Exam Solutions

1a)
$$\theta(z) = T\,(P/P_0)^{-x}$$
$$= T\left[\tfrac{1}{P_0} P_1 e^{-(z-z_1)/H}\right]^{-x}$$

$$\theta(z_2) = T\left[(P_1/P_0)\, e^{-(z_2-z_1)/H}\right]^{-x}$$
$$\simeq (273-80)\left[\left(\tfrac{250\text{ mBAR}}{1013\text{ mBAR}}\right) e^{-(12\text{km}-10\text{km})/5.7\text{km}}\right]^{-2/7}$$
$$\simeq 193\left[(0.247)\, e^{-0.351}\right]^{-2/7} \simeq 193\,[0.174]^{-2/7}$$
$$\simeq 193\,(1.65) \simeq \boxed{318\text{ K}}$$

b) Since adiabatic descent, $\theta_2 \simeq 318\text{ K}$ is unchanged
So temperature is
$$T = \theta_2\,(P_1/P_0)^{+x}$$
$$\simeq 318\left[\left(\tfrac{250\text{ mBAR}}{1013\text{ mBAR}}\right)\right]^{+2/7} \simeq 318\,[0.247]^{+2/7}$$
$$\simeq 318\,(0.671) \simeq 213\text{ K}$$
$$\Rightarrow \boxed{T_2 \simeq -60^\circ\text{C}}$$

c) Density is given by $\rho = P/(R_a T)$
Density difference is
$$\Delta\rho = \rho_2 - \rho_1$$
$$= P_1/R_aT_2 - P_1/R_aT_1$$
$$= P_1/R_a\left(\tfrac{1}{T_2} - \tfrac{1}{T_1}\right)$$
$$\simeq \left(0.250\times 10^5\text{ Pa}/287\,\tfrac{\text{J}}{\text{kg K}}\right)\left(\tfrac{1}{213\text{K}} - \tfrac{1}{193\text{K}}\right)$$
$$\simeq (87.1)(-4.87\times 10^{-4})$$
$$\boxed{\Delta\rho \simeq -0.042\ \text{kg/m}^3}$$

Since negative, the air will feel and upward buoyancy force.

2 a) $\rho = \rho_0 [1 - \beta_T (T - T_0) + \beta_S (S - S_0) + \beta_P (P - P_0)]$

$\simeq 998 \, [1 - 2.1 \times 10^{-4} (20 - 20)$
$+ 7.4 \times 10^{-4} (35 - 0)]$ (IGNORING PRESSURE)

$\simeq \boxed{1025 \text{ kg/m}^3}$

b) $\rho_{TOP} = \rho_0 = 998 \text{ kg/m}^3$
$\rho_{BOT} = 1025 \text{ kg/m}^3$ (FROM RESULT IN a))

So $N^2 = -\frac{g}{\rho_0} \frac{d\bar{\rho}}{dz} \simeq -\frac{9.81}{998} \frac{(998 - 1025)}{50} \simeq 5.31 \times 10^{-3} \, s^{-2}$

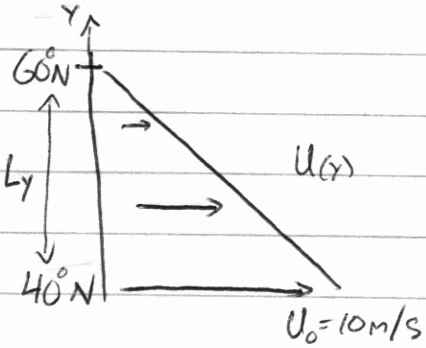
BUOYANCY FREQUENCY IS $\boxed{N \simeq 7.29 \times 10^{-2} \, s^{-1}}$
BUOYANCY PERIOD IS $\boxed{T = \frac{2\pi}{N} \simeq 86 \, s}$

c) $\rho_{POT} \equiv \rho + \rho_{0,} \frac{z}{H_\rho} = \rho = 1025 \text{ kg/m}^3$ AT SURFACE (WHERE $z = 0$)

WHEN BROUGHT TO 4 km DEPTH ρ_{POT} IS THE SAME SINCE DESCENT IS ADIABATIC.

So AT $z = -4$ km, $1025 = \rho_{4km} + 998 \frac{(-4000)}{200000}$
(ALSO OK TO CHOOSE 1025 HERE)

$\Rightarrow \rho_{4km} = 1025 + 998 \left(\frac{1}{50}\right) \simeq \boxed{1045 \text{ kg/m}^3}$

3) 

a) $$W_E = \frac{1}{\rho_o f_o}(\nabla \times \underline{\tau}) \cdot \hat{z}$$

$\tau_{yz} = 0$

$$(\nabla \times \underline{\tau}) \cdot \hat{z} = -\frac{\partial}{\partial y}(\tau_{xz}) = -\frac{\tau_{60N} - \tau_{40N}}{L_y}$$

WHERE $\tau_{60N} = 0$

$\tau_{40N} = C_D \rho_a U_o^2 = 10^{-3} \cdot 1.29 \cdot (10)^2 \simeq 0.129 \underbrace{N/m^2}_{Pa}$

$L_y = (60 - 40) \times 111 \text{ km/DEG} \simeq 2.22 \times 10^6 \text{ m}$

$\Rightarrow (\nabla \times \underline{\tau}) \cdot \hat{z} \simeq -(0 - 0.129)/2.22 \times 10^6 \simeq 5.8 \times 10^{-8}$ Pa/m

$\Rightarrow W_E = \frac{1}{(1000)(10^{-4})}[5.8 \times 10^{-8}] \approx \boxed{5.8 \times 10^{-7} \text{ m/s}}$

NOTE, $W_E > 0 \Rightarrow$ EKMAN SUCTION DRAWS FLUID UPWARD

b) $$V_E = \frac{1}{\rho_o \beta}(\nabla \times \underline{\tau}) \cdot \hat{z}$$

ESTIMATE β AT MIDLATITUDES $\sim \frac{1}{2}(40 + 60) = 50°N$

$\Rightarrow \beta = \frac{1}{R_e}(2\Omega_e \cos\phi_o)$

$\simeq \frac{1}{6.4 \times 10^6}(2 \times 7.3 \times 10^{-5} \cos(50°))$

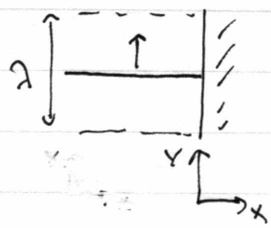
$\simeq 1.5 \times 10^{-11} \ (m \cdot s)^{-1}$

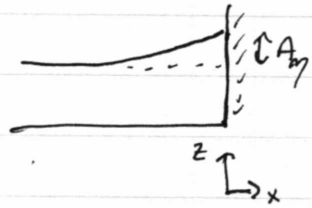
FROM a)

$\Rightarrow V_E = \frac{1}{(1000)(1.5 \times 10^{-11})}[5.8 \times 10^{-8}]$

$\approx \boxed{3.9 \text{ m}^2/\text{s}}$

4)





a) NO MOTION IN DIRECTION TOWARD COAST $\Rightarrow u = 0$

i.e. MAXIMUM ZONAL VELOCITY IS $\boxed{A_u = 0}$

From $\frac{\partial v}{\partial t} + f_0 u = -g\frac{\partial \eta}{\partial y} \Rightarrow -i\omega A_v = -g i l A_\eta$

$\Rightarrow A_v = \frac{gl}{\omega} A_\eta$

FROM DISPERSION RELATION $\omega = cl$, $c = \sqrt{gH}$

$\Rightarrow A_v = \frac{g}{c} A_\eta = \sqrt{\frac{g}{H}} A_\eta$

$\simeq [9.81/4000]^{1/2} (0.10 m)$

$\Rightarrow \boxed{A_v \simeq 4.95 \times 10^{-3} \ m/s}$ IS MAXIMUM MERIDIONAL VELOCITY

b) $\eta = A e^{x/L_D} \cos(ly - \omega t)$ WITH $x \leq 0$

WHERE $L_D = C/f_0$:

$c = \sqrt{gH} \simeq [(9.81)(4000)]^{1/2} \simeq 198 \ m/s$

$f_0 \simeq 2\left(\frac{2\pi}{(60)(60)(24)s}\right) \sin 33 \simeq (1.45 \times 10^{-4} s^{-1})(0.545) \simeq 7.92 \times 10^{-5}$

$\Rightarrow L_D \simeq 198/7.92 \times 10^{-5} = 2.50 \times 10^{6} m$

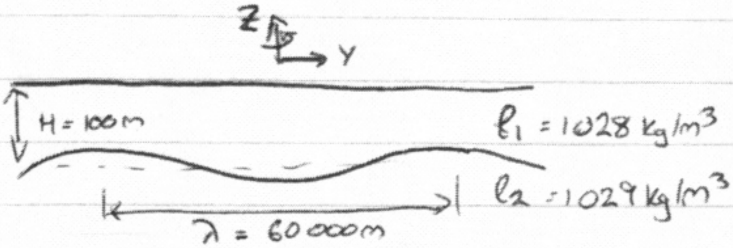
So MAXIMUM DISPLACEMENT AT $x = -1000 km = -10^{6} m$ IS

$A e^{-10^6/2.50 \times 10^6} \simeq (0.10 m) e^{-0.4}$

$\simeq 0.067$

$\Rightarrow$ MAXIMUM DISPLACEMENT IS 6.7 cm

5)



a) $g' \equiv g\dfrac{\rho_2-\rho_1}{\rho_0} \simeq 9.8 \times \dfrac{1029-1028}{1000} \simeq 9.8\times10^{-3}\ m/s^2$

(OR COULD USE ρ_2 OR ρ_1)

$\Rightarrow c = \sqrt{g'H} = \sqrt{(9.8\times10^{-3})(100)} \simeq 0.99\ m/s$

AT $45°N$, $f_0 \approx 10^{-4}\ s^{-1}$

$\Rightarrow L_D \equiv c/f_0 \approx (0.99\ m/s)/(10^{-4}s^{-1}) \simeq 9.9\times10^3 m \boxed{= 9.9 km}$

b) $l = \dfrac{2\pi}{\lambda} = \dfrac{2\pi}{60\,000} \approx 1.05\times10^{-4}\ m^{-1}$

SO FROM DISPERSION RELATION (WITH $k=0$)

$\omega^2 = c^2 l^2 + f_0^2 \simeq (0.99)^2(1.05\times10^{-4})^2 + (10^{-4})^2 \approx 2.08\times10^{-8}$

$\Rightarrow \boxed{\omega \approx 1.4\times10^{-4}\ s^{-1}}$

c) PHASE VELOCITY COMPONENTS: $\quad \underline{c}_p = \dfrac{\omega}{|\underline{K}|}\dfrac{(k,l)}{|\underline{K}|} = \dfrac{\omega}{|\underline{K}|^2}(0,l) \overset{|K|^2=l^2}{=} \left(0, \dfrac{\omega}{l}\right)$

$$\boxed{\begin{aligned} C_{px} &= \left(\frac{\omega}{|K|}\right)\hat{k} = 0 \quad \text{(SINCE NORTHWARD)} \\ C_{py} &= \frac{\omega}{l} = \frac{1.4\times10^{-4}}{1.05\times10^{-4}} \simeq 1.4\ m/s \end{aligned}}$$

GROUP VELOCITY: $\quad \underline{c}_g = \left(\dfrac{\partial\omega}{\partial k}, \dfrac{\partial\omega}{\partial l}\right) = \left(0, \dfrac{\partial}{\partial l}\left(\overbrace{\sqrt{c^2l^2+f_0^2}}^{\omega}\right)\right) = \left(0, \dfrac{1}{2\omega}2c^2l\right)$

$\boxed{C_{gx} = 0}$

$C_{gy} = \dfrac{c^2 l}{\omega} \approx \dfrac{(0.99)^2(1.05\times10^{-4})}{1.4\times10^{-4}} \boxed{\approx 0.71\ m/s}$

d) BY DEFINITION ZONAL MOMENTUM IS $\rho_0 u$. SO ITS MERIDIONAL TRANSPORT IS $\langle \rho_0 u v\rangle = \rho_0\langle uv\rangle$

BUT u & v ARE $90°$ OUT OF PHASE

$\Rightarrow \boxed{\rho_0\langle uv\rangle = 0}$.