algorithm for type II, we shall provide some examples of type II symmetric tight framelet filter banks with three high-pass filters in Section 4.

## 2. Type I Symmetric Tight Framelet Filter Banks with Three High-pass Filters

In this section we first show that there are two types of tight framelet filter banks $\{a; b_1, b_2, b_3\}_\Theta$ with [complex] symmetry, according to the relations between symmetry centers of the filters. Then we provide a detailed analysis and a complete algorithm for type I tight framelet filter banks $\{a; b_1, b_2, b_3\}_\Theta$ with [complex] symmetry. As a byproduct, our results in this section not only generalize the method in [1] and provide a natural explanation for the method in [1] (see (1.8) for detail) but also enable us to settle the question Q1 in Section 1.

Since we shall extensively discuss filters having symmetry or complex symmetry in this paper, it is convenient for us to record the symmetry pattern of a filter by using symmetry operators. For a nontrivial filter $u$, we define the *symmetry operator* $\mathsf{S}$ and the *complex symmetry operator* $\mathbb{S}$ to be

$$\mathsf{S}\mathsf{u}(z) := \frac{\mathsf{u}(z)}{\mathsf{u}(z^{-1})} \qquad \text{and} \qquad \mathbb{S}\mathsf{u}(z) := \frac{\mathsf{u}(z)}{\mathsf{u}^\star(z)}.$$

It is obvious that a filter $u$ has symmetry in (1.3) if and only if $\mathsf{S}\mathsf{u}(z) = \epsilon z^c$. Similarly, a filter $u$ has complex symmetry in (1.4) if and only if $\mathbb{S}\mathsf{u}(z) = \epsilon z^c$.

We now investigate the relations between symmetry centers of filters in a tight framelet filter bank $\{a; b_1, b_2, b_3\}_\Theta$ with [complex] symmetry. According to the following result, there are two types of tight framelet filter banks $\{a; b_1, b_2, b_3\}_\Theta$ with [complex] symmetry.

**Proposition 1.** *Let* $\{a; b_1, b_2, b_3\}_\Theta$ *be a tight framelet filter bank such that all the filters* $\Theta, a, b_1, b_2, b_3 \in l_0(\mathbb{Z})$ *are not identically zero and have symmetry (or complex symmetry by replacing* $\mathsf{S}$ *with* $\mathbb{S}$ *below):*

$$\mathsf{S}\Theta(z) = 1, \qquad \mathsf{S}\mathsf{a}(z) = \epsilon z^c, \quad \mathsf{S}\mathsf{b}_1(z) = \epsilon_1 z^{c_1}, \quad \mathsf{S}\mathsf{b}_2(z) = \epsilon_2 z^{c_2}, \quad \mathsf{S}\mathsf{b}_3(z) = \epsilon_3 z^{c_3} \tag{2.1}$$

*for some* $\epsilon, \epsilon_1, \epsilon_2, \epsilon_3 \in \{-1, 1\}$ *and* $c, c_1, c_2, c_3 \in \mathbb{Z}$*. Up to reordering of* $b_1, b_2, b_3$*, one of the following two types must hold:*

Type I: $c_3 - c$ *is even and* $c_1 - c, c_2 - c$ *are odd. Moreover, the following identities must hold:*

$$\Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(-z) + \mathsf{b}_3(z)\mathsf{b}_3^\star(-z) = 0, \quad \mathsf{b}_1(z)\mathsf{b}_1^\star(-z) + \mathsf{b}_2(z)\mathsf{b}_2^\star(-z) = 0; \tag{2.2}$$

Type II: *all* $c_3 - c$, $c_1 - c, c_2 - c$ *are even.*

*Proof.* Using the fact that $\mathsf{S}(\mathsf{u}(z)\mathsf{v}(z)) = \mathsf{S}\mathsf{u}(z)\mathsf{S}\mathsf{v}(z)$, we see that

$$\mathsf{S}(\Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(-z)) = (-1)^c, \qquad \mathsf{S}(\mathsf{b}_\ell(z)\mathsf{b}_\ell^\star(-z)) = (-1)^{c_\ell}, \qquad \ell = 1, 2, 3.$$

If $(-1)^{c_\ell} \neq (-1)^c$ for all $\ell = 1, 2, 3$, moving all the terms involving $\mathsf{b}_1, \mathsf{b}_2, \mathsf{b}_3$ to the right side of (1.6), then we must have $\Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(-z) = 0$, which is a contradiction to our assumption. By the same argument, we see that there are exactly either one or three of $(-1)^{c_1}, (-1)^{c_2}, (-1)^{c_3}$ having the same value as $(-1)^c$. Without loss of generality, we assume $(-1)^{c_3} = (-1)^c \neq (-1)^{c_1} = (-1)^{c_2}$ which is type I, or $(-1)^{c_1} = (-1)^{c_2} = (-1)^{c_3} = (-1)^c$ which is type II.

For type I, by (1.6), since the two sides of

$$\Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(-z) + \mathsf{b}_3(z)\mathsf{b}_3^\star(-z) = -\mathsf{b}_1(z)\mathsf{b}_1^\star(-z) - \mathsf{b}_2(z)\mathsf{b}_2^\star(-z)$$

have different symmetry patterns $(-1)^c$ and $(-1)^{c+1}$, we see that (2.2) must hold. $\square$

The filter $a$ in a tight framelet filter bank $\{a; b_1, b_2, b_3\}_\Theta$ is often a low-pass filter satisfying $\mathsf{a}(1) = 1$, which will force $\epsilon = 1$ in Proposition 1. The constructed symmetric tight framelet filter banks with three high-pass filters in [1] can be either Type I or Type II, but all of them satisfy the special condition in (2.2), which can be easily seen from (1.8). However, all the symmetric tight framelet filter banks $\{a; b_1, b_2, b_3\}_\Theta$ constructed in [10, 11] must be Type II, since (2.2) cannot hold due to (1.12).

In the rest of this section, we provide a detailed study for all symmetric tight framelet filter banks $\{a; b_1, b_2, b_3\}_\Theta$ satisfying the condition in (2.2).

Let us first investigate moment correcting filters in a symmetric tight framelet filter bank. For a Laurent polynomial $\mathsf{p}$ and $z_0 \in \mathbb{C}\backslash\{0\}$, by $\mathsf{Z}(\mathsf{p}, z_0)$ we denote the multiplicity of zeros of $\mathsf{p}(z)$ at $z = z_0$. The following result provides a necessary and sufficient condition for $\Theta(z^2) = \boldsymbol{\theta}(z)\boldsymbol{\theta}^\star(-z)$ and its proof provides an algorithm for constructing all the possible desired Laurent polynomials $\boldsymbol{\theta}$ satisfying $\Theta(z^2) = \boldsymbol{\theta}(z)\boldsymbol{\theta}^\star(-z)$.

**Theorem 2.** *Let $\Theta \in l_0(\mathbb{Z})$ such that $\Theta$ is not identically zero.*

(i) *There exists a Laurent polynomial $\boldsymbol{\theta}$ such that $\Theta(z^2) = \boldsymbol{\theta}(z)\boldsymbol{\theta}^\star(-z)$ for all $z \in \mathbb{C}\backslash\{0\}$ if and only if $\Theta^\star = \Theta$. Moreover, if $\Theta^\star = \Theta$, then there must exist a Laurent polynomial $\boldsymbol{\theta}$ with complex symmetry such that $\Theta(z^2) = \boldsymbol{\theta}(z)\boldsymbol{\theta}^\star(-z)$.*

(ii) *There exists a Laurent polynomial $\boldsymbol{\theta}$ with symmetry such that $\Theta(z^2) = \boldsymbol{\theta}(z)\boldsymbol{\theta}^\star(-z)$ if and only if $\Theta^\star = \Theta$, $\Theta$ has real coefficients, and $\mathsf{Z}(\Theta, x)$ is an even integer for every $-1 \leqslant x < 0$. Moreover, if the three conditions on $\Theta$ are satisfied, then there must exist a Laurent polynomial $\boldsymbol{\theta}$ with symmetry and real coefficients such that $\Theta(z^2) = \boldsymbol{\theta}(z)\boldsymbol{\theta}^\star(-z)$.*

*For both (i) and (ii), the symmetry center of $\boldsymbol{\theta}$ satisfying $\Theta(z^2) = \boldsymbol{\theta}(z)\boldsymbol{\theta}^\star(-z)$ must be an integer.*

*Proof.* We first show that $\Theta(z^2) = \boldsymbol{\theta}(z)\boldsymbol{\theta}^\star(-z)$ always implies $\Theta^\star = \Theta$, and if $\boldsymbol{\theta}$ has symmetry or complex symmetry, then the symmetry center of $\boldsymbol{\theta}$ must be an integer. Indeed, we have $\Theta^\star(z^2) = \boldsymbol{\theta}^\star(z)\boldsymbol{\theta}(-z)$. Replacing $z$ by $-z$, we conclude that $\Theta^\star(z^2) = \boldsymbol{\theta}^\star(-z)\boldsymbol{\theta}(z) = \Theta(z^2)$. Hence, $\Theta^\star = \Theta$. Suppose that $\boldsymbol{\theta}$ has symmetry or complex symmetry and $\frac{c}{2}$, with $c \in \mathbb{Z}$, is the symmetry center of $\boldsymbol{\theta}$. Then $\mathrm{fsupp}(\boldsymbol{\theta}) = [c - n_{\boldsymbol{\theta}}, n_{\boldsymbol{\theta}}]$ for some integer $n_{\boldsymbol{\theta}}$. Now it is trivial to see that $\mathrm{fsupp}(\boldsymbol{\theta}(z)\boldsymbol{\theta}^\star(-z)) = [c - 2n_{\boldsymbol{\theta}}, 2n_{\boldsymbol{\theta}} - c]$. Since the endpoints of the interval $\mathrm{fsupp}(\Theta(z^2))$ must be even integers, we conclude from $\mathrm{fsupp}(\boldsymbol{\theta}(z)\boldsymbol{\theta}^\star(-z)) = \mathrm{fsupp}(\Theta(z^2))$ that $c$ must be an even integer; that is, the symmetry center $\frac{c}{2}$ of $\boldsymbol{\theta}$ must be an integer.

We just proved the necessity part of item (i). We now prove the sufficiency part of item (i) with complex symmetry. We construct $\boldsymbol{\theta}$ as a product of some selected factors from $\Theta(z^2)$. Since $\Theta$ has the complex symmetry $\Theta^\star = \Theta$, by [7, Proposition 2.2], according to the location of the roots of $\Theta$, the Laurent polynomial $\Theta$ has two types of factors:

Type 1: $(z - \zeta)^{\mathsf{Z}(\Theta,\zeta)}$ with $\zeta \in \mathbb{C}$ and $|\zeta| = 1$ (that is, $\zeta \in \mathbb{T}$). $\boldsymbol{\theta}$ takes the factor

$$(z - \sqrt{\zeta})^m (z + \sqrt{\zeta})^n \quad \text{with} \quad m + n = \mathsf{Z}(\Theta, \zeta),$$

where $m$ and $n$ are nonnegative integers and $\sqrt{\zeta} \in \mathbb{C}$ is a solution to $(\sqrt{\zeta})^2 = \zeta$.

Type 2: $[(z - \zeta)(z^{-1} - \overline{\zeta})]^{\mathsf{Z}(\Theta,\zeta)}$ with $\zeta \in \mathbb{C}$ and $0 < |\zeta| < 1$. $\boldsymbol{\theta}$ takes the factor

$$[(z - \sqrt{\zeta})(z^{-1} - \overline{\sqrt{\zeta}})]^m [(z + \sqrt{\zeta})(z^{-1} + \overline{\sqrt{\zeta}})]^n \quad \text{with} \quad m + n = \mathsf{Z}(\Theta, \zeta),$$

where $m$ and $n$ are nonnegative integers.

Then $\mathsf{Z}(\boldsymbol{\theta}, z) = \mathsf{Z}(\boldsymbol{\theta}, \overline{z}^{-1})$ for all $z \in \mathbb{C}\backslash\{0\}$. By [7, Proposition 2.2], after multiplying some number from $\mathbb{T}$ with $\boldsymbol{\theta}$, $\boldsymbol{\theta}$ has complex symmetry. By construction, the two Laurent polynomials $\Theta(z^2)$ and $\boldsymbol{\theta}(z)\boldsymbol{\theta}^\star(-z)$ have the same zeros on $\mathbb{C}\backslash\{0\}$. Hence, $\Theta(z^2) = \lambda z^m \boldsymbol{\theta}(z)\boldsymbol{\theta}^\star(-z)$ for some constant $\lambda \in \mathbb{C}\backslash\{0\}$

and $m \in \mathbb{Z}$. By $\Theta^\star(z^2)=\Theta(z^2)$ and noting $\lambda(-z)^m\boldsymbol{\theta}(-z)\boldsymbol{\theta}^\star(z)=\Theta((-z)^2)=\Theta(z^2)=\lambda z^m\boldsymbol{\theta}(z)\boldsymbol{\theta}^\star(-z)$, we have $\boldsymbol{\theta}^\star(z)\boldsymbol{\theta}(-z)=(-1)^m\boldsymbol{\theta}(z)\boldsymbol{\theta}^\star(-z)$ and

$$\lambda z^m\boldsymbol{\theta}(z)\boldsymbol{\theta}^\star(-z)=\Theta(z^2)=\Theta^\star(z^2)=\overline{\lambda}z^{-m}\boldsymbol{\theta}^\star(z)\boldsymbol{\theta}(-z)=\overline{\lambda}(-1)^m z^{-m}\boldsymbol{\theta}(z)\boldsymbol{\theta}^\star(-z).$$

Thus, $\lambda z^m=\overline{\lambda}(-1)^m z^{-m}$ which implies $m=0$ and $\lambda\in\mathbb{R}\backslash\{0\}$. If $\lambda>0$, replace $\boldsymbol{\theta}$ by $\sqrt{\lambda}\boldsymbol{\theta}$; if $\lambda<0$, replace $\boldsymbol{\theta}$ by $\sqrt{|\lambda|}z\boldsymbol{\theta}(z)$. Now it is straightforward to check that $\Theta(z^2)=\boldsymbol{\theta}(z)\boldsymbol{\theta}^\star(-z)$.

Necessity of item (ii). We proved that $\Theta^\star=\Theta$ and hence $\mathbb{S}\Theta=1$. Since $\boldsymbol{\theta}$ has symmetry, say, $\mathsf{S}\boldsymbol{\theta}(z)=\epsilon_{\boldsymbol{\theta}}z^{c_{\boldsymbol{\theta}}}$, then $\Theta$ must have symmetry $\mathsf{S}\Theta(z^2)=\mathsf{S}\boldsymbol{\theta}(z)\mathsf{S}\boldsymbol{\theta}^\star(-z)=(-1)^{c_{\boldsymbol{\theta}}}$. Since we proved that $c_{\boldsymbol{\theta}}$ must be an even integer, we have $\mathsf{S}\Theta=\mathbb{S}\Theta=1$. Therefore, by [7, Lemma 2.3], $\Theta$ must have real coefficients.

Plugging $z=ix, x\in\mathbb{R}\backslash\{0\}$ into $\Theta(z^2)=\boldsymbol{\theta}(z)\boldsymbol{\theta}^\star(-z)=\boldsymbol{\theta}(z)\overline{\boldsymbol{\theta}(-\bar{z}^{-1})}$, we deduce that

$$\Theta(-x^2)=\boldsymbol{\theta}(ix)\overline{\boldsymbol{\theta}((ix)^{-1})}=\boldsymbol{\theta}(ix)\overline{\boldsymbol{\theta}(ix)}/\overline{\mathsf{S}\boldsymbol{\theta}(ix)}=|\boldsymbol{\theta}(ix)|^2/\overline{\mathsf{S}\boldsymbol{\theta}(ix)}.$$

Hence, $\mathsf{Z}(\Theta,-x^2)$ must be an even integer for all $x>0$. This proves the necessity part of item (ii).

Sufficiency of item (ii). Since $\Theta$ has symmetry and real coefficients, $\Theta$ has six types of factors:
Type 1: $(z-1)^{\mathsf{Z}(\Theta,1)}$. $\boldsymbol{\theta}$ takes the factor $(z-1)^m(z+1)^n$ with $m+n=\mathsf{Z}(\Theta,1)$, where $m$ and $n$ are nonnegative integers.
Type 2: $(z+1)^{\mathsf{Z}(\Theta,-1)}$. By our assumption, $\mathsf{Z}(\Theta,-1)$ is an even integer. $\boldsymbol{\theta}$ takes the factor $(z^2+1)^{\mathsf{Z}(\Theta,-1)/2}$.
Type 3: $[(z-\zeta)(z^{-1}-\zeta)]^{\mathsf{Z}(\Theta,\zeta)}$ with $0<\zeta<1$. $\boldsymbol{\theta}$ takes the factor

$$[(z-\sqrt{\zeta})(z^{-1}-\sqrt{\zeta})]^m[(z+\sqrt{\zeta})(z^{-1}+\sqrt{\zeta})]^n \quad\text{with}\quad m+n=\mathsf{Z}(\Theta,\zeta).$$

Type 4: $[(z-\zeta)(z^{-1}-\zeta)]^{\mathsf{Z}(\Theta,\zeta)}$ with $-1<\zeta<0$. By our assumption, $\mathsf{Z}(\Theta,\zeta)$ is an even integer. $\boldsymbol{\theta}$ takes the factor $[(z^2-\zeta)(z^{-2}-\zeta)]^{\mathsf{Z}(\Theta,\zeta)/2}$.
Type 5: $[(z-\zeta)(z-\overline{\zeta})]^{\mathsf{Z}(\Theta,\zeta)}$ with $\zeta\in\mathbb{T}\backslash\{-1,1\}$. $\boldsymbol{\theta}$ takes the factor $[(z-\sqrt{\zeta})(z-\overline{\sqrt{\zeta}})]^m[(z+\sqrt{\zeta})(z+\overline{\sqrt{\zeta}})]^n$ with $m+n=\mathsf{Z}(\Theta,\zeta)$.
Type 6: $[(z^2-2\operatorname{Re}(\zeta)z+|\zeta|^2)(z^{-2}-2\operatorname{Re}(\zeta)z^{-1}+|\zeta|^2)]^{\mathsf{Z}(\Theta,\zeta)}$ with $0<|\zeta|<1$ and $\zeta\notin\mathbb{R}$. $\boldsymbol{\theta}$ takes the factor

$$[(z^2-2\operatorname{Re}(\sqrt{\zeta})z+|\zeta|)(z^{-2}-2\operatorname{Re}(\sqrt{\zeta})z^{-1}+|\zeta|)]^m[(z^2+2\operatorname{Re}(\sqrt{\zeta})z+|\zeta|)(z^{-2}+2\operatorname{Re}(\sqrt{\zeta})z^{-1}+|\zeta|)]^n$$

with $m+n=\mathsf{Z}(\Theta,\zeta)$. Then $\mathsf{Z}(\boldsymbol{\theta},z)=\mathsf{Z}(\boldsymbol{\theta},z^{-1})$ for all $z\in\mathbb{C}\backslash\{0\}$ and $\boldsymbol{\theta}$ has real coefficients. By [7, Proposition 2.2], $\boldsymbol{\theta}$ must have symmetry. By the same argument as in item (i), after multiplying a monomial with $\boldsymbol{\theta}$, we have $\Theta(z^2)=\boldsymbol{\theta}(z)\boldsymbol{\theta}^\star(-z)$. ☐

If $\Theta=1$, then it is easy to see that all the solutions $\boldsymbol{\theta}$ to $\Theta(z^2)=\boldsymbol{\theta}(z)\boldsymbol{\theta}^\star(-z)$ are $\boldsymbol{\theta}(z)=\lambda z^{2k}$ with $\lambda\in\mathbb{T}$ and $k\in\mathbb{Z}$.

To study (2.2), we need the following result.

**Lemma 3.** *For filters $a,b,\Theta\in l_0(\mathbb{Z})$ which are not identically zero,*

$$\Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(-z)+\mathsf{b}(z)\mathsf{b}^\star(-z)=0 \tag{2.3}$$

*if and only if*

$$\Theta(z^2)=\boldsymbol{\theta}(z)\boldsymbol{\theta}^\star(-z),\quad \mathsf{a}(z)=\mathsf{d_a}(z)\mathring{\mathsf{a}}(z),\quad \mathsf{b}(z)=\mathsf{d_a}(z)\boldsymbol{\theta}(z)z\mathring{\mathsf{a}}^\star(-z) \tag{2.4}$$

*for some Laurent polynomials $\boldsymbol{\theta},\mathsf{d_a}$, and $\mathring{\mathsf{a}}$. Moreover, if all $a,b,\Theta$ have [complex] symmetry (and/or real coefficients), then all Laurent polynomials $\boldsymbol{\theta},\mathsf{d_a},\mathring{\mathsf{a}}$ have [complex] symmetry (and/or real coefficients).*

*Proof.* The sufficiency part can be directly verified. We only prove the necessity part. Define $\mathsf{d_a}:=\gcd(\mathsf{a},\mathsf{b})$, the greatest common divisor of $\mathsf{a}$ and $\mathsf{b}$. Then we can write $\mathsf{a}(z)=\mathsf{d_a}(z)\mathring{\mathsf{a}}(z)$ and $\mathsf{b}(z)=\mathsf{d_a}(z)\mathring{\mathsf{b}}(z)$ for some Laurent polynomials $\mathring{\mathsf{a}}$ and $\mathring{\mathsf{b}}$. Now (2.3) holds if and only if

$$\Theta(z^2)\mathring{\mathsf{a}}(z)\mathring{\mathsf{a}}^\star(-z)+\mathring{\mathsf{b}}(z)\mathring{\mathsf{b}}^\star(-z)=0. \tag{2.5}$$

Since $\gcd(\mathring{a}, \mathring{b}) = 1$, we must have $\mathring{b}(z) \mid \Theta(z^2)\mathring{a}^\star(-z)$. Hence, $\Theta(z^2)\mathring{a}^\star(-z) = z^{-1}\boldsymbol{\theta}^\star(-z)\mathring{b}(z)$ for some Laurent polynomial $\boldsymbol{\theta}$. Plugging this relation back into (2.5), we see that $\mathring{b}(z)[z^{-1}\boldsymbol{\theta}^\star(-z)\mathring{a}(z)+\mathring{b}^\star(-z)] = 0$. Therefore, we must have $\mathring{b}(z) = z\boldsymbol{\theta}(z)\mathring{a}^\star(-z)$ and hence $b(z) = d_a(z)\mathring{b}(z) = d_a(z)\boldsymbol{\theta}(z)z\mathring{a}^\star(-z)$. Now

$$\Theta(z^2)\mathring{a}^\star(-z) = z^{-1}\boldsymbol{\theta}^\star(-z)\mathring{b}(z) = \boldsymbol{\theta}(z)\boldsymbol{\theta}^\star(-z)\mathring{a}^\star(-z),$$

from which we see that $\Theta(z^2) = \boldsymbol{\theta}(z)\boldsymbol{\theta}^\star(-z)$.

If all filters $a, b, \Theta$ have [complex] symmetry (and/or real coefficients), then $d_a := \gcd(a, b)$ can have [complex] symmetry (and/or real coefficients), see [7, Lemma 2.4]. Now it is easy to check that all filters $\boldsymbol{\theta}, d_a, \mathring{a}$ have [complex] symmetry (and/or real coefficients). ☐

We now have a complete picture of (2.2) in Proposition 1 which covers all type I tight framelet filter banks $\{a; b_1, b_2, b_3\}_\Theta$ with [complex] symmetry.

**Theorem 4.** *Let $\Theta, a, b_1, b_2, b_3 \in l_0(\mathbb{Z})$ be filters which are not identically zero. Then $\{a; b_1, b_2, b_3\}_\Theta$ is a tight framelet filter bank with [complex] symmetry (and real coefficients) satisfying* (2.2) *if and only if*

$$\Theta(z^2) = \boldsymbol{\theta}(z)\boldsymbol{\theta}^\star(-z), \quad a(z) = d_a(z)\mathring{a}(z), \quad b_3(z) = d_a(z)\boldsymbol{\theta}(z)z\mathring{a}^\star(-z) \tag{2.6}$$

*and*

$$b_1(z)b_1^\star(z) + b_2(z)b_2^\star(z) = p(z), \qquad b_1(z)b_1^\star(-z) + b_2(z)b_2^\star(-z) = 0, \tag{2.7}$$

*where $\boldsymbol{\theta}, d_a, \mathring{a}$ are Laurent polynomials with [complex] symmetry (and real coefficients) and*

$$p(z) := \Theta(z) - d_a(z)d_a^\star(z)\boldsymbol{\theta}(z)[\mathring{a}(z)\mathring{a}^\star(z)\boldsymbol{\theta}^\star(-z) + \mathring{a}(-z)\mathring{a}^\star(-z)\boldsymbol{\theta}^\star(z)]. \tag{2.8}$$

*Proof.* (2.6) follows directly from Lemma 3. The second part of (2.7) is from (2.2). By (1.5) and (2.6), we have

$$b_1(z)b_1^\star(z) + b_2(z)b_2^\star(z) = \Theta(z) - \Theta(z^2)a(z)a^\star(z) - b_3(z)b_3^\star(z) = p(z).$$

The sufficiency part can be directly verified. ☐

We have the following necessary and sufficient condition to completely solve (2.7) as follows:

**Theorem 5.** *If there exist $b_1, b_2$ of Laurent polynomials with [complex] symmetry (and real coefficients) such that* (2.7) *holds, then*

(i) *$p(z) \geqslant 0$ for all $z \in \mathbb{T}$. For the case of symmetry (or real coefficients), $p$ has real coefficients;*

(ii) *$\frac{p(z)}{q(z^2)} = d_p(z)d_p^\star(z)$ for some Laurent polynomial $d_p$ with [complex] symmetry (and real coefficients), where $q(z^2) := \gcd(p(z), p(-z))$.*

*Conversely, if items (i) and (ii) are satisfied, then there exists a solution $\{b_1, b_2\}$ of Laurent polynomials to* (2.7) *and all the solutions $\{b_1, b_2\}$ of Laurent polynomials to* (2.7), *with $b_1, b_2$ having [complex] symmetry (and real coefficients), are given by*

$$b_1(z) = d_p(z)d_q(z)b(z), \qquad b_2(z) = d_p(z)d_q(z)\lambda z^{2k+1}b^\star(-z), \tag{2.9}$$

*where $k \in \mathbb{Z}$, $\lambda \in \mathbb{T}$ ($\lambda \in \{\pm 1, \pm i\}$ for the case of complex symmetry; $\lambda \in \{\pm 1\}$ for the case of real coefficients), $d_q$ and $b$ are Laurent polynomials with [complex] symmetry (and real coefficients) satisfying*

$$d_q(z)d_q^\star(z) = d_q(-z)d_q^\star(-z), \qquad d_q(z)d_q^\star(z) \mid q(z^2) \tag{2.10}$$

*and*

$$b(z)b^\star(z) + b(-z)b^\star(-z) = \frac{q(z^2)}{d_q(z)d_q^\star(z)}. \tag{2.11}$$

*For complex symmetry (and real coefficients), without loss of any solutions, we can take $d_q = 1$.*

*Proof.* Obviously, (2.7) implies $p(z) = b_1(z)b_1^\star(z) + b_2(z)b_2^\star(z) \geqslant 0$ for all $z \in \mathbb{T}$ and $\mathbb{S}p = 1$ by $p^\star = p$. For the case of symmetry, since $b_1$ and $b_2$ have symmetry, we have $S(b_1(z)b_1^\star(z)) = Sb_1(z)Sb_1^\star(z) = 1 = S(b_2(z)b_2^\star(z))$. Therefore, $Sp = 1$. By [7, Lemma 2.3] and $Sp = \mathbb{S}p = 1$, the Laurent polynomial $p$ must have real coefficients.

Applying Lemma 3 with $\Theta = 1$ to the second identity in (2.7), we have $\boldsymbol{\theta}(z) = \lambda z^{2k}$ and

$$b_1(z) = \mathring{d}(z)\mathring{b}(z), \qquad b_2(z) = \mathring{d}(z)\lambda z^{2k+1}\mathring{b}^\star(-z), \tag{2.12}$$

where $k \in \mathbb{Z}$, $\lambda \in \mathbb{T}$ ($\lambda \in \{\pm 1, \pm i\}$ for the case of complex symmetry; $\lambda \in \{\pm 1\}$ for the case of real coefficients), and $\mathring{\mathsf{d}}, \mathring{\mathsf{b}}$ are Laurent polynomials with [complex] symmetry (and real coefficients). Hence, the first identity in (2.7) becomes

$$\mathring{\mathsf{d}}(z)\mathring{\mathsf{d}}^\star(z)\mathsf{C}(z^2) = \mathsf{p}(z) \quad \text{with} \quad \mathsf{C}(z^2) := \mathring{\mathsf{b}}(z)\mathring{\mathsf{b}}^\star(z) + \mathring{\mathsf{b}}(-z)\mathring{\mathsf{b}}^\star(-z). \tag{2.13}$$

By (2.13) and the definition $\mathsf{q}(z^2) = \gcd(\mathsf{p}(z), \mathsf{p}(-z))$, we see that

$$\mathsf{q}(z^2) = \gcd(\mathsf{p}(z), \mathsf{p}(-z)) = \mathsf{C}(z^2)\mathsf{D}(z^2) \quad \text{with} \quad \mathsf{D}(z^2) := \gcd(\mathring{\mathsf{d}}(z)\mathring{\mathsf{d}}^\star(z), \mathring{\mathsf{d}}(-z)\mathring{\mathsf{d}}^\star(-z)). \tag{2.14}$$

Hence, by (2.13) again,

$$\frac{\mathsf{p}(z)}{\mathsf{q}(z^2)} = \frac{\mathring{\mathsf{d}}(z)\mathring{\mathsf{d}}^\star(z)\mathsf{C}(z^2)}{\mathsf{C}(z^2)\mathsf{D}(z^2)} = \frac{\mathring{\mathsf{d}}(z)\mathring{\mathsf{d}}^\star(z)}{\mathsf{D}(z^2)}. \tag{2.15}$$

For the case of symmetry (and real coefficients), since $\mathring{\mathsf{d}}$ has symmetry, by the trivial relation $\mathring{\mathsf{d}}^\star(z) = \overline{\mathring{\mathsf{d}}(\bar{z}^{-1})}$, we have $\mathsf{Z}(\mathring{\mathsf{d}}^\star, x) = \mathsf{Z}(\mathring{\mathsf{d}}, x)$ for all $x \in \mathbb{R}\backslash\{0\}$. Therefore, by the definition of $\mathsf{D}$ in (2.14), for all $x \in \mathbb{R}\backslash\{0\}$,

$$\mathsf{Z}(\mathsf{D}(z^2), x) = \min(\mathsf{Z}(\mathring{\mathsf{d}}\mathring{\mathsf{d}}^\star, x), \mathsf{Z}(\mathring{\mathsf{d}}(-\cdot)\mathring{\mathsf{d}}^\star(-\cdot), x)) = 2\min(\mathsf{Z}(\mathring{\mathsf{d}}, x), \mathsf{Z}(\mathring{\mathsf{d}}, -x)), \qquad \forall\, x \in \mathbb{R}\backslash\{0\}.$$

Thus, by (2.15) we see that

$$\mathsf{Z}(\tfrac{\mathsf{p}(z)}{\mathsf{q}(z^2)}, x) = \mathsf{Z}(\mathring{\mathsf{d}}\mathring{\mathsf{d}}^\star, x) - \mathsf{Z}(\mathsf{D}(z^2), x) = 2\max(0, \mathsf{Z}(\mathring{\mathsf{d}}, x) - \mathsf{Z}(\mathring{\mathsf{d}}, -x)) \in 2\mathbb{Z}. \tag{2.16}$$

For the case of symmetry, we proved that the Laurent polynomial $\mathsf{p}$ must have real coefficients. Hence, both $\mathsf{q}$ and $\frac{\mathsf{p}(z)}{\mathsf{q}(z^2)}$ also have real coefficients. By item (i) and [7, Lemma 2.4], we have $\frac{\mathsf{p}(z)}{\mathsf{q}(z^2)} \geqslant 0$ for all $z \in \mathbb{T}$. Applying [7, Theorem 2.9], we see that item (ii) holds for some Laurent polynomial $\mathsf{d}_\mathsf{p}$ with symmetry.

For the case of complex symmetry (and real coefficients), we have $\mathsf{Z}(\mathring{\mathsf{d}}^\star, x) = \mathsf{Z}(\mathring{\mathsf{d}}, x)$ for all $x \in \mathbb{C}\backslash\{0\}$ since $\mathring{\mathsf{d}}$ has complex symmetry. Therefore, (2.16) must hold for all $x \in \mathbb{C}\backslash\{0\}$. Applying [7, Theorem 2.8], we see that item (ii) holds for some Laurent polynomial $\mathsf{d}_\mathsf{p}$ with complex symmetry (and real coefficients).

Sufficiency. If items (i) and (ii) are satisfied, one can directly check that $\{\mathsf{b}_1, \mathsf{b}_2\}$ given in (2.9), with $\mathsf{d}_\mathsf{q}$ and $\mathsf{b}$ satisfying (2.10) and (2.11), is indeed a solution to (2.7). Taking $\mathsf{d}_\mathsf{q} = 1$ and noting that $\mathsf{q}(z) \geqslant 0$ for all $z \in \mathbb{T}$, by [7, Theorems 2.6 and 2.7], we see that there always exists a desired $\mathsf{b}$ satisfying (2.11).

We now show that all the solutions to (2.7) can be obtained in this way. By the above argument in the proof of items (i) and (ii), all (2.12)–(2.15) must hold.

For the case of complex symmetry (and real coefficients), since $\mathring{\mathsf{d}}$ has complex symmetry, we have $\mathsf{Z}(\mathring{\mathsf{d}}\mathring{\mathsf{d}}^\star, z) = 2\mathsf{Z}(\mathring{\mathsf{d}}, z)$ for all $z \in \mathbb{C}\backslash\{0\}$. Consequently, by the definition of $\mathsf{D}$, we have $\mathsf{D}(z) = \tilde{\mathsf{d}}(z)\tilde{\mathsf{d}}^\star(z)$ with $\tilde{\mathsf{d}}(z^2) := \gcd(\mathring{\mathsf{d}}(z), \mathring{\mathsf{d}}(-z))$. Hence, $\mathring{\mathsf{d}}(z) = \mathsf{d}_\mathsf{p}(z)\tilde{\mathsf{d}}(z^2)$ for some Laurent polynomial $\mathsf{d}_\mathsf{p}$ with complex symmetry (and real coefficients). It follows directly from (2.15) that such $\mathsf{d}_\mathsf{p}$ indeed satisfies

$$\frac{\mathsf{p}(z)}{\mathsf{q}(z^2)} = \frac{\mathring{\mathsf{d}}(z)\mathring{\mathsf{d}}^\star(z)}{\mathsf{D}(z^2)} = \frac{\mathring{\mathsf{d}}(z)}{\tilde{\mathsf{d}}(z^2)}\frac{\mathring{\mathsf{d}}^\star(z)}{\tilde{\mathsf{d}}^\star(z^2)} = \mathsf{d}_\mathsf{p}(z)\mathsf{d}_\mathsf{p}^\star(z).$$

Now define $\mathsf{b}(z) := \tilde{\mathsf{d}}(z^2)\mathring{\mathsf{b}}(z)$ and $\tilde{\lambda} z^{2\tilde{k}+1} := \lambda z^{2k+1}\mathbb{S}\tilde{\mathsf{d}}(z^2)$. By $\mathring{\mathsf{d}}(z) = \mathsf{d}_\mathsf{p}(z)\tilde{\mathsf{d}}(z^2)$, (2.12) becomes $\mathsf{b}_1(z) = \mathsf{d}_\mathsf{p}(z)\mathsf{b}(z)$ and $\mathsf{b}_2(z) = \mathsf{d}_\mathsf{p}(z)\tilde{\lambda} z^{2\tilde{k}+1}\mathsf{b}^\star(-z)$. Moreover, the first identity of (2.7) implies that (2.11) holds with $\mathsf{d}_\mathsf{q} = 1$.

For the case of symmetry, applying [7, Algorithm 1] with $\mathsf{p}_1(z) = \frac{\mathsf{p}(z)}{\mathsf{q}(z^2)}$ and $\mathsf{p}_2(z) = \mathring{\mathsf{d}}(z)$, we can construct a Laurent polynomial $\mathsf{d}_\mathsf{p}$ with symmetry such that

$$[\mathsf{d}_\mathsf{p}(z)\mathsf{d}_\mathsf{p}^\star(z)] \mid \tfrac{\mathsf{p}(z)}{\mathsf{q}(z^2)}, \quad \mathsf{d}_\mathsf{p} \mid \mathring{\mathsf{d}} \quad \text{and} \quad \gcd(\tfrac{\mathsf{p}(z)}{\mathsf{q}(z^2)\mathsf{d}_\mathsf{p}(z)\mathsf{d}_\mathsf{p}^\star(z)}, \tfrac{\mathring{\mathsf{d}}(z)}{\mathsf{d}_\mathsf{p}(z)}) = 1. \tag{2.17}$$

Hence, $\mathsf{d}_\mathsf{q}(z) := \frac{\mathring{\mathsf{d}}(z)}{\mathsf{d}_\mathsf{p}(z)}$ is a well-defined Laurent polynomial with symmetry. It follows from (2.15) that

$$\mathsf{D}(z^2)\frac{\mathsf{p}(z)}{\mathsf{q}(z^2)\mathsf{d}_\mathsf{p}(z)\mathsf{d}_\mathsf{p}^\star(z)} = \mathsf{D}(z^2)\frac{\mathsf{p}(z)}{\mathsf{q}(z^2)}\frac{1}{\mathsf{d}_\mathsf{p}(z)\mathsf{d}_\mathsf{p}^\star(z)} = \frac{\mathring{\mathsf{d}}(z)}{\mathsf{d}_\mathsf{p}(z)}\frac{\mathring{\mathsf{d}}^\star(z)}{\mathsf{d}_\mathsf{p}^\star(z)} = \mathsf{d}_\mathsf{q}(z)\mathsf{d}_\mathsf{q}^\star(z). \tag{2.18}$$

From the above identity and the relation in (2.17), we see that $\frac{\mathsf{p}(z)}{\mathsf{q}(z^2)\mathsf{d}_\mathsf{q}(z)\mathsf{d}_\mathsf{q}^\star(z)}$ must be a monomial and hence must be a positive constant by its nonnegativity. Therefore, by multiplying a proper positive constant with $\mathsf{d}_\mathsf{p}$, we can assume that this monomial is 1, that is, $\frac{\mathsf{p}(z)}{\mathsf{q}(z^2)} = \mathsf{d}_\mathsf{p}(z)\mathsf{d}_\mathsf{p}^\star(z)$. By the definition $\mathsf{d}_\mathsf{q}(z) = \frac{\mathring{\mathsf{d}}(z)}{\mathsf{d}_\mathsf{p}(z)}$, we have $\mathring{\mathsf{d}}(z) = \mathsf{d}_\mathsf{p}(z)\mathsf{d}_\mathsf{q}(z)$. Now it follows from (2.18) that $\mathsf{d}_\mathsf{q}(z)\mathsf{d}_\mathsf{q}^\star(z) = \mathsf{D}(z^2)$. Thus, by (2.14) we see that (2.10) is satisfied. Defining $\mathsf{b}(z) := \mathring{\mathsf{b}}(z)$, we see from (2.12) that

$$\mathsf{d}_\mathsf{q}(z)\mathsf{d}_\mathsf{q}^\star(z)[\mathsf{b}(z)\mathsf{b}^\star(z) + \mathsf{b}(-z)\mathsf{b}^\star(-z)] = \frac{\mathsf{p}(z)}{\mathsf{d}_\mathsf{p}(z)\mathsf{d}_\mathsf{p}^\star(z)} = \mathsf{q}(z^2).$$

Now it is easy to deduce from the above relation that (2.11) must hold. $\square$

The problem in (2.11) has been well studied in [7, 9] and (2.11) always has a desired solution (see [7, Theorems 2.6 and 2.7]). In the following, we outline two ways of constructing all desired solutions $\mathsf{b}$ to (2.11). Define $\mathring{\mathsf{q}}(z^2) := \frac{\mathsf{q}(z^2)}{\mathsf{d}_\mathsf{q}(z)\mathsf{d}_\mathsf{q}^\star(z)}$. By (2.10) and item (i) of Theorem 5, $\mathring{\mathsf{q}}$ is well defined and $\mathring{\mathsf{q}}(z) \geqslant 0$ for all $z \in \mathbb{T}$. We now state the first method using Fejér-Riesz lemma. Since $\mathring{\mathsf{q}}(z) \geqslant 0$ for all $z \in \mathbb{T}$, using Fejér-Riesz lemma, we can obtain a Laurent polynomial $\mathsf{u}$ such that $\mathsf{u}(z)\mathsf{u}^\star(z) = \mathring{\mathsf{q}}(z)$. For the case of complex symmetry, define

$$\mathsf{b}(z) := [\mathsf{u}(z^2) + \epsilon_\mathsf{b} z^{c_\mathsf{b}}\mathsf{u}^\star(z^2)]/2 \qquad \text{with} \qquad \epsilon_\mathsf{b} \in \{-1, 1\} \quad \text{and} \quad c_\mathsf{b} \text{ being an odd integer.} \tag{2.19}$$

Then $\mathsf{b}$ is a solution to (2.11) with the complex symmetry $\mathbb{S}\mathsf{b}(z) = \epsilon_\mathsf{b} z^{c_\mathsf{b}}$. For the case of symmetry (and real coefficients), since $\mathring{\mathsf{q}}$ must have real coefficients, we can further require $\mathsf{u}$ to have real coefficients. Define

$$\mathsf{b}(z) := [\mathsf{u}(z^2) + \epsilon_\mathsf{b} z^{c_\mathsf{b}}\mathsf{u}(z^{-2})]/2 \qquad \text{with} \qquad \epsilon_\mathsf{b} \in \{-1, 1\} \quad \text{and} \quad c_\mathsf{b} \text{ being an odd integer.} \tag{2.20}$$

Then $b$ in (2.20) with $\mathsf{u}$ having real coefficients provides a solution to (2.11) with symmetry and real coefficients. Define $[-m, m] := \mathrm{fsupp}(\mathring{\mathsf{q}})$. We often take $c_\mathsf{b} = 2m + 1$ and $\mathrm{fsupp}(\mathsf{u}) \subseteq [0, m]$ so that the constructed filter $b$ has the shortest possible filter support contained inside $[0, 2m + 1]$. We now discuss the second method using sum of squares of Laurent polynomials. By [7, Theorem 2.6], there exist Laurent polynomials $\mathsf{u}_1$ and $\mathsf{u}_2$ with complex symmetry such that

$$\mathsf{u}_1(z)\mathsf{u}_1^\star(z) + \mathsf{u}_2(z)\mathsf{u}_2^\star(z) = \mathring{\mathsf{q}}(z) \quad \text{with} \quad \frac{\mathbb{S}\mathsf{u}_1(z)}{\mathbb{S}\mathsf{u}_2(z)} = z. \tag{2.21}$$

For the case of symmetry (and real coefficients), under the additional assumption that $\mathsf{Z}(\mathring{\mathsf{q}}, x) \in 2\mathbb{Z}$ for all $x \in (0, 1)$, by [7, Theorem 2.7] or [9, Lemma 4.4], there exist Laurent polynomials $\mathsf{u}_1$ and $\mathsf{u}_2$ with symmetry and real coefficients such that

$$\mathsf{u}_1(z)\mathsf{u}_1^\star(z) + \mathsf{u}_2(z)\mathsf{u}_2^\star(z) = \mathring{\mathsf{q}}(z) \quad \text{with} \quad \frac{\mathsf{S}\mathsf{u}_1(z)}{\mathsf{S}\mathsf{u}_2(z)} = z. \tag{2.22}$$

Now define

$$\mathsf{b}(z) := [\mathsf{u}_1(z^2) + z\mathsf{u}_2(z^2)]/\sqrt{2}. \tag{2.23}$$

Then $\mathsf{b}$ in (2.23) is also a solution to (2.11) with [complex] symmetry (and real coefficients) such that $\epsilon_\mathsf{b} z^{c_\mathsf{b}} := \mathbb{S}\mathsf{b}(z) = \mathbb{S}\mathsf{u}_1(z^2)$ (or $\epsilon_\mathsf{b} z^{c_\mathsf{b}} := \mathsf{S}\mathsf{b}(z) = \mathsf{S}\mathsf{u}_1(z^2)$ for the case of symmetry) with $c_\mathsf{b}$ being an even integer. Define $[-m, m] := \mathrm{fsupp}(\mathring{\mathsf{q}})$. Then the relation in (2.21) or (2.22) will force $\max(\mathrm{len}(\mathsf{u}_1), \mathrm{len}(\mathsf{u}_2)) = m$ and consequently, we must have $\mathrm{len}(\mathsf{b}) \leqslant 2m = \mathrm{len}(\mathring{\mathsf{q}})$. All the solutions to (2.21) or (2.22) can be obtained using [7, Theorems 2.8 and 2.9].

It is also not difficult to see that all the solutions $\mathsf{b}$ with $\mathbb{S}\mathsf{b}(z) = \epsilon_\mathsf{b} z^{c_\mathsf{b}}$ (or $\mathsf{S}\mathsf{b}(z) = \epsilon_\mathsf{b} z^{c_\mathsf{b}}$) to (2.11) are obtained by either (2.19) or (2.23) according to the parity of $c_\mathsf{b}$.

Summarizing the above results on (2.2), we have the following algorithm to construct all tight framelet filter banks $\{a; b_1, b_2, b_3\}_\Theta$ with [complex] symmetry (and real coefficients) satisfying (2.2).

**Algorithm 1.** *Let $a, \Theta \in l_0(\mathbb{Z})$ be filters having [complex] symmetry (and real coefficients) such that $\mathsf{S}a(z) = \epsilon z^c$ and $\mathsf{S}\Theta = \mathbb{S}\Theta = 1$, where $\epsilon \in \{-1, 1\}$ and $c \in \mathbb{Z}$ (for complex symmetry, replace $\mathsf{S}$ by $\mathbb{S}$).*

(S1) *Construct a Laurent polynomial* $\boldsymbol{\theta}$ *with [complex] symmetry (and real coefficients) by Theorem 2 such that* $\Theta(z^2) = \boldsymbol{\theta}(z)\boldsymbol{\theta}^\star(-z)$*;*

(S2) *Select a Laurent polynomial* $\mathsf{d_a}$ *with [complex] symmetry (and real coefficients) such that* $\mathsf{d_a} \mid \mathsf{a}$*. Write* $\mathsf{a}(z) = \mathsf{d_a}(z)\mathring{\mathsf{a}}(z)$ *for a unique Laurent polynomial* $\mathring{\mathsf{a}}$*;*

(S3) *Define* $\mathsf{p}$ *as in* (2.8) *and* $\mathsf{q}(z^2) := \gcd(\mathsf{p}(z), \mathsf{p}(-z))$*. If the necessary and sufficient condition in items (i) and (ii) of Theorem 5 is satisfied, then find a Laurent polynomial* $\mathsf{d_p}$ *with [complex] symmetry satisfying* $\mathsf{d_p}(z)\mathsf{d_p^\star}(z) = \frac{\mathsf{p}(z)}{\mathsf{q}(z^2)}$*. If item (i) or item (ii) of Theorem 5 fails, then stop and restart the algorithm by selecting other choices of* $\boldsymbol{\theta}$ *in (S1) and* $\mathsf{d_a}$ *in (S2);*

(S4) *Select a Laurent polynomial* $\mathsf{d_q}$ *with [complex] symmetry (and real coefficients) satisfying* (2.10)*. Without loss of any generality, we can take* $\mathsf{d_q} = 1$ *for complex symmetry (and real coefficients).*

(S5) *Define* $\mathring{\mathsf{q}}(z^2) := \frac{\mathsf{q}(z^2)}{\mathsf{d_q}(z)\mathsf{d_q^\star}(z)}$*. Since item (i) of Theorem 5 implies* $\mathring{\mathsf{q}}(z) \geqslant 0$ *for all* $z \in \mathbb{T}$*, we can always construct a Laurent polynomial* $\mathsf{b}$ *with [complex] symmetry (and real coefficients) by* (2.20) *(or by* (2.19) *for complex symmetry) or by* (2.23) *such that* (2.11) *holds.*

*Define*

$$\mathsf{b}_1(z) = \mathsf{d_p}(z)\mathsf{d_q}(z)\mathsf{b}(z), \quad \mathsf{b}_2(z) = \mathsf{d_p}(z)\mathsf{d_q}(z)z\mathsf{b}^\star(-z), \quad \mathsf{b}_3(z) := \mathsf{d_a}(z)\boldsymbol{\theta}(z)z\mathring{\mathsf{a}}^\star(-z). \tag{2.24}$$

*Then* $\{a; b_1, b_2, b_3\}_\Theta$ *is a tight framelet filter bank with [complex] symmetry (and real coefficients).*

Since we have essentially finitely many choices of $\boldsymbol{\theta}$ in (S1) and $\mathsf{d_a}$ in (S2) as well as $\mathsf{d_q}$ in (S4) (we only have to consider all those $\boldsymbol{\theta}$ in (S1) with symmetry center from $\{0, 1\}$ and $\mathsf{d_a}$ in (S2) with symmetry center from $\{0, \frac{1}{2}\}$), Algorithm 1 can be used to find all tight framelet filter banks with [complex] symmetry satisfying (2.2). As a consequence, all type I symmetric tight framelet filter banks with three high-pass filters can be obtained via Algorithm 1, since (2.2) must hold for type I tight framelet filter banks by Proposition 1. Note that Algorithm 1 can be also used to construct all type II symmetric tight framelet filter banks satisfying the additional assumption in (2.2).

Under the extra condition $\mathsf{p}(-z) = \mathsf{p}(z)$ (for example, this holds if $\boldsymbol{\theta} = \mathsf{d_a} = 1$), we have $\mathsf{q}(z^2) = \gcd(\mathsf{p}(z), \mathsf{p}(-z)) = \mathsf{p}(z)$. Therefore, item (ii) of Theorem 5 is automatically true by taking $\mathsf{d_p} = 1$ and we only need to check the condition in item (i) of Theorem 5. For the particular case of $\boldsymbol{\theta} = \mathsf{d_a} = 1$, item (i) of Theorem 5 becomes (1.7).

For all $B$-spline filters, we are now ready to completely resolve the question Q1 on symmetric real-valued tight framelet filter banks $\{a_m^B; b_1, b_2, b_3\}$ with the shortest possible filter supports.

**Theorem 6.** *Let* $a \in l_0(\mathbb{Z})$ *be a real-valued filter with symmetry such that* (1.7) *is satisfied. If* $\operatorname{len}(a)$ *is an odd integer or if*

$$\mathsf{Z}(\mathsf{p}, x) \in 2\mathbb{Z}, \qquad \forall\, x \in (0,1) \qquad \text{with} \quad \mathsf{p}(z) := 1 - \mathsf{a}(z)\mathsf{a}^\star(z) - \mathsf{a}(-z)\mathsf{a}^\star(-z), \tag{2.25}$$

*then there always exist real-valued filters* $b_1, b_2, b_3 \in l_0(\mathbb{Z})$ *with symmetry such that* $\{a; b_1, b_2, b_3\}$ *is a tight framelet filter bank with symmetry and*

$$\max(\operatorname{len}(b_1), \operatorname{len}(b_2), \operatorname{len}(b_3)) = \operatorname{len}(a). \tag{2.26}$$

*In particular, both* (1.7) *and* (2.25) *hold for every B-spline filter* $a = a_m^B$ *of order* $m \in \mathbb{N}$*.*

*Proof.* Take $\boldsymbol{\theta} = 1$, $\mathsf{d_a} = 1$, and $\mathsf{d_q} = 1$ in Algorithm 1. Then $\mathsf{q}(z^2) = \mathsf{p}(z) \geqslant 0$ for all $z \in \mathbb{T}$ and $\mathsf{d_p} = 1$.

If $\operatorname{len}(a)$ is odd, then $\operatorname{fsupp}(\mathsf{q}) \subseteq [\frac{1-\operatorname{len}(a)}{2}, \frac{\operatorname{len}(a)-1}{2}]$. Therefore, by Fejér-Riesz lemma, there exists a polynomial $\mathsf{u}$ with real coefficients such that $\operatorname{fsupp}(\mathsf{u}) \subseteq [0, \frac{\operatorname{len}(a)-1}{2}]$ and $\mathsf{u}(z)\mathsf{u}^\star(z) = \mathsf{q}(z)$. Define $\mathsf{b}$ as in (2.20) with $\epsilon_{\mathsf{b}} = 1$ and $c_{\mathsf{b}} = \operatorname{len}(a)$, where we used the assumption that $\operatorname{len}(a)$ is an odd integer. Then $\operatorname{fsupp}(\mathsf{b}) \subseteq [0, \operatorname{len}(a)]$ and $\mathsf{Sb}(z) = z^{\operatorname{len}(a)}$. Define

$$\mathsf{b}_1(z) = \mathsf{b}(z), \qquad \mathsf{b}_2(z) = z\mathsf{b}^\star(-z), \qquad \mathsf{b}_3(z) = z\mathsf{a}^\star(-z). \tag{2.27}$$

It is straightforward to check that $\operatorname{len}(b_1) = \operatorname{len}(b_2) = \operatorname{len}(b) \leqslant \operatorname{len}(a)$ and $\operatorname{len}(b_3) = \operatorname{len}(a)$. Therefore, $\{a; b_1, b_2, b_3\}$ is a desired real-valued tight framelet filter bank with symmetry and satisfies (2.26). Moreover, it is a type II symmetric tight framelet filter bank satisfying (2.2), since

$$\mathsf{Sb}_1(z) = z^{\operatorname{len}(a)}, \qquad \mathsf{Sb}_2(z) = -z^{2-\operatorname{len}(a)}, \qquad \mathsf{Sb}_3(z) = z^2\mathsf{Sa}^\star(-z).$$

If $\mathrm{len}(a)$ is an even integer, we can still use (2.20) to derive a real-valued tight framelet filter bank $\{a; b_1, b_2, b_3\}$ with symmetry. However, we will end up with $\mathrm{len}(b_1) = \mathrm{len}(b_2) = \mathrm{len}(b) \geqslant \mathrm{len}(a) + 1$.

If (2.25) holds, we use (2.23) instead of (2.20) to achieve short filter supports. Since $\mathsf{q}(z^2) = \mathsf{p}(z)$, by the relation $\mathsf{Z}(\mathsf{q}, x) = \mathsf{Z}(\mathsf{p}, \sqrt{x})$ for all $x \in (0, 1)$, we conclude that $\mathsf{Z}(\mathsf{q}, x) \in 2\mathbb{Z}$ for all $x \in (0, 1)$. Note that $\mathrm{fsupp}(\mathsf{q}) \subseteq [-m, m]$, where $m := \mathrm{len}(a) - \frac{1-(-1)^{\mathrm{len}(a)}}{2}$. Thus, it is guaranteed by [7, Theorem 2.7] or [9, Lemma 4.4] that (2.23) has a solution with real coefficients and $\max(\mathrm{len}(\mathsf{u}_1), \mathrm{len}(\mathsf{u}_2)) \leqslant \frac{\mathrm{len}(\mathsf{q})}{2} = \frac{m}{2}$. Define $\epsilon_1 z^{c_1} := \mathsf{S}\mathsf{u}_1(z)$. Due to the symmetry constraint $\mathsf{S}\mathsf{u}_1(z) = z\mathsf{S}\mathsf{u}_2(z)$, we have $\mathsf{S}\mathsf{u}_2(z) = \epsilon_1 z^{c_1+1}$. If $c_1$ and $\frac{m}{2}$ have the same parity (that is, $c_1 - \frac{m}{2} \in 2\mathbb{Z}$), then we must have $\mathrm{len}(\mathsf{u}_2) < \mathrm{len}(\mathsf{u}_1) = \frac{m}{2}$, from which and the definition of $\mathsf{b}$ in (2.23) we conclude that $\mathrm{len}(\mathsf{b}) \leqslant 2\,\mathrm{len}(\mathsf{u}_1) = m$. If $c_1$ and $\frac{m}{2}$ have different parity, then we must have $\mathrm{len}(\mathsf{u}_1) < \mathrm{len}(\mathsf{u}_2) = \frac{m}{2}$, from which and the definition of $\mathsf{b}$ in (2.23) we conclude that $\mathrm{len}(\mathsf{b}) \leqslant 2\,\mathrm{len}(\mathsf{u}_2) = m$. Now by (2.27), we see that the condition in (2.26) is satisfied. In fact, if $\mathrm{len}(a)$ is odd, then $\mathrm{len}(b_1) = \mathrm{len}(b_2) \leqslant m < \mathrm{len}(a)$.

Since $|\mathsf{a}^B_m(z)|^2 = \cos^{2m}(\xi/2) \leqslant \cos^2(\xi/2)$ with $z = e^{-i\xi}$, it is trivial to see that $1 - \mathsf{a}^B_m(z)(\mathsf{a}^B_m(z))^\star - \mathsf{a}^B_m(-z)(\mathsf{a}^B_m(-z))^\star \geqslant 1 - \cos^2(\xi/2) - \sin^2(\xi/2) = 0$ for all $\xi \in \mathbb{R}$ and $m \in \mathbb{N}$. Hence, (1.7) is satisfied with $a = a^B_m$. We now prove that (2.25) is satisfied. (2.25) holds for $a = a^B_1$ since $\mathsf{p} = 0$. Define $f(x) := 1 - x^m - (1-x)^m, x \in \mathbb{R}$. For $m > 1$ and $x < 0$, we have $1 - x > -x > 0$ and

$$f'(x) = m[(1-x)^{m-1} - x^{m-1}] \geqslant m[(1-x)^{m-1} - (-x)^{m-1}] > 0.$$

Therefore, $f$ is a strictly increasing function on $(-\infty, 0]$ and hence, $f(x) < f(0) = 0$ for all $x < 0$. Noting that $\frac{1}{2} - \frac{x+x^{-1}}{2} < 0$ for all $x \in (0, 1)$, we conclude that $\mathsf{p}(x) = f(\frac{1}{2} - \frac{x+x^{-1}}{2}) < 0$ for all $x \in (0, 1)$. Hence, $\mathsf{Z}(\mathsf{p}, x) = 0$ for all $x \in (0, 1)$ and (2.25) holds. $\square$

For the case of complex symmetry, using the same argument as in Theorem 6 for the case of (2.25) but employing [7, Theorem 2.6] instead of [7, Theorem 2.7], we have

**Theorem 7.** *Let $a \in l_0(\mathbb{Z})$ be a filter with complex symmetry such that* (1.7) *is satisfied. Then there always exist (complex-valued) filters $b_1, b_2, b_3 \in l_0(\mathbb{Z})$ with complex symmetry such that $\{a; b_1, b_2, b_3\}$ is a tight framelet filter bank with complex symmetry and* (2.26) *holds for the shortest possible filter support.*

## 3. Examples of Symmetric Tight Framelet Filter Banks Using Algorithm 1

In this section we present several examples of symmetric tight framelet filter banks $\{a; b_1, b_2, b_3\}_\Theta$ which are constructed using Algorithm 1 in Section 2.

For a function $f : \mathbb{R} \to \mathbb{C}$, we use the following notation:

$$f_{\lambda;k,n}(x) := [\![\lambda; k, n]\!] f(x) := |\lambda|^{1/2} e^{-in\lambda x} f(\lambda x - k), \qquad \lambda, k, n, x \in \mathbb{R}.$$

In particular, we define $f_{\lambda;k} := f_{\lambda;k,0} = |\lambda|^{1/2} f(\lambda \cdot -k)$. Suppose that $\{a; b_1, \dots, b_s\}_\Theta$ is a finitely supported tight framelet filter bank with $\mathsf{a}(1) = \Theta(1) = 1$. The standard refinable function/distribution $\phi^a$ associated with the low-pass filter $a$ is defined to be

$$\widehat{\phi^a}(\xi) := \prod_{j=1}^{\infty} \mathsf{a}(e^{-i2^{-j}\xi}), \qquad \xi \in \mathbb{R}. \tag{3.1}$$

Since $\mathsf{a}(1) = 1$ and $a \in l_0(\mathbb{Z})$, the function $\widehat{\phi^a}$ is a well-defined continuous function. Let $\boldsymbol{\theta}$ be a Laurent polynomial satisfying $\Theta(z) = \boldsymbol{\theta}(z)\boldsymbol{\theta}^\star(z)$. We define functions $\eta, \psi^{a,b_1}, \dots, \psi^{a,b_s}$ by

$$\widehat{\eta}(\xi) := \boldsymbol{\theta}(e^{-i\xi})\widehat{\phi^a}(\xi) \qquad \text{and} \qquad \widehat{\psi^{a,b_\ell}}(\xi) := \mathsf{b}_\ell(e^{-i\xi/2})\widehat{\phi^a}(\xi/2), \qquad \ell = 1, \dots, s, \;\; \xi \in \mathbb{R}.$$

It is guaranteed by [6, Theorem 2 and Corollary 10] that all functions $\eta, \psi^{a,b_1}, \dots, \psi^{a,b_s} \in L_2(\mathbb{R})$ and both $\{\eta(\cdot - k) \;:\; k \in \mathbb{Z}\} \cup \{\psi^{a,b_\ell}_{2^j;k} \;:\; j \in \mathbb{N} \cup \{0\}, k \in \mathbb{Z}, \ell = 1, \dots, s\}$ and $\{\psi^{a,b_\ell}_{2^j;k} \;:\; j \in \mathbb{Z}, k \in \mathbb{Z}, \ell = 1, \dots, s\}$ are (normalized) tight frames for $L_2(\mathbb{R})$, that is,

$$\|f\|^2_{L_2(\mathbb{R})} = \sum_{k\in\mathbb{Z}} |\langle f, \eta(\cdot - k)\rangle|^2 + \sum_{j=0}^{\infty}\sum_{\ell=1}^{s}\sum_{k\in\mathbb{Z}} |\langle f, \psi^{a,b_\ell}_{2^j;k}\rangle|^2 = \sum_{j\in\mathbb{Z}}\sum_{\ell=1}^{s}\sum_{k\in\mathbb{Z}} |\langle f, \psi^{a,b_\ell}_{2^j;k}\rangle|^2, \qquad \forall\, f \in L_2(\mathbb{R}).$$

It is interesting to point out that there is indeed a one-to-one correspondence between filter banks and frequency-based framelets, recently established in [6, Theorem 2]. Hence, in this paper we mainly concentrate on filter banks and Laurent polynomials. For references on tight wavelet frames, see [2, 3, 5, 6, 14] and references therein.

For a filter $u$, if $\mathsf{u}(z) = (z-1)^m \mathsf{v}(z)$ for some Laurent polynomial $\mathsf{v}$ with $\mathsf{v}(1) \neq 0$, then we define $\mathrm{vm}(u) := m$, called the order of vanishing moments of $u$. To study the frequency information of a filter $u$, we also use the notation $\widehat{u}(\xi) := \mathsf{u}(e^{-i\xi})$ in this paper.

We now present a few examples using Algorithm 1. When $\Theta = \boldsymbol{\delta}$, without further mention we always take $\boldsymbol{\theta} = 1$ such that $\boldsymbol{\theta}(z)\boldsymbol{\theta}^\star(-z) = 1$ in Theorem 2.

**Example 1.** Let $a = a_4^B(\cdot - 2) = \{\frac{1}{16}, \frac14, \underline{\mathbf{\frac{3}{8}}}, \frac14, \frac{1}{16}\}_{[-2,2]}$ be the centered B-spline filter of order 4 and $\Theta = \boldsymbol{\delta}$. Then $\boldsymbol{\theta} = 1$. Picking $\mathsf{d_a} = 1$ in Algorithm 1, we have $\mathring{\mathsf{a}}(z) = \mathsf{a}(z)$ and $\mathsf{q}(z^2) = \mathsf{p}(z) = \frac{1}{128}(z^{-1} - z)^2(z^{-2} + 30 + z^2)$. Then $\mathsf{d_p}(z) = 1$ satisfies $\mathsf{d_p}(z)\mathsf{d_p^\star}(z) = 1 = \frac{\mathsf{p}(z)}{\mathsf{q}(z^2)}$. Selecting $\mathsf{d_q} = 1$, by Theorem 6, we can use (2.23) with $\mathring{\mathsf{q}} = \mathsf{q}$ to have $\mathsf{b}(z) = \frac{1}{16}(1 - z^2)(z^{-1} + 2\sqrt{7} + z)$. Hence,

$$\mathsf{b}_1(z) = \mathsf{d_p}(z)\mathsf{d_q}(z)\mathsf{b}(z) = \mathsf{b}(z) = \{\tfrac{1}{16}, \underline{\mathbf{\tfrac{\sqrt{7}}{8}}}, 0, -\tfrac{\sqrt{7}}{8}, -\tfrac{1}{16}\}_{[-1,3]},$$

$$\mathsf{b}_2(z) = \mathsf{d_p}(z)\mathsf{d_q}(z) z\mathsf{b}^\star(-z) = z\mathsf{b}^\star(-z) = \{\tfrac{1}{16}, -\tfrac{\sqrt{7}}{8}, \underline{\mathbf{0}}, \tfrac{\sqrt{7}}{8}, -\tfrac{1}{16}\}_{[-2,2]},$$

$$\mathsf{b}_3(z) = \mathsf{d_a}(z)\boldsymbol{\theta}(z) z\mathring{\mathsf{a}}^\star(-z) = \tfrac{1}{16} z^3(1 - z^{-1})^4 = \{\tfrac{1}{16}, \underline{\mathbf{-\tfrac{1}{4}}}, \tfrac38, -\tfrac14, \tfrac{1}{16}\}_{[-1,3]}.$$

Then $\mathsf{Sb}_1(z) = -z^2, \mathsf{Sb}_2(z) = -1, \mathsf{Sb}_3(z) = z^2$ (type II) and $\mathrm{vm}(b_1) = \mathrm{vm}(b_2) = 1, \mathrm{vm}(b_3) = 4$. A slightly different example is obtained in [1, Example 6] through ad hoc construction.

We can also take $\mathsf{d_a}(z) = 1 + z$ in Algorithm 1. Then $\mathring{\mathsf{a}}(z) = \frac{1}{16} z^{-2}(z+1)^3$, $\mathsf{p}(z) = \frac{1}{64}(1-z)(1-z^{-1})[3(z^{-2} + z^2) + 12(z^{-1} + z) + 34]$, and $\mathsf{q}(z) = 1$. Hence, $\mathsf{d_p}(z) = \frac{\sqrt{3}}{24}(z^{-1} - 1)[3z^{-1} + 6 + 4\sqrt{3}i + 3z]$ satisfies $\mathsf{d_p}(z)\mathsf{d_p^\star}(z) = \mathsf{p}(z) = \frac{\mathsf{p}(z)}{\mathsf{q}(z^2)}$. Take $\mathsf{d_q} = 1$ and the trivial choice $\mathsf{b} = \frac{\sqrt{2}}{2}$ satisfying $\mathsf{b}(z)\mathsf{b}^\star(z) + \mathsf{b}(-z)\mathsf{b}^\star(-z) = 1$. Then

$$\mathsf{b}_1(z) = \mathsf{d_p}(z)\mathsf{d_q}(z)\mathsf{b}(z) = \tfrac{\sqrt{2}}{2}\mathsf{d_p}(z) = \{-\tfrac{\sqrt{6}}{16}, -\tfrac{\sqrt{6}+4\sqrt{2}i}{16}, \underline{\mathbf{\tfrac{\sqrt{6}+4\sqrt{2}i}{16}}}, \tfrac{\sqrt{6}}{16}\}_{[-2,1]},$$

$$\mathsf{b}_2(z) = \mathsf{d_p}(z)\mathsf{d_q}(z) z\mathsf{b}^\star(-z) = z\mathsf{b}_1(z) = \{-\tfrac{\sqrt{6}}{16}, \underline{\mathbf{-\tfrac{\sqrt{6}+4\sqrt{2}i}{16}}}, \tfrac{\sqrt{6}+4\sqrt{2}i}{16}, \tfrac{\sqrt{6}}{16}\}_{[-1,2]},$$

$$\mathsf{b}_3(z) = \mathsf{d_a}(z)\boldsymbol{\theta}(z) z\mathring{\mathsf{a}}^\star(-z) = \tfrac{1}{16}(1+z)(z-1)^3 = \{\underline{\mathbf{-\tfrac{1}{16}}}, \tfrac18, 0, -\tfrac18, \tfrac{1}{16}\}_{[0,4]}.$$

Then $\mathsf{Sb}_1(z) = -z^{-1}, \mathsf{Sb}_2(z) = -z, \mathsf{Sb}_3(z) = -z^4$ (type I) and $\mathrm{vm}(b_1) = \mathrm{vm}(b_2) = 1, \mathrm{vm}(b_3) = 3$. See Figure 3.1 for graphs of the associated refinable function $\phi^a$ and framelet functions $\psi^{a,b_1}, \psi^{a,b_2}$, and $\psi^{a,b_3}$.

**Example 2.** Let $a = a_5^B(\cdot - 2) = \{\frac{1}{32}, \frac{5}{32}, \underline{\mathbf{\frac{5}{16}}}, \frac{5}{16}, \frac{5}{32}, \frac{1}{32}\}_{[-2,3]}$ be the shifted B-spline filter of order 5 and $\Theta = \boldsymbol{\delta}$. Then $\boldsymbol{\theta} = 1$. Picking $\mathsf{d_a} = 1$ in Algorithm 1, we have $\mathring{\mathsf{a}}(z) = \mathsf{a}(z)$, $\mathsf{q}(z^2) = \mathsf{p}(z) = \frac{5}{256}(1 - z^{-2})(1 - z^2)(z^{-2} + 14 + z^2)$. Then $\mathsf{d_p} = 1$. Setting $\mathsf{d_q}(z) = 1$ and using (2.19) with $\mathring{\mathsf{q}} = \mathsf{q}$, we have $\mathsf{b}(z) = \frac{\sqrt{5}(\sqrt{3}-2)}{32}(z-1)^2(z^{-1}+1)(-z^{-1} + 4\sqrt{3} + 6 - z)$. Hence,

$$\mathsf{b}_1(z) = \mathsf{d_p}(z)\mathsf{d_q}(z)\mathsf{b}(z) = \mathsf{b}(z) = \{\tfrac{2\sqrt{5}-\sqrt{15}}{32}, -\tfrac{2\sqrt{5}+\sqrt{15}}{32}, \underline{\mathbf{\tfrac{\sqrt{15}}{16}}}, \tfrac{\sqrt{15}}{16}, -\tfrac{2\sqrt{5}+\sqrt{15}}{32}, \tfrac{2\sqrt{5}-\sqrt{15}}{32}\}_{[-2,3]},$$

$$\mathsf{b}_2(z) = \mathsf{d_p}(z)\mathsf{d_q}(z) z\mathsf{b}^\star(-z) = z\mathsf{b}^\star(-z) = \{\tfrac{\sqrt{15}-2\sqrt{5}}{32}, -\tfrac{2\sqrt{5}+\sqrt{15}}{32}, \underline{\mathbf{-\tfrac{\sqrt{15}}{16}}}, \tfrac{\sqrt{15}}{16}, \tfrac{2\sqrt{5}+\sqrt{15}}{32}, \tfrac{2\sqrt{5}-\sqrt{15}}{32}\}_{[-2,3]},$$

$$\mathsf{b}_3(z) = \mathsf{d_a}(z)\boldsymbol{\theta}(z) z\mathring{\mathsf{a}}^\star(-z) = \tfrac{1}{32} z^{-2}(z-1)^5 = \{-\tfrac{1}{32}, \tfrac{5}{32}, \underline{\mathbf{-\tfrac{5}{16}}}, \tfrac{5}{16}, -\tfrac{5}{32}, \tfrac{1}{32}\}_{[-2,3]}.$$

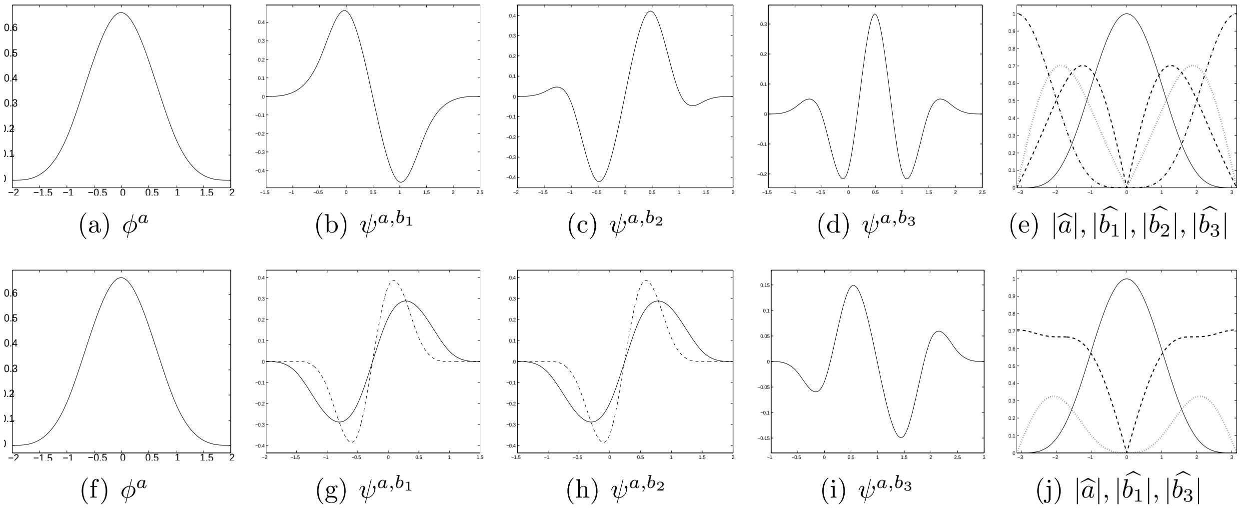
Then $\mathsf{Sb}_1(z) = z, \mathsf{Sb}_2(z) = -z, \mathsf{Sb}_3(z) = -z$ (type II) and $\mathrm{vm}(b_1) = 2, \mathrm{vm}(b_2) = 1, \mathrm{vm}(b_3) = 5$.

Since (2.25) is satisfied (see Theorem 6), we can also use (2.23) with $\mathring{\mathsf{q}} = \mathsf{q}$ to have $\mathsf{b}(z) = \frac{\sqrt{10}}{32}(z^{-2} - 1)(z + \sqrt{3} + \sqrt{2})(z + \sqrt{3} - \sqrt{2})$. Hence,

$$\mathsf{b}_1(z) = \mathsf{d_p}(z)\mathsf{d_q}(z)\mathsf{b}(z) = \mathsf{b}(z) = \{\tfrac{\sqrt{10}}{32}, \tfrac{\sqrt{30}}{16}, \underline{\mathbf{0}}, -\tfrac{\sqrt{30}}{16}, -\tfrac{\sqrt{10}}{32}\}_{[-2,2]},$$

$$\mathsf{b}_2(z) = \mathsf{d_p}(z)\mathsf{d_q}(z) z\mathsf{b}^\star(-z) = z\mathsf{b}^\star(-z) = \{-\tfrac{\sqrt{10}}{32}, \underline{\mathbf{\tfrac{\sqrt{30}}{16}}}, 0, -\tfrac{\sqrt{30}}{16}, \tfrac{\sqrt{10}}{32}\}_{[-1,3]},$$

$$\mathsf{b}_3(z) = \mathsf{d_a}(z)\boldsymbol{\theta}(z) z\mathring{\mathsf{a}}^\star(-z) = \tfrac{1}{32} z^{-2}(z-1)^5 = \{-\tfrac{1}{32}, \tfrac{5}{32}, \underline{\mathbf{-\tfrac{5}{16}}}, \tfrac{5}{16}, -\tfrac{5}{32}, \tfrac{1}{32}\}_{[-2,3]}.$$



(a) $\phi^a$ (b) $\psi^{a,b_1}$ (c) $\psi^{a,b_2}$ (d) $\psi^{a,b_3}$ (e) $|\widehat{a}|, |\widehat{b_1}|, |\widehat{b_2}|, |\widehat{b_3}|$

(f) $\phi^a$ (g) $\psi^{a,b_1}$ (h) $\psi^{a,b_2}$ (i) $\psi^{a,b_3}$ (j) $|\widehat{a}|, |\widehat{b_1}|, |\widehat{b_3}|$

FIGURE 3.1. (a)–(e) for the tight framelet filter bank $\{a; b_1, b_2, b_3\}$ with symmetry constructed by Algorithm 1 in Example 1 with $a = a_4^B(\cdot - 2)$ and $\mathsf{d_a} = 1$. (a), (b), (c), (d) are the graphs of the standard refinable function $\phi^a$ and the framelet functions $\psi^{a,b_1}$, $\psi^{a,b_2}$, $\psi^{a,b_3}$, respectively. (e) is the magnitudes of $\widehat{a}$ (in solid line), $\widehat{b_1}$ (in dashed line), $\widehat{b_2}$ (in dotted line), and $\widehat{b_3}$ (in dashed-dotted line) on the interval $[-\pi, \pi]$. (f)–(j) for the tight framelet filter bank $\{a; b_1, b_2, b_3\}$ with symmetry constructed by Algorithm 1 in Example 1 with the choice $\mathsf{d_a}(z) = 1 + z$. For the graphs in (g) and (h), the solid lines are for the real parts and the dotted lines are for the imaginary parts of the framelet functions $\psi^{a,b_1}$ and $\psi^{a,b_2}$. (j) is the magnitudes of $\widehat{a}$ (in solid line), $\widehat{b_1}$ (in dashed line), and $\widehat{b_3}$ (in dotted line), on the interval $[-\pi, \pi]$. Note that $|\widehat{b_1}(\xi)| = |\widehat{b_2}(\xi)|$.

Then $\mathsf{S}\mathsf{b}_1(z) = -1, \mathsf{S}\mathsf{b}_2(z) = -z^2, \mathsf{S}\mathsf{b}_3(z) = -z$ (type I) and $\mathrm{vm}(b_1) = \mathrm{vm}(b_2) = 1, \mathrm{vm}(b_3) = 5$. A slightly different example is obtained in [1, Example 7].

We can also take $\mathsf{d_a}(z) = z^{-1}(1+z)^3$ (or $\mathsf{d_a}(z) = 1 + z$) in Algorithm 1. Then $\mathring{\mathsf{a}}(z) = \frac{1}{32} z^{-1}(z+1)^2$, $\mathsf{q}(z) = z^{-1} + 14 + z$, and $\mathsf{d_p}(z) = \frac{\sqrt{2}}{32}(z-1)(z^{-2} + (4+2i)z^{-1} + 1)$. Setting $\mathsf{d_q} = 1$ and using (2.23), we have $\mathsf{b}(z) = \frac{\sqrt{2}}{2} z^{-1}(z + \sqrt{2} + \sqrt{3})(z + \sqrt{3} - \sqrt{2})$ satisfying $\mathsf{b}(z)\mathsf{b}^\star(z) + \mathsf{b}(-z)\mathsf{b}^\star(-z) = \mathsf{q}(z^2)$. Then

$$\mathsf{b}_1(z) = \mathsf{d_p}(z)\mathsf{d_q}(z)\mathsf{b}(z) = \{-\tfrac{1}{32}, -\tfrac{2\sqrt{3}+3+2i}{32}, \tfrac{1-3\sqrt{3}+(1-2\sqrt{3})i}{16}, \underline{\tfrac{\mathbf{3\sqrt{3}-1+(2\sqrt{3}-1)i}}{\mathbf{16}}}, \tfrac{3+2\sqrt{3}+2i}{32}, \tfrac{1}{32}\}_{[-3,2]},$$

$$\mathsf{b}_2(z) = \mathsf{d_p}(z)\mathsf{d_q}(z)z\mathsf{b}^\star(-z) = \{\tfrac{1}{32}, \tfrac{3-2\sqrt{3}+2i}{32}, \underline{-\tfrac{\mathbf{3\sqrt{3}+1+(2\sqrt{3}+1)i}}{\mathbf{16}}}, \tfrac{3\sqrt{3}+1+(2\sqrt{3}+1)i}{16}, \tfrac{2\sqrt{3}-3-2i}{32}, -\tfrac{1}{32}\}_{[-2,3]},$$

$$\mathsf{b}_3(z) = \mathsf{d_a}(z)\boldsymbol{\theta}(z)z\mathring{\mathsf{a}}^\star(-z) = -\tfrac{1}{32}z(1-z^{-1})^2(1+z)^3 = \{-\tfrac{1}{32}, \underline{-\tfrac{\mathbf{1}}{\mathbf{32}}}, \tfrac{1}{16}, \tfrac{1}{16}, -\tfrac{1}{32}, -\tfrac{1}{32}\}_{[-1,4]}.$$

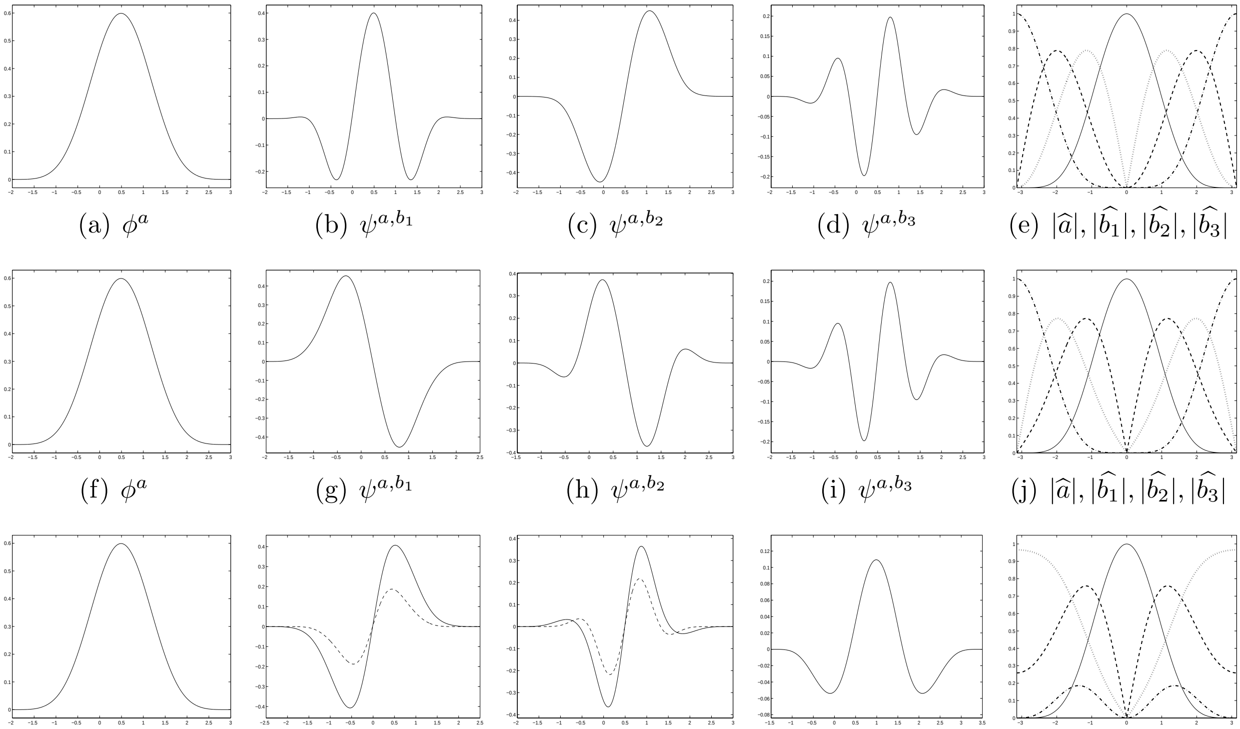
Then $\mathsf{S}\mathsf{b}_1(z) = -z^{-1}, \mathsf{S}\mathsf{b}_2(z) = -z, \mathsf{S}\mathsf{b}_3(z) = -z^3$ (type II) and $\mathrm{vm}(b_1) = \mathrm{vm}(b_2) = 1, \mathrm{vm}(b_3) = 2$. See Figure 3.2 for graphs of the associated refinable function $\phi^a$ and framelet functions $\psi^{a,b_1}, \psi^{a,b_2}$, and $\psi^{a,b_3}$.

**Example 3.** Let $a = a_6^B(\cdot - 3) = \{\frac{1}{64}, \frac{3}{32}, \frac{15}{64}, \underline{\frac{\mathbf{5}}{\mathbf{16}}}, \frac{15}{64}, \frac{3}{32}, \frac{1}{64}\}_{[-3,3]}$ be the centered B-spline filter of order 6 and $\Theta = \boldsymbol{\delta}$. Then $\boldsymbol{\theta} = 1$. Picking $\mathsf{d_a} = 1$ in Algorithm 1, we have $\mathring{\mathsf{a}}(z) = \mathsf{a}(z)$ and $\mathsf{q}(z^2) = \mathsf{p}(z) = \frac{1}{2048}(1-z^{-2})(1-z^2)(z^{-4} + 68z^{-2} + 630 + 68z^2 + z^4)$. Hence $\mathsf{d_p}(z) = 1$ satisfies $\mathsf{d_p}(z)\mathsf{d_p}^\star(z) = 1 = \frac{\mathsf{p}(z)}{\mathsf{q}(z^2)}$. Setting $\mathsf{d_q}(z) = 1$, by Theorem 6, we can use (2.23) with $\mathring{\mathsf{q}} = \mathsf{q}$ to have $\mathsf{b}(z) = (z^{-1} - z)[\frac{1-2\sqrt{31}}{32} + \frac{\sqrt{16+2\sqrt{31}}}{32}(z^{-1} + z) + \frac{1}{64}(z^{-2} + z^2)]$. Hence,

$$\mathsf{b}_1(z) = \mathsf{d_p}(z)\mathsf{d_q}(z)\mathsf{b}(z) = \mathsf{b}(z) = \{\tfrac{1}{64}, \tfrac{\sqrt{16+2\sqrt{31}}}{32}, \tfrac{1-4\sqrt{31}}{64}, \underline{\mathbf{0}}, \tfrac{4\sqrt{31}-1}{64}, -\tfrac{\sqrt{16+2\sqrt{31}}}{32}, -\tfrac{1}{64}\}_{[-3,3]},$$

$$\mathsf{b}_2(z) = \mathsf{d_p}(z)\mathsf{d_q}(z)z\mathsf{b}^\star(-z) = z\mathsf{b}^\star(-z) = \{\tfrac{1}{64}, -\tfrac{\sqrt{16+2\sqrt{31}}}{32}, \underline{\tfrac{\mathbf{1-4\sqrt{31}}}{\mathbf{64}}}, 0, \tfrac{4\sqrt{31}-1}{64}, \tfrac{\sqrt{16+2\sqrt{31}}}{32}, -\tfrac{1}{64}\}_{[-2,4]},$$

$$\mathsf{b}_3(z) = \mathsf{d_a}(z)\boldsymbol{\theta}(z)z\mathring{\mathsf{a}}^\star(-z) = -\tfrac{1}{64}z^{-2}(z-1)^6 = \{-\tfrac{1}{64}, \tfrac{3}{32}, \underline{-\tfrac{\mathbf{15}}{\mathbf{64}}}, \tfrac{5}{16}, -\tfrac{15}{64}, \tfrac{3}{32}, -\tfrac{1}{64}\}_{[-2,4]}.$$



(a) $\phi^a$ (b) $\psi^{a,b_1}$ (c) $\psi^{a,b_2}$ (d) $\psi^{a,b_3}$ (e) $|\widehat{a}|, |\widehat{b_1}|, |\widehat{b_2}|, |\widehat{b_3}|$

(f) $\phi^a$ (g) $\psi^{a,b_1}$ (h) $\psi^{a,b_2}$ (i) $\psi^{a,b_3}$ (j) $|\widehat{a}|, |\widehat{b_1}|, |\widehat{b_2}|, |\widehat{b_3}|$

(k) $\phi^a$ (l) $\psi^{a,b_1}$ (m) $\psi^{a,b_2}$ (n) $\psi^{a,b_3}$ (o) $|\widehat{a}|, |\widehat{b_1}|, |\widehat{b_2}|, |\widehat{b_3}|$

FIGURE 3.2. (a)–(e) for the tight framelet filter bank $\{a; b_1, b_2, b_3\}$ with symmetry constructed by Algorithm 1 in Example 2 with $a = a_5^B(\cdot - 2)$ and using (2.19). (a), (b), (c), (d) are the graphs of the refinable function $\phi^a$ and the framelet functions $\psi^{a,b_1}$, $\psi^{a,b_2}$, $\psi^{a,b_3}$, respectively. (e) is the magnitudes of $\widehat{a}$ (in solid line), $\widehat{b_1}$ (in dashed line), $\widehat{b_2}$ (in dotted line), and $\widehat{b_3}$ (in dashed-dotted line) on the interval $[-\pi, \pi]$. (f)–(j) for the tight framelet filter bank $\{a; b_1, b_2, b_3\}$ with symmetry constructed by Algorithm 1 in Example 2 with $a = a_5^B(\cdot - 2)$ and using (2.23). (k)–(o) for the tight framelet filter bank $\{a; b_1, b_2, b_3\}$ with symmetry constructed by Algorithm 1 in Example 2 with the choice $\mathsf{d_a}(z) = z^{-1}(1+z)^3$. For the graphs in (l) and (m), the solid lines are for the real parts and the dotted lines are for the imaginary parts of the framelet functions $\psi^{a,b_1}$ and $\psi^{a,b_2}$. (o) is the magnitudes of $\widehat{a}$ (in solid line), $\widehat{b_1}$ (in dashed line), $\widehat{b_2}$ (in dotted line), and $\widehat{b_3}$ (in dashed-dotted line) on the interval $[-\pi, \pi]$.

Then $\mathsf{Sb}_1(z) = -1, \mathsf{Sb}_2(z) = -z^2, \mathsf{Sb}_3(z) = z^2$ (type II) and $\mathrm{vm}(b_1) = \mathrm{vm}(b_2) = 1, \mathrm{vm}(b_3) = 6$. A slightly different example is obtained in [1, Example 8].

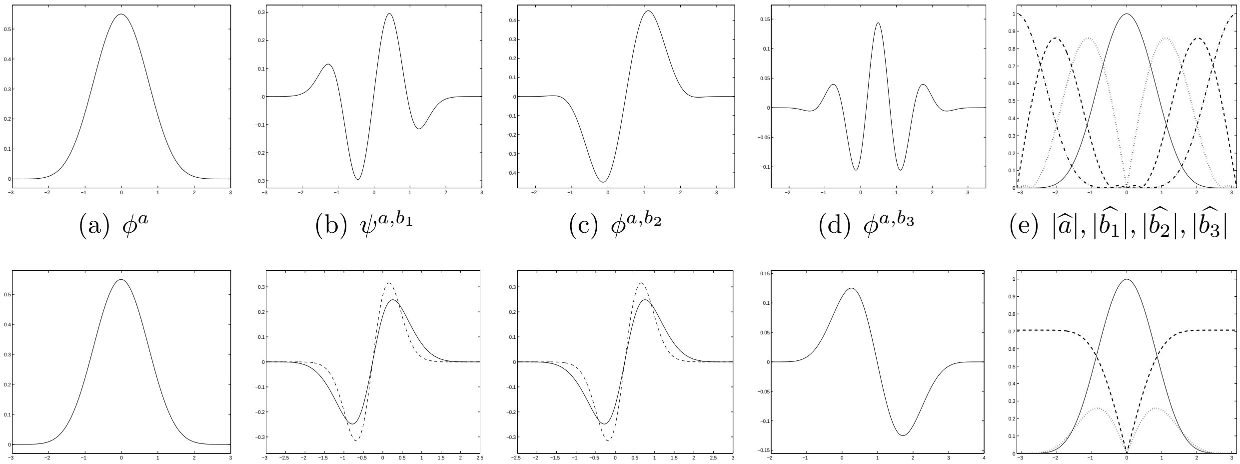
We can also take $\mathsf{d_a}(z) = z^{-2}(1+z)^5$ (or $\mathsf{d_a}(z) = (1+z)$ or $(1+z)^3$) in Algorithm 1. Then $\mathring{\mathsf{a}}(z) = \frac{1}{64}(z^{-1}+1)$, $\mathsf{q}(z) = 1$, and

$$\mathsf{d_p}(z) = \tfrac{1}{32}(z^2 - z^{-3}) + \tfrac{5+2\sqrt{5-2\sqrt{5}}i}{32}(z - z^{-2}) + \tfrac{2\sqrt{5}+\sqrt{5-2\sqrt{5}}(7+2\sqrt{5})i}{16}(1 - z^{-1}).$$

Taking $\mathsf{d_q} = 1$ and the trivial choice $\mathsf{b} = \frac{\sqrt{2}}{2}$, we get $\mathsf{b}_1(z) = \frac{\sqrt{2}}{2}\mathsf{d_p}(z)$ and

$$\begin{aligned}
\mathsf{b}_1(z) &= \{-\tfrac{\sqrt{2}}{64}, -\tfrac{5\sqrt{2}+2\sqrt{10-4\sqrt{5}}i}{64}, -\tfrac{2\sqrt{10}+\sqrt{10-4\sqrt{5}}(7+2\sqrt{5})i}{32}, \tfrac{2\sqrt{10}+\sqrt{10-4\sqrt{5}}(7+2\sqrt{5})i}{32}, \tfrac{5\sqrt{2}+2\sqrt{10-4\sqrt{5}}i}{64}, \tfrac{\sqrt{2}}{64}\}_{[-3,2]},\\
\mathsf{b}_2(z) &= \mathsf{d_p}(z)\mathsf{d_q}(z)z\mathsf{b}^\star(-z) = z\mathsf{b}_1(z),\\
\mathsf{b}_3(z) &= \mathsf{d_a}(z)\boldsymbol{\theta}(z)z\mathring{\mathsf{a}}^\star(-z) = \tfrac{1}{64}(z^{-1}-1)(1+z)^5 = \{\tfrac{1}{64}, \underline{\boldsymbol{\tfrac{1}{16}}}, \tfrac{5}{64}, 0, -\tfrac{5}{64}, -\tfrac{1}{16}, -\tfrac{1}{64}\}_{[-1,5]}.
\end{aligned}$$

Then $\mathsf{Sb}_1(z) = -z^{-1}, \mathsf{Sb}_2(z) = -z, \mathsf{Sb}_3(z) = -z^4$ (type I) and $\mathrm{vm}(b_1) = \mathrm{vm}(b_2) = \mathrm{vm}(b_3) = 1$. See Figure 3.3 for graphs of the associated refinable function $\phi^a$ and framelet functions $\psi^{a,b_1}, \psi^{a,b_2}$, and $\psi^{a,b_3}$.



(a) $\phi^a$ (b) $\psi^{a,b_1}$ (c) $\phi^{a,b_2}$ (d) $\phi^{a,b_3}$ (e) $|\widehat{a}|, |\widehat{b_1}|, |\widehat{b_2}|, |\widehat{b_3}|$

(f) $\phi^a$ (g) $\psi^{a,b_1}$ (h) $\psi^{a,b_2}$ (i) $\psi^{a,b_3}$ (j) $|\widehat{a}|, |\widehat{b_1}|, |\widehat{b_3}|$

FIGURE 3.3. (a)–(e) for the tight framelet filter bank $\{a; b_1, b_2, b_3\}$ with symmetry constructed by Algorithm 1 in Example 3 with $a = a_6^B(\cdot - 3)$ and $\mathsf{d_a}(z) = 1$. (a), (b), (c), (d) are the graphs of the refinable function $\phi^a$ and the framelet functions $\psi^{a,b_1}$, $\psi^{a,b_2}$, $\psi^{a,b_3}$, respectively. (e) is the magnitudes of $\widehat{a}$ (in solid line), $\widehat{b_1}$ (in dashed line), $\widehat{b_2}$ (in dotted line), and $\widehat{b_3}$ (in dashed-dotted line) on the interval $[-\pi, \pi]$. (f)–(j) for the tight framelet filter bank $\{a; b_1, b_2, b_3\}$ with symmetry constructed by Algorithm 1 in Example 3 with the choice $\mathsf{d_a}(z) = 1 + z$. For the graphs in (g) and (h), the solid lines are for the real parts and the dotted lines are for the imaginary parts of the framelet functions $\psi^{a,b_1}$ and $\psi^{a,b_2}$. (j) is the magnitudes of $\widehat{a}$ (in solid line), $\widehat{b_1}$ (in dashed line), $\widehat{b_3}$ (in dotted line), on the interval $[-\pi, \pi]$. Note that $|\widehat{b_1}(\xi)| = |\widehat{b_2}(\xi)|$.

**Example 4.** Let $a = a_4^I = \{-\frac{1}{32}, 0, \frac{9}{32}, \underline{\boldsymbol{\frac{1}{2}}}, \frac{9}{32}, 0, -\frac{1}{32}\}_{[-3,3]}$ and $\Theta = \boldsymbol{\delta}$. Then $\boldsymbol{\theta} = 1$. Setting $\mathsf{d_a} = 1$ in Algorithm 1, we have $\mathring{\mathsf{a}}(z) = \mathsf{a}(z)$ and $\mathsf{q}(z^2) = \mathsf{p}(z) = \frac{1}{512}(1 - z^{-2})^2(1 - z^2)^2(14 - z^{-2} - z^2)$. Hence, $\mathsf{d_p} = 1$. We observe that $\mathsf{Z}(\mathsf{q}, 7 - 4\sqrt{3}) = 1$ and $7 - 4\sqrt{3} \in (0, 1)$. Hence, we cannot use (2.23) to obtain a filter $\mathsf{b}$ with symmetry. Setting $\mathsf{d_q} = 1$ and using (2.20) via Fejér-Riesz lemma with $\mathring{\mathsf{q}} = \mathsf{q}$, we have $\mathsf{b}(z) = \frac{\sqrt{2}(2+\sqrt{3})}{64}(1 - z^{-1})^2(1 + z)^3(z^{-1} + 4\sqrt{3} - 8 + z)$. Hence,

$$\mathsf{b}_1(z) = \mathsf{b}(z) = \{\tfrac{\sqrt{6}+2\sqrt{2}}{64}, \tfrac{\sqrt{6}-2\sqrt{2}}{64}, -\tfrac{\sqrt{6}+6\sqrt{2}}{64}, \underline{\boldsymbol{\tfrac{6\sqrt{2}-\sqrt{6}}{64}}}, \tfrac{6\sqrt{2}-\sqrt{6}}{64}, -\tfrac{\sqrt{6}+6\sqrt{2}}{64}, \tfrac{\sqrt{6}-2\sqrt{2}}{64}, \tfrac{\sqrt{6}+2\sqrt{2}}{64}\}_{[-3,4]},$$

$$\mathsf{b}_2(z) = z\mathsf{b}^\star(-z) = \{\tfrac{\sqrt{6}+2\sqrt{2}}{64}, \tfrac{2\sqrt{2}-\sqrt{6}}{64}, -\tfrac{\sqrt{6}+6\sqrt{2}}{64}, \underline{\boldsymbol{\tfrac{\sqrt{6}-6\sqrt{2}}{64}}}, \tfrac{6\sqrt{2}-\sqrt{6}}{64}, \tfrac{\sqrt{6}+6\sqrt{2}}{64}, \tfrac{\sqrt{6}-2\sqrt{2}}{64}, -\tfrac{\sqrt{6}+2\sqrt{2}}{64}\}_{[-3,4]},$$

$$\mathsf{b}_3(z) = \tfrac{1}{32}z^{-1}(z - 1)^4(z^{-1} + 4 + z) = \{\tfrac{1}{32}, 0, \underline{\boldsymbol{-\tfrac{9}{32}}}, \tfrac{1}{2}, -\tfrac{9}{32}, 0, \tfrac{1}{32}\}_{[-2,4]}.$$

Then $\mathsf{Sb}_1(z) = z, \mathsf{Sb}_2(z) = -z, \mathsf{Sb}_3(z) = z^2$ (type I) and $\mathrm{vm}(b_1) = 2, \mathrm{vm}(b_2) = 3, \mathrm{vm}(b_3) = 4$. This example is given in [1, Example 9] with $\mathrm{len}(b_1) = \mathrm{len}(b_2) = 7 > \mathrm{len}(a) = 6$. Noting $\mathsf{a}(z) = z^{-3}(z + 1)^4(z - 2 - \sqrt{3})(z - 2 + \sqrt{3})$ and searching all the possible finitely many choices of $\boldsymbol{\theta}$ and $\mathsf{d_a}$ in Algorithm 1, we find that there is no tight framelet filter bank $\{a_4^I; b_1, b_2, b_3\}$ with symmetry which can satisfy both $\max(\mathrm{len}(b_1), \mathrm{len}(b_2), \mathrm{len}(b_3)) = \mathrm{len}(a_4^I)$ and (2.2) with $\Theta = \boldsymbol{\delta}$.

However, we can still use complex symmetry in Theorem 7 to achieve the shortest possible filter supports. We have $\mathsf{b}(z) = \frac{1}{32}(z - z^{-1})^2(z - 2\sqrt{3}i - z^{-1})$ with complex symmetry $\mathbb{S}\mathsf{b}(z) = -1$ and

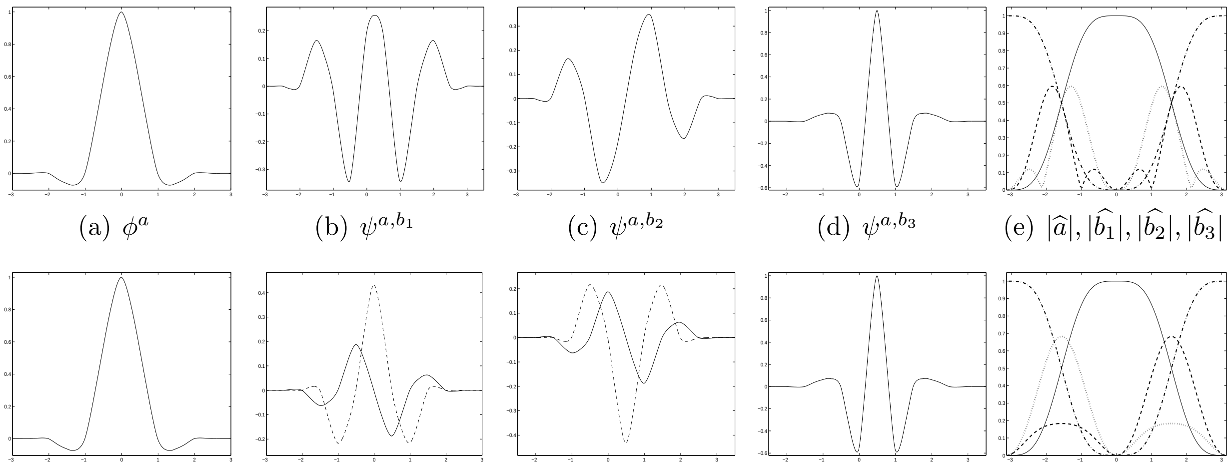
$$\mathsf{b}_1(z) = \mathsf{d_p}(z)\mathsf{d_q}(z)\mathsf{b}(z) = \mathsf{b}(z) = \{-\tfrac{1}{32}, -\tfrac{\sqrt{3}i}{16}, \tfrac{3}{32}, \underline{\boldsymbol{\tfrac{\sqrt{3}i}{8}}}, -\tfrac{3}{32}, -\tfrac{\sqrt{3}i}{16}, \tfrac{1}{32}\}_{[-3,3]},$$

$$\mathsf{b}_2(z) = \mathsf{d_p}(z)\mathsf{d_q}(z)z\mathsf{b}^\star(-z) = z\mathsf{b}^\star(-z) = \{-\tfrac{1}{32}, \tfrac{\sqrt{3}i}{16}, \underline{\boldsymbol{\tfrac{3}{32}}}, -\tfrac{\sqrt{3}i}{8}, -\tfrac{3}{32}, \tfrac{\sqrt{3}i}{16}, \tfrac{1}{32}\}_{[-2,4]},$$

$$\mathsf{b}_3(z) = \mathsf{d_a}(z)\boldsymbol{\theta}(z)z\mathring{\mathsf{a}}(z) = \tfrac{1}{32}z^{-1}(z - 1)^4(z^{-1} + 4 + z) = \{\tfrac{1}{32}, 0, \underline{\boldsymbol{-\tfrac{9}{32}}}, \tfrac{1}{2}, -\tfrac{9}{32}, 0\tfrac{1}{32}\}_{[-2,4]}.$$

Then $\mathbb{S}\mathsf{b}_1(z) = -1, \mathbb{S}\mathsf{b}_2(z) = -z^2, \mathbb{S}\mathsf{b}_3(z) = z^2$ (type II) and $\mathrm{vm}(b_1) = 2, \mathrm{vm}(b_2) = 2, \mathrm{vm}(b_3) = 4$. See Figure 3.4 for graphs of the associated refinable function $\phi^a$ and framelet functions $\psi^{a,b_1}, \psi^{a,b_2}$, and

$\psi^{a,b_3}$. We also point out here that if the condition (2.2) is dropped (and therefore, Algorithm 1 cannot be used), we shall show in Example 9 that there indeed exists a real-valued tight framelet filter bank $\{a_4^I; b_1, b_2, b_3\}$ with symmetry and $\max(\mathrm{len}(b_1), \mathrm{len}(b_2), \mathrm{len}(b_3)) = \mathrm{len}(a_4^I)$.



(a) $\phi^a$ (b) $\psi^{a,b_1}$ (c) $\psi^{a,b_2}$ (d) $\psi^{a,b_3}$ (e) $|\widehat{a}|, |\widehat{b_1}|, |\widehat{b_2}|, |\widehat{b_3}|$

(f) $\phi^a$ (g) $\psi^{a,b_1}$ (h) $\psi^{a,b_2}$ (i) $\psi^{a,b_3}$ (j) $|\widehat{a}|, |\widehat{b_1}|, |\widehat{b_2}|, |\widehat{b_3}|$

FIGURE 3.4. (a)–(e) for the tight framelet filter bank $\{a; b_1, b_2, b_3\}$ with symmetry constructed by Algorithm 1 in Example 4 with $a = a_4^I$ and (2.20). (a), (b), (c), (d) are the graphs of the refinable function $\phi^a$ and the framelet functions $\psi^{a,b_1}$, $\psi^{a,b_2}$, $\psi^{a,b_3}$, respectively. (e) is the magnitudes of $\widehat{a}$ (in solid line), $\widehat{b_1}$ (in dashed line), $\widehat{b_2}$ (in dotted line), and $\widehat{b_3}$ (in dashed-dotted line) on the interval $[-\pi, \pi]$. (f)–(j) for the tight framelet filter bank $\{a; b_1, b_2, b_3\}$ with symmetry constructed by Algorithm 1 in Example 4 with $a = a_4^I$ and (2.23). (f), (g), (h), (i) are the graphs of the refinable function $\phi^a$ and the framelet functions $\psi^{a,b_1}$, $\psi^{a,b_2}$, $\psi^{a,b_3}$, respectively. For the graphs in (g) and (h), the solid lines are for the real parts and the dotted lines are for the imaginary parts of the framelet functions $\psi^{a,b_1}$ and $\psi^{a,b_2}$. (j) is the magnitudes of $\widehat{a}$ (in solid line), $\widehat{b_1}$ (in dashed line), $\widehat{b_2}$ (in dotted line), and $\widehat{b_3}$ (in dashed-dotted line) on the interval $[-\pi, \pi]$.

**Example 5.** Let $a = a_4^B(\cdot - 2) = \{\frac{1}{16}, \frac{1}{4}, \underline{\mathbf{\frac{3}{8}}}, \frac{1}{4}, \frac{1}{16}\}_{[0,4]}$ be the centered B-spline filter of order 4 and $\Theta = \{-\frac{1}{3}, \underline{\mathbf{\frac{5}{3}}}, -\frac{1}{3}\}_{[-1,1]}$. Then $\boldsymbol{\theta}(z) = \frac{\sqrt{21}}{3} - \frac{\sqrt{3}}{3}(z^{-1} + z)$ by Theorem 2. Selecting $\mathsf{d_a}(z) = 1 + z$ in Algorithm 1, we have $\mathring{\mathsf{a}}(z) = \frac{1}{16}(z^{-2} + 3z^{-1} + 3 + z)$, $\mathsf{q}(z) = 1$, and

$$\mathsf{d_p}(z) = \tfrac{\sqrt{6}}{384} z^{-1}(z-1)^2[8(z^{-2} + z^2) + (24 - 4\sqrt{7} + 8\lambda i)(z^{-1} + z) + 101 + 12\sqrt{7} - 4\lambda^2 - 8\lambda_1 i],$$

where $\lambda \approx 3.971226296945828$ is a real root of

$$64\lambda^6 + (-2032 - 576\sqrt{7})\lambda^4 + (9196 + 8224\sqrt{7})\lambda^2 + 82819 + 23580\sqrt{7} = 0$$

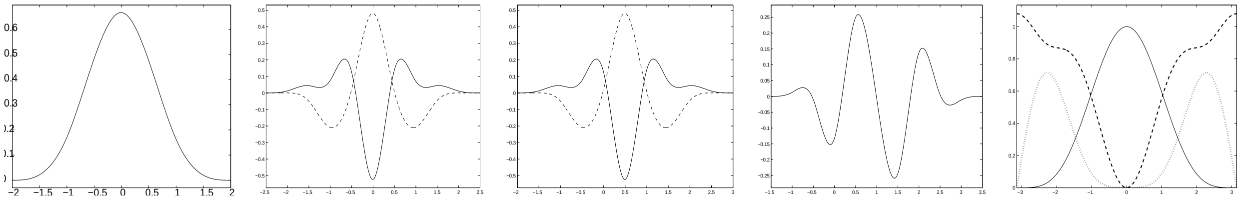
and $\lambda_1 = \frac{262+45\sqrt{7}}{16}\lambda^{-1} + \frac{6-\sqrt{7}}{4}\lambda \approx -9.327304339851380$. Take $\mathsf{d_q} = 1$ and $\mathsf{b}(z) = \frac{\sqrt{2}}{2}$. Hence,

$$\begin{aligned}\mathsf{b}_1(z) = \mathsf{d_p}(z)\mathsf{d_q}(z)\mathsf{b}(z) = \tfrac{\sqrt{2}}{2}\mathsf{d_p}(z) = \{&0.0360843918242, -0.0116507716644 - 0.143299285722\, i,\\ &0.229282359208 - 0.0499715330192\, i, -0.507431958733 + 0.386541637482\, i,\\ &0.229282359208 - 0.0499715330192\, i, -0.0116507716644 - 0.143299285722\, i,\\ &0.0360843918242\}_{[-3,3]},\end{aligned}$$

$$\mathsf{b}_2(z) = \mathsf{d_p}(z)\mathsf{d_q}(z)z\mathsf{b}^\star(z) = z\mathsf{b}_1(z),$$

$$\mathsf{b}_3(z) = \mathsf{d_a}(z)\boldsymbol{\theta}(z)z\mathring{\mathsf{a}}(z) = \{\tfrac{\sqrt{3}}{48}, \underline{-\tfrac{\boldsymbol{\sqrt{21}+2\sqrt{3}}}{\mathbf{48}}}, \tfrac{2\sqrt{21}+\sqrt{3}}{48}, 0, -\tfrac{2\sqrt{21}+\sqrt{3}}{48}, \tfrac{\sqrt{21}+2\sqrt{3}}{48}, -\tfrac{\sqrt{3}}{48}\}_{[-1,5]}.$$

Then $\mathsf{Sb}_1(z) = 1, \mathsf{Sb}_2(z) = z^2, \mathsf{Sb}_3(z) = -z^4$ (type II) and $\mathrm{vm}(b_1) = \mathrm{vm}(b_2) = 2, \mathrm{vm}(b_3) = 3$. See Figure 3.5 for graphs of the refinable function $\phi^a$ and framelet functions $\psi^{a,b_1}, \psi^{a,b_2}$, and $\psi^{a,b_3}$.



(a) $\phi^a$ (b) $\psi^{a,b_1}$ (c) $\psi^{a,b_2}$ (d) $\psi^{a,b_3}$ (e) $|\widehat{a}|, |\widehat{b_1}|, |\widehat{b_3}|$

FIGURE 3.5. The tight framelet filter bank $\{a; b_1, b_2, b_3\}_\Theta$ with symmetry is constructed by Algorithm 1 in Example 5 with $a = a_4^B(\cdot - 2)$. (a), (b), (c), (d) are the graphs of the refinable function $\phi^a$ and the framelet functions $\psi^{a,b_1}$, $\psi^{a,b_2}$, $\psi^{a,b_3}$, respectively. For the graphs in (b) and (c), the solid lines are for the real parts and the dotted lines are for the imaginary parts of the framelet functions $\psi^{a,b_1}$ and $\psi^{a,b_2}$. (e) is the magnitudes of $\widehat{a}$ (in solid line), $\widehat{b_1}$ (in dashed line), $\widehat{b_3}$ (in dotted line) on the interval $[-\pi, \pi]$. Note that $|\widehat{b_1}(\xi)| = |\widehat{b_2}(\xi)|$.

## 4. Type II Symmetric Tight Framelet Filter Banks with Three High-pass Filters

Though Algorithm 1 can be used to construct all type II symmetric tight framelet filter banks satisfying the condition in (2.2), there are many type II symmetric tight framelet filter banks which do not satisfy the condition in (2.2). The construction of all type II tight framelet filter banks $\{a; b_1, b_2, b_3\}_\Theta$ such that the extra assumption in (2.2) fails is much more involved. Using results in [7], in this section we propose an algorithm to construct all symmetric tight framelet filter banks $\{a; b_1, b_2, b_3\}_\Theta$ having the shortest possible filter support in (1.13). To do so, we first examine in detail the symmetry patterns of a tight framelet filter bank $\{a; b_1, b_2, b_3\}_\Theta$ with [complex] symmetry.

**Theorem 8.** *Let $\{a; b_1, b_2, b_3\}_\Theta$ be a tight framelet filter bank such that all the filters $\Theta, a, b_1, b_2, b_3 \in l_0(\mathbb{Z})$ are not identically zero and have symmetry in* (2.1) *(or complex symmetry by replacing* $\mathsf{S}$ *with* $\mathbb{S}$*). If*

$$\max(\mathrm{len}(b_1), \mathrm{len}(b_2), \mathrm{len}(b_3)) \leqslant \mathrm{len}(a) + \mathrm{len}(\Theta) \neq 0, \tag{4.1}$$

*then up to reordering of $b_1, b_2, b_3$, we must have one of the following four cases:*

(1) $\max(\mathrm{len}(b_1), \mathrm{len}(b_2)) < \mathrm{len}(b_3) = \mathrm{len}(a) + \mathrm{len}(\Theta)$ *with* $\frac{c_3}{2} - (\frac{c}{2} + n_\Theta) \in 2\mathbb{Z}$*;*
(2) $\mathrm{len}(b_1) < \mathrm{len}(b_2) = \mathrm{len}(b_3) = \mathrm{len}(a) + \mathrm{len}(\Theta)$ *with* $\frac{c_\ell}{2} - (\frac{c}{2} + n_\Theta) \in 2\mathbb{Z}$ *for* $\ell = 2, 3$*;*
(3) $\mathrm{len}(b_1) = \mathrm{len}(b_2) = \mathrm{len}(b_3) = \mathrm{len}(a) + \mathrm{len}(\Theta)$ *with* $\frac{c_\ell}{2} - (\frac{c}{2} + n_\Theta) \in 2\mathbb{Z}$ *for* $\ell = 1, 2, 3$*;*
(4) $\mathrm{len}(b_1) = \mathrm{len}(b_2) = \mathrm{len}(b_3) = \mathrm{len}(a) + \mathrm{len}(\Theta)$ *with* $\frac{c_3}{2} - (\frac{c}{2} + n_\Theta) \in 2\mathbb{Z}$ *and* $\frac{c_\ell}{2} - (\frac{c+1}{2} + n_\Theta) \in 2\mathbb{Z}$ *for* $\ell = 1, 2$.

*For the case of symmetry, we always have $\epsilon_3 = -\epsilon sgn(\Theta(n_\Theta))$ and for item (4) we additionally have $\epsilon_1\epsilon_2 = -1$, where $\Theta(n_\Theta)$ is the leading coefficient of $\Theta$, and $sgn(\Theta(n_\Theta)) = 1$ if $\Theta(n_\Theta) > 0$ and $sgn(\Theta(n_\Theta)) = -1$ if $\Theta(n_\Theta) < 0$.*

*Proof.* To prove the claim, we compare the leading terms in the prefect reconstruction conditions in (1.5) and (1.6). Define

$$[c - n_0, n_0] := \mathrm{fsupp}(a), \quad [c_\ell - n_\ell, n_\ell] := \mathrm{fsupp}(b_\ell), \qquad \ell = 1, 2, 3.$$

For the case of complex symmetry, we define

$$\lambda_0 := \epsilon\Theta(n_\Theta)(a(n_0))^2, \qquad \lambda_\ell := \epsilon_\ell(b_\ell(n_\ell))^2, \qquad \ell = 1, 2, 3.$$

For the case of symmetry, noting that $\Theta$ must have real coefficients by $\mathsf{S}\Theta = \mathbb{S}\Theta = 1$, we define

$$\lambda_0 := \epsilon\Theta(n_\Theta)|a(n_0)|^2, \qquad \lambda_\ell := \epsilon_\ell|b_\ell(n_\ell)|^2, \qquad \ell = 1, 2, 3. \tag{4.2}$$

Then the leading terms of each addent in (1.5) are

$$\lambda_0 z^{2n_\Theta + 2n_0 - c}, \quad \lambda_1 z^{2n_1 - c_1}, \quad \lambda_2 z^{2n_2 - c_2}, \quad \lambda_3 z^{2n_3 - c_3}, \quad \Theta(n_\Theta) z^{n_\Theta}$$

and the leading terms of each addent in (1.6) are

$$(-1)^{n_0-c}\lambda_0 z^{2n_\Theta+2n_0-c}, \quad (-1)^{n_1-c_1}\lambda_1 z^{2n_1-c_1}, \quad (-1)^{n_2-c_2}\lambda_2 z^{2n_2-c_2}, \quad (-1)^{n_3-c_3}\lambda_3 z^{2n_3-c_3},$$

respectively. Note that $n_\Theta < 2n_\Theta + 2n_0 - c$ by $\mathrm{len}(a)+\mathrm{len}(\Theta) \neq 0$. Now there must exist $1 \leqslant L \leqslant 3$ such that $\mathrm{len}(b_\ell) = \mathrm{len}(a) + \mathrm{len}(\Theta)$ (that is, $2n_\ell - c_\ell = 2n_\Theta + 2n_0 - c$) for all $L \leqslant \ell \leqslant 3$, and $\mathrm{len}(b_\ell) < \mathrm{len}(a) + \mathrm{len}(\Theta)$ for all $1 \leqslant \ell < L$. Therefore,

$$\lambda_0 + \sum_{\ell=L}^{3} \lambda_\ell = 0, \qquad \lambda_0 + \sum_{\ell=L}^{3} (-1)^{n_\ell - n_0 - c_\ell + c}\lambda_\ell = 0. \tag{4.3}$$

Without loss of generality, there exists $L' \geqslant L$ such that $(-1)^{n_\ell-n_0-c_\ell+c} = 1$ for all $L' \leqslant \ell \leqslant 3$ and $(-1)^{n_\ell-n_0-c_\ell+c} = -1$ for all $L \leqslant \ell < L'$. Now it follows from (4.3) that

$$\lambda_0 + \sum_{\ell=L'}^{3} \lambda_\ell = 0, \qquad \sum_{\ell=L}^{L'-1} \lambda_\ell = 0. \tag{4.4}$$

Clearly, we must have $L' \leqslant 3$, otherwise, $\lambda_0 = 0$, which is a contradiction. Since $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ are nonzero numbers and $L \leqslant L'$, we deduce from (4.4) that there are only four possible cases.

Case 1: $L = 3$ and $L' = 3$. Then (4.4) becomes $\lambda_0 + \lambda_3 = 0$. For the case of symmetry, $\lambda_0 + \lambda_3 = 0$ implies that $\lambda_0\lambda_3 < 0$ and therefore, we must have $\epsilon_3 = -\epsilon\,\mathrm{sgn}(\Theta(n_\Theta))$. By the definition of $L$ and $L'$, we have $\max(\mathrm{len}(b_1), \mathrm{len}(b_2)) < \mathrm{len}(b_3) = \mathrm{len}(a) + \mathrm{len}(\Theta)$ and $n_3 - n_0 - c_3 + c \in 2\mathbb{Z}$. Noting that $2n_3 - c_3 = \mathrm{len}(b_3)$ and $2n_0 - c = \mathrm{len}(a)$, we see that this case leads to item (1) since

$$c_3 - (c + 2n_\Theta) = c_3 - c - \mathrm{len}(\Theta) = c_3 - c - (\mathrm{len}(b_3) - \mathrm{len}(a)) = -2(n_3 - n_0 - c_3 + c) \in 4\mathbb{Z}.$$

Case 2: $L = 2$ and $L' = 2$. By the same argument as in Case 1, this case leads to item (2).

Case 3: $L = 1$ and $L' = 1$. This case leads to item (3).

Case 4: $L = 1$ and $L' = 3$. This case leads to item (4). ☐

Note that $\{a; b_1, b_2, b_3\}_\Theta$ is a tight framelet filter bank if and only if $\{a; \lambda_1 b_1(\cdot - 2n_1), \lambda_2 b_2(\cdot - 2n_2), \lambda_3 b_3(\cdot - 2n_3)\}_\Theta$ is a tight framelet filter bank, where $|\lambda_1| = |\lambda_2| = |\lambda_3| = 1$ and $n_1, n_2, n_3 \in \mathbb{Z}$. It is also important to notice the trivial relation $\mathbb{S}(i\mathsf{b}) = -\mathbb{S}\mathsf{b}$ for any filter $b$ with complex symmetry, that is, for a complex-valued filter $b_\ell$ in a tight framelet filter bank $\{a; b_1, b_2, b_3\}_\Theta$ such that $\mathbb{S}\mathsf{b}_\ell(z) = \epsilon_\ell z^{c_\ell}$ with $\epsilon_\ell \in \{-1, 1\}$ and $c_\ell \in \mathbb{Z}$, the sign of $\epsilon_\ell$ can be easily flipped and therefore plays no significant role in a complex-valued tight framelet filter bank $\{a; b_1, b_2, b_3\}_\Theta$ with complex symmetry.

Basically, Theorem 8 says that under the condition in (4.1) the symmetry center of a high-pass filter $b_\ell$ satisfying $\mathrm{len}(b_\ell) = \mathrm{len}(a) + \mathrm{len}(\Theta)$ is uniquely determined by the filters $a$ and $\Theta$. Moreover, up to an even integer shift and reordering, there must be a high-pass filter $b_3$ having the symmetry pattern $-\epsilon\,\mathrm{sgn}(\Theta(n_\Theta))z^{c+2n_\Theta}$. Using this observation and [7] on symmetric tight framelet filter banks with two high-pass filters, we shall propose an algorithm to construct all symmetric tight framelet filter banks with three high-pass filters. To do so, let us recall some notation from [7].

For a filter $u = \{u(k)\}_{k\in\mathbb{Z}} \in l_0(\mathbb{Z})$ and $\gamma \in \mathbb{Z}$, the $\gamma$-coset of $u$ is defined to be

$$u^{[\gamma]}(k) := u(\gamma + 2k), \qquad k \in \mathbb{Z}, \quad \text{or equivalently,} \quad \mathsf{u}^{[\gamma]}(z) = \sum_{k\in\mathbb{Z}} u(\gamma + 2k) z^k. \tag{4.5}$$

We often write

$$\mathsf{b}_1(z) = (1 - z^{-1})^{n_b}\mathring{\mathsf{b}}_1(z), \quad \ldots, \quad \mathsf{b}_s(z) = (1 - z^{-1})^{n_b}\mathring{\mathsf{b}}_s(z).$$

Using the coset sequences in (4.5) and the relation $\mathring{\mathsf{b}}(z) = \mathring{\mathsf{b}}^{[0]}(z^2) + z\mathring{\mathsf{b}}^{[1]}(z^2)$, one can easily check that the condition in (1.1) for a tight framelet filter bank $\{a; b_1, \ldots, b_s\}_\Theta$ can be equivalently rewritten as

$$\begin{bmatrix} \mathring{\mathsf{b}}_1^{[0]}(z) & \cdots & \mathring{\mathsf{b}}_s^{[0]}(z) \\ \mathring{\mathsf{b}}_1^{[1]}(z) & \cdots & \mathring{\mathsf{b}}_s^{[1]}(z) \end{bmatrix} \begin{bmatrix} \mathring{\mathsf{b}}_1^{[0]}(z) & \cdots & \mathring{\mathsf{b}}_s^{[0]}(z) \\ \mathring{\mathsf{b}}_1^{[1]}(z) & \cdots & \mathring{\mathsf{b}}_s^{[1]}(z) \end{bmatrix}^{\star} = \mathcal{N}_{a,\Theta|n_b}(z) \tag{4.6}$$

with

$$\mathcal{N}_{a,\Theta|n_b}(z) := \frac{1}{2} \begin{bmatrix} \mathsf{A}^{[0]}(z) + \mathsf{B}^{[0]}(z) & z\big(\mathsf{A}^{[1]}(z) + \mathsf{B}^{[1]}(z)\big) \\ \mathsf{A}^{[1]}(z) - \mathsf{B}^{[1]}(z) & \mathsf{A}^{[0]}(z) - \mathsf{B}^{[0]}(z) \end{bmatrix} \tag{4.7}$$

and

$$\mathsf{A}(z) := \frac{\Theta(z) - \Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(z)}{(1-z)^{n_b}(1-z^{-1})^{n_b}}, \qquad \mathsf{B}(z) := -\Theta(z^2)\frac{\mathsf{a}(-z)}{(1-z)^{n_b}}\frac{\mathsf{a}^\star(z)}{(1+z^{-1})^{n_b}}. \tag{4.8}$$

For a filter $u$, if $\mathsf{u}(z) = (1+z)^m \mathsf{v}(z)$ for some Laurent polynomial $\mathsf{v}$ such that $\mathsf{v}(-1) \neq 0$, then we define $\operatorname{sr}(u) := m$, called the order of sum rules of $u$. To guarantee that both $\mathsf{A}$ and $\mathsf{B}$ are well-defined Laurent polynomials, it is natural to require that

$$0 \leqslant n_b \leqslant \min\big(\operatorname{sr}(a), \tfrac{1}{2}\operatorname{vm}(\Theta(z) - \Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(z))\big). \tag{4.9}$$

We are now ready to present an algorithm to construct all symmetric tight framelet filter banks $\{a; b_1, b_2, b_3\}_\Theta$ with [complex] symmetry.

**Algorithm 2.** *Let $a, \Theta \in l_0(\mathbb{Z})$ be filters having [complex] symmetry (and real coefficients) such that $\mathsf{S}\mathsf{a}(z) = \epsilon z^c$ and $\mathsf{S}\Theta = \mathbb{S}\Theta = 1$, where $\epsilon \in \{-1, 1\}$ and $c \in \mathbb{Z}$ (for complex symmetry, replace $\mathsf{S}$ by $\mathbb{S}$). Choose a nonnegative integer $n_b$ such that* (4.9) *holds.*

(S1) *Select $c_3 \in \{c, c+2\}$ and $\epsilon_3 \in \{-1, 1\}$. For example, by Theorem 8 we can take $c_3 = c + 2n_\Theta$ and $\epsilon_3 = -\epsilon sgn(\Theta(n_\Theta))$, where $\Theta(n_\Theta)$ is the leading coefficient of $\Theta$ (set $\epsilon_3 = 1$ if $\Theta(n_\Theta)$ is a complex number);*

(S2) *Parameterize a filter $\mathring{\mathsf{b}}_3$ having symmetry $\mathsf{S}\mathring{\mathsf{b}}_3(z) = (-1)^{n_b}\epsilon_3 z^{c_3+n_b}$ as follows:*

$$\mathring{\mathsf{b}}_3(z) := \begin{cases} z^{\frac{c_3+n_b}{2}} \sum_{j=0}^{\ell-1}(\lambda_j z^j + (-1)^{n_b}\epsilon_3\lambda_j z^{-j}), & \text{if } c_3+n_b \text{ is even;} \\ z^{\frac{c_3+n_b-1}{2}} \sum_{j=0}^{\ell-1}(\lambda_j z^j + (-1)^{n_b}\epsilon_3\lambda_j z^{-1-j}), & \text{if } c_3+n_b \text{ is odd,} \end{cases} \tag{4.10}$$

*where $\lambda_0, \dots, \lambda_{\ell-1} \in \mathbb{C}$ (for the case of complex symmetry, replace the first $\lambda_j$ in each row by $\overline{\lambda_j}$ in* (4.10)*; for the case of real coefficients, $\lambda_0, \dots, \lambda_{\ell-1} \in \mathbb{R}$). To have short filter supports, we often choose $\ell \in \mathbb{N}$ such that $\operatorname{len}(\mathring{\mathsf{b}}_3) \leqslant \operatorname{len}(a) + \operatorname{len}(\Theta) - n_b$;*

(S3) *Define a $2 \times 2$ matrix $\mathcal{N}_{a,\Theta;b_3|n_b}$ by*

$$\mathcal{N}_{a,\Theta;b_3|n_b}(z) := \mathcal{N}_{a,\Theta|n_b}(z) - \begin{bmatrix} \mathring{\mathsf{b}}_3^{[0]}(z) \\ \mathring{\mathsf{b}}_3^{[1]}(z) \end{bmatrix} [(\mathring{\mathsf{b}}_3^{[0]}(z))^\star \quad (\mathring{\mathsf{b}}_3^{[1]}(z))^\star] \tag{4.11}$$

*and $\mathsf{p}(z) := \det(\mathcal{N}_{a,\Theta;b_3|n_b}(z))$. For the case of symmetry, apply* [7, Theorem 2.9] *to derive a Laurent polynomial $\mathsf{q_p}$ with symmetry from $\mathsf{p}$. For the case of complex symmetry (and real coefficients), apply* [7, Theorem 2.8] *to derive a Laurent polynomial $\mathsf{q_p}$ with complex symmetry from $\mathsf{p}$. Let $X$ denote the set of all equations by taking all the coefficients in $\mathsf{p}(z) - \mathsf{q_p}(z)\mathsf{q_p^\star}(z)$ to be zero. Use Gröbner basis method to solve the set $X$ of equations to determine the unknowns $\lambda_0, \dots, \lambda_{\ell-1}$.*

*If items (i) and (iii) of* [7, Theorem 4.2] *are satisfied with $\mathcal{N}_{a,\Theta|n_b}$ being replaced by $\mathcal{N}_{a,\Theta;b_3|n_b}$, then apply* [7, Algorithms 2 and 3] *with $\mathcal{N}_{a,\Theta|n_b}$ being replaced by $\mathcal{N}_{a,\Theta;b_3|n_b}$ to construct high-pass filters $\mathring{b}_1, \mathring{b}_2$ with [complex] symmetry (and real coefficients) such that*

$$\begin{bmatrix} \mathring{\mathsf{b}}_1^{[0]}(z) & \mathring{\mathsf{b}}_2^{[0]}(z) \\ \mathring{\mathsf{b}}_1^{[1]}(z) & \mathring{\mathsf{b}}_2^{[1]}(z) \end{bmatrix} \begin{bmatrix} \mathring{\mathsf{b}}_1^{[0]}(z) & \mathring{\mathsf{b}}_2^{[0]}(z) \\ \mathring{\mathsf{b}}_1^{[1]}(z) & \mathring{\mathsf{b}}_2^{[1]}(z) \end{bmatrix}^\star = \mathcal{N}_{a,\Theta;b_3|n_b}(z). \tag{4.12}$$

*Then $\{a; b_1, b_2, b_3\}_\Theta$ is a tight framelet filter bank with [complex] symmetry (and real coefficients), where the high-pass filters $b_1, b_2, b_3$ are defined by*

$$\mathsf{b}_1(z) := (1-z^{-1})^{n_b}\mathring{\mathsf{b}}_1(z), \quad \mathsf{b}_2(z) := (1-z^{-1})^{n_b}\mathring{\mathsf{b}}_2(z), \quad \mathsf{b}_3(z) := (1-z^{-1})^{n_b}\mathring{\mathsf{b}}_3(z). \tag{4.13}$$

*Proof.* To apply [7, Algorithms 2 and 3] with $\mathcal{N}_{a,\Theta|n_b}$ being replaced by $\mathcal{N}_{a,\Theta;b_3|n_b}$, we need to check that all the entries in the matrix $\mathcal{N}_{a,\Theta;b_3|n_b}$ defined in (4.7) have the same [complex] symmetry as these in $\mathcal{N}_{a,\Theta|n_b}$. By the definition of $b_3$, we have $\mathsf{S}\mathsf{b}_3(z) = \epsilon_3 z^{c_3}$. Since $c - c_3 \in 2\mathbb{Z}$, we have

$$\mathsf{S}(\mathsf{a}(-z)\mathsf{a}^\star(z)) = \mathsf{S}\mathsf{a}(-z)(\mathsf{S}\mathsf{a}(z))^\star = (-1)^c = (-1)^{c_3} = \mathsf{S}(\mathsf{b}_3(-z)\mathsf{b}_3^\star(z)).$$

We now can directly check that the symmetry pattern of $\mathcal{N}$ in the proof of [7, Theorem 4.2] obtained with $\mathcal{N}_{a,\Theta|n_b}$ being replaced by $\mathcal{N}_{a,\Theta;b_3|n_b}$ is the same as before. $\square$

By Proposition 1, we must have $c_3 - c \in 2\mathbb{Z}$ and therefore, all tight framelet filter banks $\{a; b_1, b_2, b_3\}_\Theta$ with [complex] symmetry (and real coefficients) can be theoretically constructed by Algorithm 2. However, solving a system of nonlinear quadratic algebraic equations is computationally expensive. Hence, we often choose a small integer $\ell$ in (S2) of Algorithm 2 to handle type II symmetric tight framelet filter banks $\{a; b_1, b_2, b_3\}_\Theta$. Further investigation is needed in order to reduce the computational complexity of Algorithm 2. In the following, we show that both methods in [1, 11] are special cases of Algorithm 2.

Take the moment correcting filter $\Theta = \boldsymbol{\delta}$. Let $n_b := 0$ and $\mathsf{b}_3(z) := z\mathsf{a}^\star(-z)$. By $\epsilon z^c = \mathsf{Sa}(z)$, we have $\epsilon_3 z^{c_3} = \mathsf{Sb}_3(z) = \epsilon(-1)^c z^{2-c}$. Therefore, $c_3 = 2 - c$ and $c_3 - c \in 2\mathbb{Z}$. Moreover, by $\mathsf{b}_3^{[0]}(z) = -(\mathsf{a}^{[1]}(z))^\star$ and $\mathsf{b}_3^{[1]}(z) = (\mathsf{a}^{[0]}(z))^\star$, we have

$$\mathcal{N}_{a,\Theta;b_3|0}(z) = \frac{1}{2}\begin{bmatrix} \mathsf{p}(z) & 0 \\ 0 & \mathsf{p}(z) \end{bmatrix} \qquad \text{with} \qquad \mathsf{p}(z^2) := 1 - \mathsf{a}(z)\mathsf{a}^\star(z) - \mathsf{a}(-z)\mathsf{a}^\star(-z),$$

where we used the identity $\mathsf{p}(z) = 2 - \mathsf{a}^{[0]}(z)(\mathsf{a}^{[0]}(z))^\star - \mathsf{a}^{[1]}(z)(\mathsf{a}^{[1]}(z))^\star$. Since $\mathsf{u}$ is obtained by Fejér-Riesz lemma such that $\mathsf{u}(z)\mathsf{u}^\star(z) = \mathsf{p}(z)$, we can trivially take

$$\mathring{\mathsf{b}}_1^{[0]}(z) = \tfrac{\mathsf{u}(z)}{2}, \qquad \mathring{\mathsf{b}}_1^{[1]}(z) = \tfrac{\mathsf{u}^\star(z)}{2}, \qquad \mathring{\mathsf{b}}_2^{[0]}(z) = \tfrac{\mathsf{u}(z)}{2}, \qquad \mathring{\mathsf{b}}_2^{[1]}(z) = -\tfrac{\mathsf{u}^\star(z)}{2}$$

so that (4.12) is satisfied. When $\mathsf{u}$ has real coefficients, we have $\mathsf{u}^\star(z) = \mathsf{u}(z^{-1})$. Now it is straightforward to see that the three high-pass filters constructed in Algorithm 2 are simply given in (1.8).

Take a moment correcting filter $\Theta$ as in (1.11). Let $n_b := 0$ and $\mathsf{b}_3(z) := z\mathsf{a}^\star(-z)\boldsymbol{\theta}^\star(z)$ in Algorithm 2. By the assumption $\boldsymbol{\theta}^\star(-z)\boldsymbol{\theta}(z) = \boldsymbol{\theta}^\star(z)\boldsymbol{\theta}(-z)$, we see that the symmetry center of $\boldsymbol{\theta}$ is an integer. Therefore, it is easy to check that we indeed have $c_3 - c \in 2\mathbb{Z}$. Using the assumption $\boldsymbol{\theta}^\star(z)\boldsymbol{\theta}(-z) = \boldsymbol{\theta}^\star(z)\boldsymbol{\theta}(-z)$, by calculation we have

$$\mathcal{N}_{a,\Theta;b_3|0}(z) = \boldsymbol{\theta}_0(z)\begin{bmatrix} \mathsf{a}^{[0]}(z) \\ \mathsf{a}^{[1]}(z) \end{bmatrix}[(\mathsf{a}^{[0]}(z))^\star \ (\mathsf{a}^{[1]}(z))^\star],$$

where $\boldsymbol{\theta}_0(z^2) = \boldsymbol{\theta}^\star(z)\boldsymbol{\theta}(-z) - \Theta(z^2)$. Since $\mathsf{v}(z)\mathsf{v}^\star(z) = \boldsymbol{\theta}_0(z)$, we can trivially take

$$\mathring{\mathsf{b}}_1^{[0]}(z) = \mathsf{a}^{[0]}(z)[\mathsf{v}(z) + \mathsf{v}^\star(z)]/2, \quad \mathring{\mathsf{b}}_1^{[1]}(z) = \mathsf{a}^{[1]}(z)[\mathsf{v}(z) + \mathsf{v}^\star(z)]/2,$$
$$\mathring{\mathsf{b}}_2^{[0]}(z) = \mathsf{a}^{[0]}(z)[\mathsf{v}(z) - \mathsf{v}^\star(z)]/2, \quad \mathring{\mathsf{b}}_2^{[1]}(z) = \mathsf{a}^{[1]}(z)[\mathsf{v}(z) - \mathsf{v}^\star(z)]/2$$

so that (4.12) is satisfied. When $\mathsf{v}$ has real coefficients, we have $\mathsf{v}^\star(z) = \mathsf{v}(z^{-1})$. Now it is straightforward to see that the three high-pass filters constructed in Algorithm 2 are simply given in (1.12). In other words, the methods in [1, 11] are special cases of Algorithm 2 with a particular choice of a moment correcting filter $\Theta$.

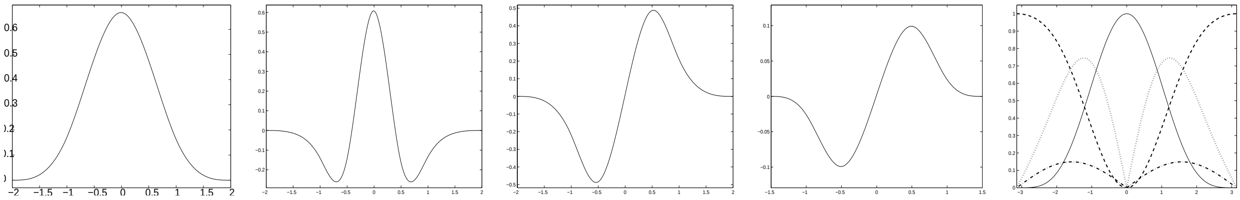
We now provide some examples using Algorithm 2. All the following presented examples are type II and cannot be obtained by Algorithm 1, since the condition in (2.2) fails.

**Example 6.** Let $a = a_4^B(\cdot - 2) = \{\frac{1}{16}, \frac{1}{4}, \underline{\mathbf{\frac{3}{8}}}, \frac{1}{4}, \frac{1}{16}\}_{[0,4]}$ be the centered B-spline filter of order 4 and $\Theta = \boldsymbol{\delta}$. Then $\epsilon z^c := \mathsf{Sa}(z) = 1$. Set $n_b = 1$, $c_3 = 0$, $\epsilon_3 = -1$, and $\mathring{\mathsf{b}}_3(z) = \lambda_0(1+z)$ in Algorithm 2. Then $\lambda_0 = \frac{1}{8}\sqrt{3-\sqrt{7}}$ and $\mathsf{q_p}(z) = \frac{\sqrt{3-\sqrt{7}}}{128} + \frac{17-3\sqrt{7}}{64\sqrt{3-\sqrt{7}}}z + \frac{\sqrt{3-\sqrt{7}}}{128}z^2$ satisfies $\mathsf{q_p}(z)\mathsf{q}_\mathsf{p}^\star(z) = \det(\mathcal{N}_{a,\Theta;b_3|n_b}(z))$. Then

$$\mathsf{b}_1(z) = \tfrac{\sqrt{7}-2}{48}(1-z^{-1})^3(3 + (14+4\sqrt{7})z + 3z^2) = \{\tfrac{2-\sqrt{7}}{16}, -\tfrac{1}{4}, \underline{\mathbf{\tfrac{2+\sqrt{7}}{8}}}, -\tfrac{1}{4}, \tfrac{2-\sqrt{7}}{16}\}_{[-2,2]},$$
$$\mathsf{b}_2(z) = \lambda_0(1-z^{-1})^2(1+z^{-1})(1 + (2+\sqrt{7})z + z^2) = \lambda_0\{-1, -2-\sqrt{7}, \underline{\mathbf{0}}, 2+\sqrt{7}, 1\}_{[-2,2]},$$
$$\mathsf{b}_3(z) = \lambda_0(1-z^{-1})(1+z) = \lambda_0\{-1, \underline{\mathbf{0}}, 1\}_{[-1,1]}.$$

Then $\mathsf{Sb}_1(z) = 1, \mathsf{Sb}_2(z) = -1, \mathsf{Sb}_3(z) = -1$ (type II) and $\mathrm{vm}(b_1) = 2, \mathrm{vm}(b_2) = \mathrm{vm}(b_3) = 1$.

If we take $c_3 = 2, \epsilon_3 = -1$ and $\mathring{\mathsf{b}}_3(z) = (1+z)(\lambda_1 + \lambda_0 z + \lambda_1 z^2)$ with $\lambda_0 = \frac{\sqrt{7}}{8}$ and $\lambda_1 = \frac{1}{16}$, we obtain the first example in Example 1. See Figure 4.1 for graphs of the associated refinable function $\phi^a$ and framelet functions $\psi^{a,b_1}, \psi^{a,b_2}$, and $\psi^{a,b_3}$.



(a) $\phi^a$ (b) $\psi^{a,b_1}$ (c) $\psi^{a,b_2}$ (d) $\psi^{a,b_3}$ (e) $|\widehat{a}|, |\widehat{b_1}|, |\widehat{b_2}|, |\widehat{b_3}|$

FIGURE 4.1. The tight framelet filter bank $\{a; b_1, b_2, b_3\}$ with symmetry is constructed by Algorithm 2 in Example 6 with $a = a_4^B(\cdot - 2)$. (a), (b), (c), (d) are the graphs of the refinable function $\phi^a$ and the framelet functions $\psi^{a,b_1}$, $\psi^{a,b_2}$, $\psi^{a,b_3}$, respectively. (e) is the magnitudes of $\widehat{a}$ (in solid line), $\widehat{b_1}$ (in dashed line), $\widehat{b_2}$ (in dotted line), and $\widehat{b_3}$ (in dashed-dotted line) on the interval $[-\pi, \pi]$.

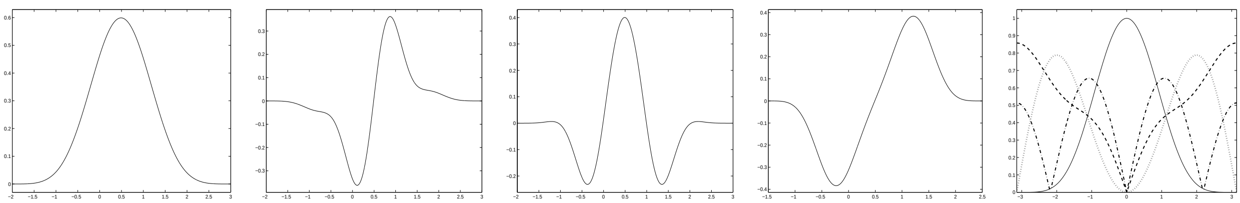
**Example 7.** Let $a = a_5^B(\cdot - 2) = \{\frac{1}{32}, \frac{5}{32}, \underline{\boldsymbol{\frac{5}{16}}}, \frac{5}{16}, \frac{5}{32}, \frac{1}{32}\}_{[-2,3]}$ be the shifted B-spline filter of order 5 and $\Theta = \boldsymbol{\delta}$. Then $\epsilon z^c := \mathsf{S}\mathsf{a}(z) = z$. Set $n_b = 1$, $c_3 = 1$, $\epsilon_3 = -1$, and $\mathring{\mathsf{b}}_3(z) = \lambda_1 + \lambda_0 z + \lambda_1 z^2$ in Algorithm 2. Then $\lambda_0 = \frac{\sqrt{3}}{24}\sqrt{15 + 5\sqrt{3}}$ and $\lambda_1 = \frac{1}{16}\sqrt{15+5\sqrt{3}}$. $\mathsf{q}_\mathsf{p}(z) = \frac{\lambda_1}{32}(z-1)(z^{-1} + 18 - 4\sqrt{3} + z)$ satisfies $\mathsf{q}_\mathsf{p}(z)\mathsf{q}_\mathsf{p}^\star(z) = \det(\mathcal{N}_{a,\Theta;b_3|n_b}(z))$. Then

$$
\begin{aligned}
\mathsf{b}_1(z) &= \frac{\sqrt{9-5\sqrt{3}}}{48}(z-1)(3z^{-2} + 3z^{-1} + 18 + 10\sqrt{3} + 3z + 3z^2),\\
\mathsf{b}_2(z) &= \frac{\sqrt{60-30\sqrt{3}}(3-\sqrt{3})}{192}(1+z^{-1})(1-z)^2(z^{-1} - 6 - 4\sqrt{3} + z),\\
\mathsf{b}_3(z) &= \frac{\sqrt{15+5\sqrt{3}}}{48}(z-1)(3z^{-1} + 2\sqrt{3} + 3z).
\end{aligned}
$$

That is,

$$
\begin{aligned}
b_1 &= \frac{\sqrt{9-5\sqrt{3}}}{48}\{-3, 0, -15 - 10\sqrt{3}, 15 + 10\sqrt{3}, 3\}_{[-2,3]},\\
b_2 &= \frac{\sqrt{60-30\sqrt{3}}(3-\sqrt{3})}{192}\{1, -7 - 4\sqrt{3}, 6 + 4\sqrt{3}, 6 + 4\sqrt{3}, -7 - 4\sqrt{3}, 1\}_{[-2,3]},\\
b_3 &= \frac{\sqrt{15+5\sqrt{3}}}{48}\{-3, 3 - 2\sqrt{3}, -3 + 2\sqrt{3}, 3\}_{[-1,2]}.
\end{aligned}
$$

Then $\mathsf{S}\mathsf{b}_1(z) = -z, \mathsf{S}\mathsf{b}_2(z) = z, \mathsf{S}\mathsf{b}_3(z) = -z$ (type II) and $\mathrm{vm}(b_1) = 1, \mathrm{vm}(b_2) = 2, \mathrm{vm}(b_3) = 1$. See Figure 4.2 for graphs of the associated refinable function $\phi^a$ and framelet functions $\psi^{a,b_1}, \psi^{a,b_2}$, and $\psi^{a,b_3}$.



(a) $\phi^a$ (b) $\psi^{a,b_1}$ (c) $\psi^{a,b_2}$ (d) $\psi^{a,b_3}$ (e) $|\widehat{a}|, |\widehat{b_1}|, |\widehat{b_2}|, |\widehat{b_3}|$

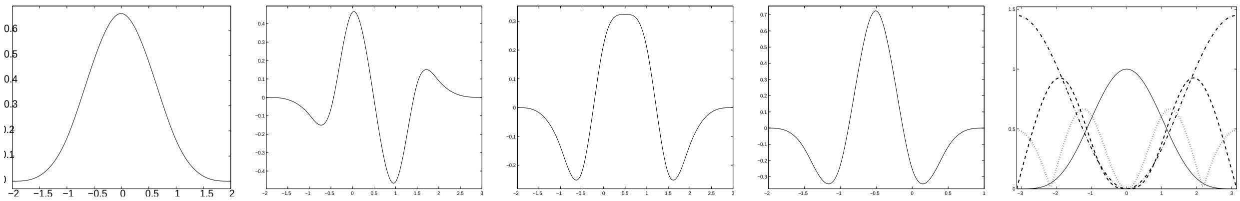
FIGURE 4.2. The tight framelet filter bank $\{a; b_1, b_2, b_3\}$ with symmetry is constructed by Algorithm 2 in Example 7 with $a = a_5^B(\cdot - 2)$. (a), (b), (c), (d) are the graphs of the refinable function $\phi^a$ and the framelet functions $\psi^{a,b_1}$, $\psi^{a,b_2}$, $\psi^{a,b_3}$, respectively. (e) is the magnitudes of $\widehat{a}$ (in solid line), $\widehat{b_1}$ (in dashed line), $\widehat{b_2}$ (in dotted line), and $\widehat{b_3}$ (in dashed-dotted line) on the interval $[-\pi, \pi]$.

**Example 8.** Let $a = a_4^B(\cdot - 2) = \{\frac{1}{16}, \frac{1}{4}, \underline{\boldsymbol{\frac{3}{8}}}, \frac{1}{4}, \frac{1}{16}\}_{[0,4]}$ be the centered B-spline filter of order 4 and $\Theta = \{-\frac{1}{3}, \underline{\boldsymbol{\frac{5}{3}}}, -\frac{1}{3}\}_{[-1,1]}$. Then $\epsilon z^c := \mathsf{S}\mathsf{a}(z) = 1$ and $\mathsf{S}\Theta(z) = 1$. Set $n_b = 2$, $c_3 = 0$, $\epsilon_3 = 1$, and $\mathring{\mathsf{b}}_3(z) = \lambda_0$

in Algorithm 2. Then $\lambda_0 = -\frac{\sqrt{9982}}{276}$ and $\mathsf{q_p}(z) = \frac{\sqrt{253}}{276}(z-1)$ satisfies $\mathsf{q_p}(z)\mathsf{q_p}^\star(z) = \frac{11}{1656} - \frac{11}{3312}(z^{-1}+z) = \det(\mathcal{N}_{a,\Theta;b_3|n_b}(z))$. Then

$$\mathsf{b}_1(z) = \tfrac{\sqrt{11}}{96}(z-z^{-1})(z-1)^2(z^{-1}+6+z) = \tfrac{\sqrt{11}}{96}\{-1,-4,\underline{\mathbf{11}},0,-11,4,1\}_{[-2,4]},$$
$$\mathsf{b}_2(z) = -\tfrac{\sqrt{23}}{2208}(z-1)^2(23z^{-2}+138z^{-1}+174+138z+23z^2) = \tfrac{\sqrt{23}}{2208}\{-23,-92,\underline{\mathbf{79}},72,79,-92,-23\}_{[-2,4]},$$
$$\mathsf{b}_3(z) = -\tfrac{\sqrt{9982}}{276}(1-z^{-1})^2 = \tfrac{\sqrt{9982}}{276}\{-1,\underline{\mathbf{2}},-1\}_{[-2,0]}.$$

Then $\mathsf{Sb}_1(z) = -z^2, \mathsf{Sb}_2(z) = z^2, \mathsf{Sb}_3(z) = z^{-2}$ (type II) and $\mathrm{vm}(b_1) = 3, \mathrm{vm}(b_2) = \mathrm{vm}(b_3) = 2$. Note that $\max(\mathrm{len}(b_1), \mathrm{len}(b_2), \mathrm{len}(b_3)) = 6 = \mathrm{len}(a) + \mathrm{len}(\Theta)$. See Figure 4.3 for graphs of the associated refinable function $\phi^a$ and framelet functions $\psi^{a,b_1}, \psi^{a,b_2}$, and $\psi^{a,b_3}$.



(a) $\phi^a$ (b) $\psi^{a,b_1}$ (c) $\psi^{a,b_2}$ (d) $\psi^{a,b_3}$ (e) $|\widehat{a}|, |\widehat{b_1}|, |\widehat{b_2}|, |\widehat{b_3}|$

FIGURE 4.3. The tight framelet filter bank $\{a; b_1, b_2, b_3\}_\Theta$ with symmetry is constructed by Algorithm 2 in Example 8 with $a = a_4^B(\cdot - 2)$ and $\Theta = \{-\frac{1}{3}, \frac{5}{3}, -\frac{1}{3}\}_{[-1,1]}$. (a), (b), (c), (d) are the graphs of the refinable function $\phi^a$ and the framelet functions $\psi^{a,b_1}$, $\psi^{a,b_2}$, $\psi^{a,b_3}$, respectively. (e) is the magnitudes of $\widehat{a}$ (in solid line), $\widehat{b_1}$ (in dashed line), $\widehat{b_2}$ (in dotted line), and $\widehat{b_3}$ (in dashed-dotted line) on the interval $[-\pi, \pi]$.

We complete the paper by positively answering the second question Q2 in Section 1 for symmetric tight framelet filter banks derived from the interpolatory filter $a_4^I$.

**Example 9.** Let $a = a_4^I = \{-\frac{1}{32}, 0, \frac{9}{32}, \underline{\mathbf{\frac{1}{2}}}, \frac{9}{32}, 0, -\frac{1}{32}\}_{[-3,3]}$ and $\Theta = \boldsymbol{\delta}$. Then $\epsilon z^c := \mathsf{Sa}(z) = 1$. Set $n_b = 2$, $c_3 = 0$, $\epsilon_3 = -1$, and $\mathring{\mathsf{b}}_3(z) = \lambda_0(1-z^{-1})(1+z)^3$ in Algorithm 2. Then $\lambda_0 = -\frac{\sqrt{2}}{16}$ and $\mathsf{q_p}(z) = \frac{\sqrt{6}}{64}(1+z)$ satisfies $\mathsf{q_p}(z)\mathsf{q_p}^\star(z) = \det(\mathcal{N}_{a,\Theta;b_3|n_b}(z))$. Then

$$\mathsf{b}_1(z) = -\tfrac{\sqrt{42}}{28}z^{-1}(1-z)^2 = \tfrac{\sqrt{42}}{28}\{-1,\underline{\mathbf{2}},-1\}_{[-1,1]},$$
$$\mathsf{b}_2(z) = \tfrac{\sqrt{7}}{224}(1-z)^2[7(z^{-3}+z)+14(z^{-2}+1)+6z^{-1}] = \tfrac{\sqrt{7}}{224}\{7,0,-15,\underline{\mathbf{16}},-15,0,7\}_{[-3,3]},$$
$$\mathsf{b}_3(z) = \tfrac{\sqrt{2}}{16}(z^{-1}-z)^3 = \tfrac{\sqrt{2}}{16}\{1,0,-3,\underline{\mathbf{0}},3,0,-1\}_{[-3,3]}.$$

Then $\mathsf{Sb}_1(z) = \mathsf{Sb}_2(z) = 1, \mathsf{Sb}_3(z) = -1$ (type II) and $\mathrm{vm}(b_1) = \mathrm{vm}(b_2) = 2, \mathrm{vm}(b_3) = 3$. See Figure 4.4 for graphs of the associated refinable function $\phi^a$ and framelet functions $\psi^{a,b_1}, \psi^{a,b_2}$, and $\psi^{a,b_3}$. Note that all the high-pass filters $b_1, b_2, b_3$ have the interesting interpolation property: $b_1(2k) = b_2(2k) = b_3(2k) = 0$ for all $k \in \mathbb{Z}\backslash\{0\}$. Consequently, $\psi^{a,b_1}(k) = \psi^{a,b_2}(k) = \psi^{a,b_3}(k) = 0$ for all $k \in \mathbb{Z}\backslash\{0\}$.

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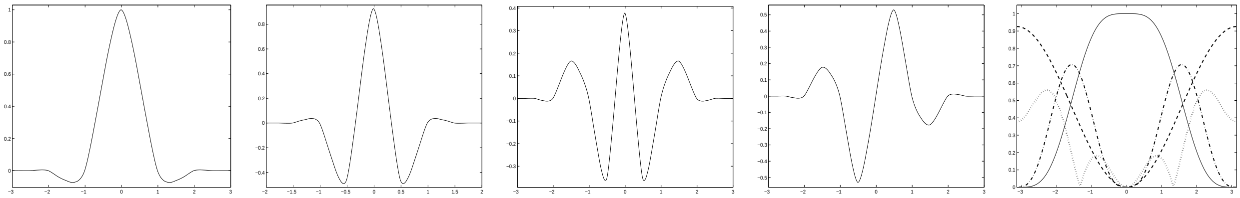

(a) $\phi^a$ (b) $\psi^{a,b_1}$ (c) $\psi^{a,b_2}$ (d) $\psi^{a,b_3}$ (e) $|\widehat{a}|, |\widehat{b_1}|, |\widehat{b_2}|, |\widehat{b_3}|$

Figure 4.4. The tight framelet filter bank $\{a; b_1, b_2, b_3\}$ with symmetry is constructed by Algorithm 2 in Example 9 with $a = a^I_4$. (a), (b), (c), (d) are the graphs of the refinable function $\phi^a$ and the framelet functions $\psi^{a,b_1}$, $\psi^{a,b_2}$, $\psi^{a,b_3}$, respectively. (e) is the magnitudes of $\widehat{a}$ (in solid line), $\widehat{b_1}$ (in dashed line), $\widehat{b_2}$ (in dotted line), and $\widehat{b_3}$ (in dashed-dotted line) on the interval $[-\pi, \pi]$.

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