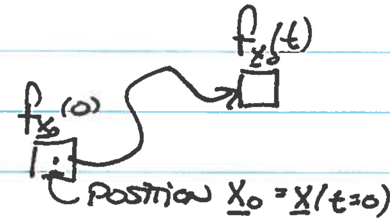
2] EQUATIONS OF MOTION

1) TIME DERIVATIVES FOR FLUIDS

A) LAGRANGIAN FRAME

WE WISH TO DESCRIBE THE PROPERTIES (eg. TEMPERATURE, VELOCITY) OF A FLUID PARCEL AS THEY EVOLVE IN TIME, t.



FOR A PARCEL INITIALLY LOCATED AT $\underline{X}_0 = \underline{X}(t=0)$, DENOTE A PROPERTY BY $f_{\underline{x}_0}(t)$. THEN $f_{\underline{x}_0}$ CHANGES IN TIME ACCORDING TO $\frac{d}{dt} f_{\underline{x}_0}$, THE LAGRANGIAN DERIVATIVE.

IF THE PROPERTY IS CONSERVED (NO FLUX IN OR OUT OF PARCEL, eg OF MASS) THEN $\frac{d}{dt} f_{\underline{x}_0} = 0$ FOR PARCELS STARTING AT ANY $\underline{x}_0$: $f_{\underline{x}_0}(t) = f_{\underline{x}_0}(0) \cdot$ CONSTANT

THIS IS SIMPLY WRITTEN AS $\frac{df}{dt} = 0$, WITH STARTING POSITION UNSPECIFIED

B) EULERIAN FRAME

- PRACTICALLY, THE LAGRANGIAN FRAME IS HARD TO WORK WITH BECAUSE WHAT MAKES f CHANGE IN TIME USUALLY DEPENDS ON PROPERTIES OF THE SURROUNDING FLUID THAT COULD HAVE ORIGINATED FROM FAR DIFFERENT LOCATIONS AT $t=0$.
- THE EULERIAN FRAME IS SET IN A FIXED COORDINATE SYSTEM $\underline{x}$. SO $f_{\underline{x}_0}(t) = f(\underline{x}(t), t)$ WITH $\underline{x}(0) = \underline{x}_0$ AND $\underline{x} = (x, y, z)$
- THEN $\frac{d}{dt} f_{\underline{x}} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt} = \frac{\partial f}{\partial t} + \underline{u} \cdot \nabla f$

  IN WHICH $\underline{u} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$ AND $\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$
- DEFINE $\underline{\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \underline{u} \cdot \nabla}$ TO BE THE "MATERIAL DERIVATIVE"
- IF A PROPERTY IS CONSERVED, THEN $\frac{Df}{Dt} = 0$ IN EULERIAN FRAME

  SO AT A FIXED LOCATION $\frac{\partial f}{\partial t} = -u \frac{\partial f}{\partial x} - v \frac{\partial f}{\partial y} - w \frac{\partial f}{\partial z}$

① B) (CONT'D)

EXAMPLE

SUPPOSE A FLUID'S INITIAL TEMPERATURE T INCREASES WITH HEIGHT z ACCORDING TO $T(z, t=0) = T_0 + \alpha z$ WITH $T_0 = 20°C$ AND $\alpha = 1°C/m$. NEGLECTING DIFFUSION, RADIATION AND THERMODYNAMIC EFFECTS, WE ASSUME TEMPERATURE IS CONSERVED. NOW SUPPOSE THE FLUID RISES AT A STEADY VERTICAL SPEED $w_0 = 1$ cm/s. WHAT IS THE TEMPERATURE AT $z=0$ AFTER A MINUTE HAS PASSED?

SOL$^{\underline{N}}$

BECAUSE T IS CONSERVED $0 = \frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \frac{\partial T}{\partial t} + w\frac{\partial T}{\partial z}$

SO $\frac{\partial T}{\partial t} = -w_0 \frac{\partial T}{\partial z} \overset{T=T_0+\alpha z}{=} -w_0(\alpha)$

BECAUSE α AND w_0 ARE CONSTANTS, WE CAN INTEGRATE IN TIME TO FIND

$= -w_0 \alpha t + \mathcal{C}_z$, WITH $\mathcal{C}_z$ CONSTANT WITH TIME

GIVEN $T(z, t=0) = T_0 + \alpha z$, THE CONSTANT IN t CAN BE FOUND

$$T(z, t=0) = -w_0\alpha(0) + \mathcal{C}_z = T_0 + \alpha z$$

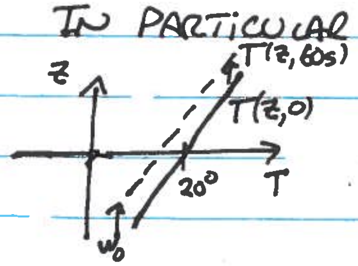
$$\Rightarrow \mathcal{C}_z = T_0 + \alpha z$$

SO $T(z,t) = -w_0 \alpha t + (T_0 + \alpha z)$

SO THE TEMPERATURE AT $z=0$ EVOLVES IN TIME AS $T(0,t) = T_0 - w_0\alpha t$

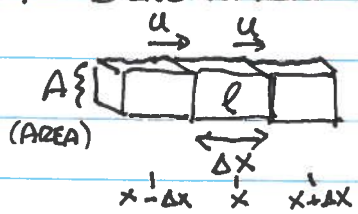
IN PARTICULAR $T(0, t=60s) = 20° - (0.01 m/s)(1°/m)(60s)$

$$= 20° - 0.60° = 19.40°$$



2 CONSERVATION OF MASS

A) DERIVATION IN 1D



DENSITY

MASS IN MIDDLE BOX : $m = \rho A \Delta x$

MASS FLUX : $\mathcal{F} = \dfrac{\text{MASS}}{\text{AREA} \cdot \text{TIME}} = \dfrac{\text{MASS}}{\text{AREA} \cdot \Delta L} \cdot \dfrac{\Delta L}{\Delta T} = \rho u$

MASS CONSERVATION : $\dfrac{\partial m}{\partial t} = \Delta(\text{AREA} \times \text{MASS FLUX})$

$$\Rightarrow \frac{\partial}{\partial t}(\rho A \Delta x) = (A\rho u)\big|_{x-\frac{\Delta x}{2}} - (A\rho u)\big|_{x+\frac{\Delta x}{2}}$$

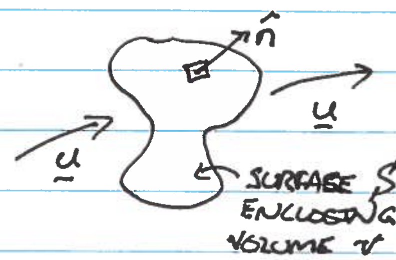
$$\Rightarrow \frac{\partial \rho}{\partial t} = \frac{1}{\Delta x}\left[(\rho u)\big|_{x-\frac{\Delta x}{2}} - (\rho u)\big|_{x+\frac{\Delta x}{2}}\right] = -\frac{1}{\Delta x}\left[(\rho u)\big|_{x+\frac{\Delta x}{2}} - (\rho u)\big|_{x-\frac{\Delta x}{2}}\right]$$

$$\xrightarrow{\Delta x \to 0} -\frac{\partial}{\partial x}(\rho u)$$

$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0} \quad (1) \quad \Rightarrow \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$$

B) DERIVATION IN 3D

$$\frac{\partial}{\partial t}\left(\iiint_V \rho \, dV\right) = -\iint_S \overset{\rho \underline{u}}{\underset{=}{\underline{\mathcal{F}}}} \cdot \overset{\text{OUTWARD UNIT NORMAL}}{\hat{n}} \, dS$$



USE DIVERGENCE THEOREM : $\iint_S \underline{\mathcal{F}} \cdot \hat{n} \, dS = \iiint_V \nabla \cdot \underline{\mathcal{F}} \, dV$

$$\Rightarrow \iiint_V \frac{\partial \rho}{\partial t} dV = -\iiint_V \nabla \cdot \underline{\mathcal{F}} \, dV \Rightarrow \iiint_V \frac{\partial \rho}{\partial t} + \nabla \cdot \underline{\mathcal{F}} \, dV = 0$$

THIS HOLDS FOR ANY VOLUME V

ASSUMING INTEGRAND IS CONTINUOUS, MUST HAVE $\dfrac{\partial \rho}{\partial t} + \nabla \cdot \underline{\mathcal{F}} = 0$

BUT $\underline{\mathcal{F}} \equiv \rho \underline{u}$

$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0} \quad (2)$$

(COMPARE WITH (1) ABOVE)

ALTERNATELY, USE $\nabla \cdot (\rho \underline{u}) = \dfrac{\partial}{\partial x}(\rho u) + \dfrac{\partial}{\partial y}(\rho v) + \dfrac{\partial}{\partial z}(\rho w)$

$$= u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\frac{\partial u}{\partial x} + \rho\frac{\partial v}{\partial y} + \rho\frac{\partial w}{\partial z}$$

$$= \underline{u} \cdot \nabla \rho + \rho \nabla \cdot \underline{u}$$

SO (2) $\Rightarrow \dfrac{\partial \rho}{\partial t} + \underline{u} \cdot \nabla \rho + \rho \nabla \cdot \underline{u} = \boxed{\dfrac{D}{Dt}(\rho) + \rho \nabla \cdot \underline{u} = 0}$ (3) $\quad \left(\dfrac{D}{Dt} = \dfrac{\partial}{\partial t} + \underline{u} \cdot \nabla\right)$ MATERIAL DERIVATIVE

CONTINUITY EQUATION

3 Conservation of Momentum

Generally, this is represented by Newtons Law

$$\underbrace{\rho}_{\frac{\text{mass}}{\text{volume}}} \underbrace{\frac{D\underline{u}}{Dt}}_{\text{acceleration}} = \sum_i \underbrace{\underline{F}_i}_{\text{forces/volume}}$$

For atmosphere - ocean flows, 4 forces are most relevant:

A) Pressure Gradient Forces

Fluid tends to move from high to low pressure $\quad \frac{\partial P}{\partial x} < 0 \quad H \xrightarrow{x} L$

Pressure is $\frac{\text{Force}}{\text{Area}}$

So pressure gradient force is $\underline{F}_{\text{pressure}} = -\nabla P_T \quad \left[\frac{1}{\text{length}} \cdot \frac{\text{force}}{\text{area}} = \frac{\text{force}}{\text{volume}}\right]$

(Here P_T is "total" pressure = background + fluctuations)

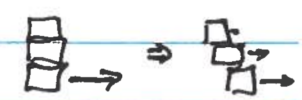
B) Buoyancy Forces

Gravity tends to pull fluid downward

$\Rightarrow \quad \underline{F}_{\text{buoyancy}} = -\rho_T g \hat{z} \qquad$ with $g \approx 9.81\ m/s^2$ near Earth

(Here ρ_T is "total" density = background + fluctuations)

C) Viscosity



Viscosity diffuses momentum from high to low

$$\Rightarrow \quad \underline{F}_{\text{viscosity}} = \mu \nabla^2 \underline{u} = \mu \left(\frac{\partial^2 \underline{u}}{\partial x^2} + \frac{\partial^2 \underline{u}}{\partial y^2} + \frac{\partial^2 \underline{u}}{\partial z^2} \right)$$

In which μ is the "molecular viscosity"

For air: $\mu \simeq 1.8 \times 10^{-5}$ kg/(m·s)

For water: $\mu \simeq 1.1 \times 10^{-3}$ kg/(m·s)

(For honey at room temperature $\mu \simeq 1$ kg/(m·s))

In most atmospheric/oceanic flows we will see that viscosity plays a negligible role. But it can be non-negligible in laboratory experiments.

③ (CONT'D)

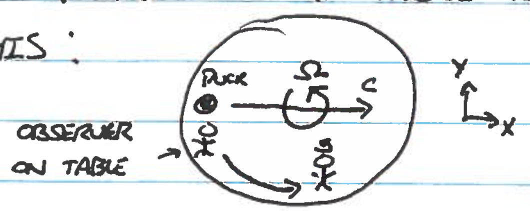
D) CORIOLIS FORCES

YOU FEEL AN EFFECTIVE FORCE IF YOU ARE IN AN ACCELERATING VEHICLE. (e.g. PUSHED BACKWARDS STANDING ON LRT AS IT LEAVES THE STATION)

ALTHOUGH THE SPEED OF THE EARTH'S ROTATION DOESN'T CHANGE, ITS ORIENTATION DOES. THIS GIVES RISE TO THE CORIOLIS FORCE

IMAGINE A PUCK SLIDING ACROSS A FRICTIONLESS TABLE THAT ROTATES COUNTERCLOCKWISE AT (ANGULAR) FREQUENCY Ω. ONE OBSERVER SITS ON THE TABLE, ANOTHER OFF IT.

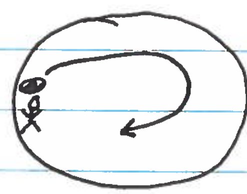
THE PERSON OFF THE TABLE (IN THE "ABSOLUTE FRAME") SEES THIS:



THE PATH OF THE PUCK AS SEEN IN THE ABSOLUTE FRAME IS

$$(x_a(t), y_a(t)) = (-L+ct, 0)$$

THE PERSON ON THE TABLE (IN THE "ROTATING FRAME") SEES THIS:



THE PATH IS CURVED NOT ONLY BECAUSE THE PERSON IS ACTUALLY GOING AROUND THE TABLE, BUT ALSO BECAUSE THEIR ORIENTATION CHANGES.

WITH SOME GEOMETRY, THE PATH OF THE PUCK IN THE ROTATING FRAME IS $(x_r(t), y_r(t)) = ((-L+ct)\cos\Omega t, -(-L+ct)\sin\Omega t)$

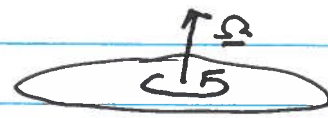
$\Rightarrow$ VELOCITY: $(u,v) = \left(\frac{dx_r}{dt}, \frac{dy_r}{dt}\right) = (c\cos\Omega t, -c\sin\Omega t) - \Omega(-y_r, x_r)$

$\Rightarrow$ ACCELERATION: $\underline{a} = \left(\frac{du}{dt}, \frac{dv}{dt}\right) = \underbrace{-2\Omega(-v,u)}_{\text{CORIOLIS}} + \underbrace{\Omega^2(x_r, y_r)}_{\text{CENTRIFUGAL}}$

IN PRACTISE, FOR EARTH THE CENTRIFUGAL ACCELERATION IS INCORPORATED INTO GRAVITY (THE EARTH BULGES AT EQUATOR DUE TO THIS ACCELERATION. BUT THE OUTWARD ACCELERATION CANCELS THE EXTRA GRAVITY DUE TO THE BULGE.)

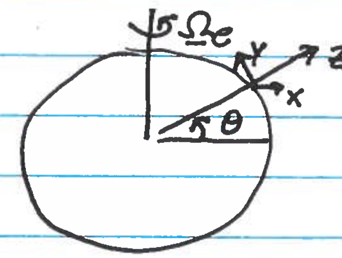
(3) D) (CONT'D)

GENERALLY, WRITE THE CORIOLIS ACCELERATION AS $-2\underline{\Omega}\times\underline{u}$ WITH $|\underline{\Omega}| = \Omega$ AND DIRECTION PERPENDICULAR TO THE PLANE OF ROTATION, UPWARD IF ROTATION IS COUNTERCLOCKWISE AS SEEN FROM ABOVE



SO, FOR COUNTERCLOCKWISE ROTATION IN X-Y PLANE, $\underline{\Omega} = \Omega\hat{z}$ AND $\underline{u} = (u, v, 0) \Rightarrow$

$$-2\underline{\Omega}\times\underline{u} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & -2\Omega \\ u & v & 0 \end{vmatrix} = (2\Omega v, -2\Omega u, 0)$$



- ON THE ROTATING EARTH THE SENSE OF ROTATION VARIES WITH LATITUDE.
- PROJECT $\underline{\Omega}_e$ ONTO (x, y, z) COORDS WITH x - EASTWARD, y - NORTHWARD, z - VERTICAL FROM SURFACE AND ORIGIN AT SOME LATITUDE θ FROM EQUATOR ($\theta > 0$ IN N.H.).

$$\Rightarrow \underline{\Omega}_e = (0, \Omega\cos\theta, \Omega\sin\theta), \quad \Omega \equiv |\underline{\Omega}_e| = \frac{2\pi}{1\text{ DAY}}$$

- SO, IN THIS CO-ORDINATE SYSTEM, THE CORIOLIS ACCELERATION IS

$$-2\underline{\Omega}_e\times\underline{u} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & -2\Omega\cos\theta & -2\Omega\sin\theta \\ u & v & w \end{vmatrix}$$

$$= (-2\Omega w\cos\theta + 2\Omega v\sin\theta, -2\Omega u\sin\theta, 2\Omega u\cos\theta)$$

- IN THE "TRADITIONAL APPROXIMATION", ASSUME $\|w\| \ll \|u\|, \|v\|$ AND NEGLECT VERTICAL ACCELERATION BY CORIOLIS FORCE.
- DEFINE $f \equiv 2\Omega\sin\theta$ THE "CORIOLIS PARAMETER"

SO THE CORIOLIS ACCELERATION IS $\approx (fv, -fu, 0)$

- THE CORIOLIS FORCE IS $\underline{F}_{CORIOLIS} = -\rho_T\overbrace{(-fv, fu, 0)}^{f\times\underline{u}_H \leftarrow \text{HORIZONTAL}}$.

(TYPICALLY, THIS IS WRITTEN ON THE LEFT-HAND SIDE OF THE MOMENTUM EQUATIONS

③ D) (CONT'D)

AT THE NORTH POLE $f = 2\left(\frac{2\pi}{1\text{ DAY}}\right)\sin 90^\circ = \frac{4\pi}{60 \times 60 \times 24\,s} \simeq 1.45 \times 10^{-4}\,s^{-1}$

AROUND EDMONTON $f \simeq 2\left(\frac{2\pi}{1\text{ DAY}}\right)\sin 54^\circ \simeq 10^{-4}\,s^{-1}$

THE "CHARACTERISTIC VALUE" OF f IS TAKEN TO BE $10^{-4}s^{-1}$

CORIOLIS FORCES DEFLECT FLOW RIGHTWARD (LEFTWARD) IN N.H. (S.H.)
$f>0$ ($f<0$)

EXAMPLES

① SUPPOSE YOU THROW A BOWLING BALL AT 3m/s DOWN A 30m LONG ALLEY. IF THERE WAS NO FRICTION, HOW MUCH WOULD IT DEFLECT BY CORIOLIS FORCES?

SOLN

THE CORIOLIS ACCELERATION IS $a = |fu| \approx 10^{-4}s^{-1} \times 3m/s = 3 \times 10^{-4}\frac{m}{s^2}$

THE TIME TO TRAVEL 30m IS $30m/(3m/s) = 10s$

SO (RIGHTWARD) DEFLECTION AT END OF ALLEY WOULD BE

$$\frac{1}{2}at^2 = \frac{1}{2}(3 \times 10^{-4}\,m/s^2)(10s)^2 = 1.5 \times 10^{-2}m = 1.5cm.$$

② SUPPOSE YOU DRIVE AT 110 km/HR FROM EDMONTON TO CALGARY (ABOUT 250 km AWAY). IF IT WEREN'T FOR FRICTION, HOW FAR TO THE RIGHT OF CALGARY WOULD YOU END UP?

SOLN

$$a \approx 10^{-4}s^{-1} \times 110\,km/HR \times \frac{1000\,m/km}{60 \times 60\,s/HR} \approx 0.0031\,m/s^2$$

TRAVEL TIME IS $\frac{250km}{110\,km/HR} \times 60 \times 60\,s/HR \simeq 8.2 \times 10^3 s$

SO DEFLECTION IS $\frac{1}{2}(0.0031)(8.2 \times 10^3)^2 \simeq 1.0 \times 10^5 m$

$\simeq 100\,km$

3 (CONT'D)

E) BACKGROUND HYDROSTATIC BALANCE

COMBINING THE FOUR FORCES TOGETHER IN THE MOMENTUM CONSERVATION EQUATION GIVES

$$\rho_T\left(\frac{D\underline{u}}{Dt} + \underline{f}\times\underline{u}_H\right) = -\nabla P_T - \rho_T g\,\hat{z} + \mu\nabla^2\underline{u} \qquad (1)$$

- NOW SUPPOSE THE FLUID IS STATIONARY ($\underline{u}=\underline{0}$, $\underline{u}_H=\underline{0}$), AND DENOTE THE PRESSURE AND DENSITY OF THIS STATIONARY FLUID BY $\bar{p}$ AND $\bar{\rho}$, RESPECTIVELY.
- FROM (1) WE HAVE:

IN x: $0 = -\frac{\partial\bar{p}}{\partial x}$ (2a)

IN y: $0 = -\frac{\partial\bar{p}}{\partial y}$ (2b)

IN z: $0 = -\frac{\partial\bar{p}}{\partial z} - \bar{\rho}g$ (2c)

(2a) AND (2b) SHOW THAT $\bar{p} \equiv \bar{p}(z)$ ... IT ONLY CHANGES VERTICALLY

- SO (2c) $\Rightarrow$ $\boxed{\frac{d\bar{p}}{dz} = -\bar{\rho}g}$ "BACKGROUND HYDROSTATIC BALANCE"

- INTEGRATING BOTH SIDES $\Rightarrow$ $\bar{p}(z) = P_0 - g\int_0^z \bar{\rho}(\tilde{z})\,d\tilde{z}$, $P_0 = P(0)$

THAT IS, BACKGROUND PRESSURE RESULTS FROM THE WEIGHT/AREA OF ALL THE FLUID ABOVE.

EXAMPLES

(1) HOW MUCH GREATER THAN ATMOSPHERIC PRESSURE IS THE PRESSURE AT THE BOTTOM OF A CUP FILLED $h = 10\,cm$ DEEP WITH WATER OF DENSITY $\rho_0 = 1000\,kg/m^3$?

SOL^N/

$$\bar{p}(-h) = P_0 - g\int_0^{-h}\rho_0\,dz = P_0 + \rho_0 g h$$

$$\Rightarrow \Delta P \equiv \bar{p}(-h) - P_0 = \rho_0 g h = \left(1000\,\tfrac{kg}{m^3}\right)\left(9.81\,\tfrac{m}{s^2}\right)(0.10\,m) = 981\ Pa.$$

(COMPARE WITH $1000\,mBAR = 10^5\,Pa$ FOR TYPICAL VALUE OF P_0)

③ (cont'd)

F) Summary

Given background density $\bar{\rho}(z)$, can find background pressure $\bar{P}$ from $\frac{d\bar{P}}{dz} = -\bar{\rho}g$.

Define fluctuation pressure and density to be difference of total and background. I.e. the change in pressure and density due to motion within the fluid!

$$p(\underline{x},t) = P_T(\underline{x},t) - \bar{P}(z); \quad \rho(\underline{x},t) = \rho_T(\underline{x},t) - \bar{\rho}(z)$$

So in ① on p.14
$$-\nabla P_T - \rho_T g\hat{z} = \left(-\frac{d\bar{P}}{dz}\hat{z} - \nabla p\right) - (\bar{\rho}+\rho)g\hat{z}$$
$$= -\nabla p - \rho g\hat{z}$$

Also in ①, it is typical to approximate $\rho_T \approx \bar{\rho}$ on the left-hand side since fluctuations in density are typically much smaller than the density itself ($\|\rho\| \ll \|\bar{\rho}\|$)

Hence the law of conservation of momentum is

$$\bar{\rho}\left(\frac{D\underline{u}}{Dt} + \underline{f} \times \underline{u}_H\right) = -\nabla p - \rho g\hat{z} + \mu\nabla^2\underline{u}$$

Explicitly in (x,y,z) co-ords:

$$\boxed{\begin{aligned} \bar{\rho}\left(\frac{Du}{Dt} - fv\right) &= -\frac{\partial p}{\partial x} + \mu\nabla^2 u \\ \bar{\rho}\left(\frac{Dv}{Dt} + fu\right) &= -\frac{\partial p}{\partial y} + \mu\nabla^2 v \\ \bar{\rho}\,\frac{Dw}{Dt} &= -\frac{\partial p}{\partial z} - \rho g + \mu\nabla^2 w \end{aligned}}$$

4 CONSERVATION OF INTERNAL ENERGY

GENERALLY, THIS REFERS TO NON-MECHANICAL ENERGY (e.g. HEAT, BUT ALSO SALINITY FOR THE OCEAN.) ITS DERIVATION COMES FROM THE EQUATION OF STATE AND THERMODYNAMICS.

A) THE ATMOSPHERE

THE EQUATION OF STATE FOR (DRY) AIR IS

$$\boxed{p = \rho R T} \qquad (1)$$

IN WHICH $R = 287 \ J/(kg \cdot K)$ IS GAS CONSTANT FOR AIR

THE FIRST LAW OF THERMODYNAMICS IS

$$dI = T d\eta - p \, d\alpha \qquad (2)$$

SPECIFIC HEAT AT FIXED VOLUME $c_v = 718 \frac{J}{kg \cdot K}$

IN WHICH I IS INTERNAL ENERGY/MASS: $dI = c_v dT$

η IS SPECIFIC ENTROPY : $T d\eta = \delta Q$ IS HEAT INPUT

α IS SPECIFIC VOLUME : $\alpha = \frac{\text{VOLUME}}{\text{MASS}} = \frac{1}{\rho}$

IF NO HEAT ENTERS OR LEAVES SYSTEM $\Rightarrow \delta Q = 0 \Rightarrow d\eta = 0$

So (2) $\Rightarrow$ $c_v dT = -p \, d\left(\frac{1}{\rho}\right) \overset{(1)}{=} -p \, d\left(\frac{RT}{p}\right) = -R dT + \frac{RT}{p} dp$

$$\Rightarrow \left(\frac{c_v}{R} + 1\right) \frac{1}{T} dT = \frac{1}{p} dp \qquad (3)$$

DEFINE $\varkappa = \left(\frac{c_v}{R} + 1\right)^{-1} = \left(\frac{718}{287} + 1\right)^{-1} = \frac{2}{7}$

INTEGRATING BOTH SIDES OF (3) GIVES

$$\ln T = \varkappa \ln P + \underbrace{C}_{\text{CONSTANT}}$$

$$\Rightarrow T = \theta \left(P/P_R\right)^{\varkappa}$$

IN WHICH CONSTANT IS SET SO THAT $T = \theta$ (REFERENCE TEMPERATURE) WHEN $P = P_R$ (REFERENCE PRESSURE)

REARRANGING GIVES DEFINITION OF "POTENTIAL TEMPERATURE":

$$\boxed{\theta \equiv T\left(\frac{P}{P_R}\right)^{-\varkappa}}$$

4) A) (cont'd)

$\theta = T\,(P/P_R)^{-\kappa}$ is the temperature air at temperature T and pressure P would have if brought to pressure P_R with no heat loss. T <u>must</u> be in Kelvin.

Example

Suppose air at tropopause (250 mBar) has temperature $-73°C$ (200K). What is its temperature if brought adiabatically to the ground where $P = P_R = 1000$ mBar?

Sol^n

Potential temperature of air at tropopause is

$$\theta = (200K)\left(\frac{250}{1000}\right)^{-2/7} \simeq 297\text{ K} \approx 24°C$$

Although T & P change during descent, θ remains constant.

At the ground $\quad T = \theta\left(\frac{P}{P_R}\right)^{+\kappa} = \theta\left(\frac{1000}{1000}\right)^{2/7} = \theta$

So temperature is $297\text{K} \approx 24°C$ //

By construction θ does not change following the motion of an air parcel. So the equation for conservation of internal energy is

$$\boxed{\frac{D\theta_T}{Dt} = 0}$$ where θ_T is <u>total</u> potential temp.

Sometimes include effect of heat diffusion: $\frac{D\theta}{Dt} = \kappa_T \nabla^2\theta$

Alternately, one could determine how density changes with . From $T \propto P^{\kappa}$ and $T = \frac{P}{\rho R}$ find

$$\rho \propto P^{1-\kappa} = P^{1/\gamma} \quad \text{with } \gamma \equiv \frac{1}{1-\kappa} = 5/7$$

$$\Rightarrow \rho = \rho_{POT}\,(P/P_R)^{1/\gamma}$$

$$\Rightarrow \boxed{\rho_{POT} = \rho\,(P/P_R)^{-1/\gamma}}$$ is "Potential Density"

Note: $\rho_{POT,T} = \left(\frac{P_R}{R}\right)\frac{1}{\theta_T}$. So usually just use θ_T for atmosphere

Also $\bar{\rho} + \rho = \left(\frac{P_R}{R}\right)\frac{1}{\bar{\theta}+\theta} \approx \left(\frac{P_R}{R}\right)\frac{1}{\bar{\theta}}\left(1 - \frac{\theta}{\bar{\theta}}\right) \Rightarrow \bar{\rho} = \frac{P_R}{R}\frac{1}{\bar{\theta}}$, $\boxed{\rho \simeq -\frac{\bar{\rho}}{\bar{\theta}}\theta}$

4 (CONT'D)

B) THE OCEAN

THE EQUATION OF STATE FOR SEA WATER IS SIGNIFICANTLY MORE COMPLICATED THAN FOR AIR. (SEE SUPPLEMENTAL MATERIAL ON WEB). FOR ONE THING, IT DEPENDS ON SALINITY AS WELL AS TEMPERATURE AND PRESSURE. IN MANY CIRCUMSTANCES, A LINEAR APPROXIMATION CAN PROVE USEFUL:

$$\rho \approx \rho_0 \left[1 - \beta_T (T - T_0) + \beta_S (S - S_0) + \beta_P (P - P_0) \right],$$

IN WHICH ρ_0, T_0, S_0, P_0 ARE REFERENCE VALUES AND $\beta_T, \beta_S, \beta_P$ ARE CONSTANT COEFFICIENTS.

PARTS PER THOUSAND = $\frac{\text{g SALT}}{\text{kg SOLUTION}}$

E.g. FOR $T_0 = 20°C$ ($\approx 293K$), $S_0 = 0$ ppt, $P_0 = 100$ kPa

HAVE $\rho_0 = 998.23$ kg/m^3

AND $\beta_T \approx 2\ (\pm 1.5) \times 10^{-4}\ K^{-1}$ THERMAL EXPANSION COEFFICIENT

$\beta_S \approx 7.6\ (\pm 0.2) \times 10^{-4}\ \text{ppt}^{-1}$ SALINE CONTRACTION COEFFICIENT

$\beta_P \approx 4.1\ (\pm 0.5) \times 10^{-10}\ \text{Pa}^{-1}$ COMPRESSIBILITY COEFFICIENT

EXAMPLE:

FOR $T = 25°C$, $S = 0$ ppt AND $P = 100$ kPa

$$\rho \approx 998.23 \left[1 - 2 \times 10^{-4} (25 - 20) + 0 + 0 \right]$$

$$\simeq 998.23 - 0.998 \simeq 997.23 \text{ kg/m}^3$$

LIKE AIR, AS WATER IS COMPRESSED UNDER PRESSURE ITS DENSITY INCREASES, BUT IT ALSO HEATS UP. THE EFFECTS ARE DESCRIBED BY INDEPENDENT EQUATIONS FOR THE POTENTIAL DENSITY AND POTENTIAL TEMPERATURE.

4 B) (CONT'D)

MAKING USE OF THE FACT THAT THE SPEED OF SOUND IN THE OCEAN IS NEARLY CONSTANT ($C_S \approx 1500\,m/s$) AND USING HYDROSTATIC BALANCE ($dp \approx -\rho_0 g\, dz$) ASSUMING $\rho \approx \rho_0$ VARIES LITTLE, FIND

$$\frac{D\rho}{Dt} - \frac{1}{C_S^2}\frac{DP}{Dt} \simeq \frac{D\rho}{Dt} + \frac{\rho_0 g}{C_S^2}\frac{Dz}{Dt} = \frac{D}{Dt}\left(\rho + \rho_0 \frac{g}{C_S^2} z\right) = 0 \qquad (1)$$

SO DEFINE POTENTIAL DENSITY FOR OCEAN: $\boxed{\rho_{POT} \equiv \rho + \rho_0 \frac{z}{H_\rho}}$

IN WHICH $\boxed{H_\rho \equiv C_S^2/g} \approx (1500\,m/s)^2/(9.81\,m/s) \approx 200\,km$

H_ρ IS THE "DENSITY SCALE HEIGHT". IT MEASURES THE DEPTH YOU WOULD HAVE TO DESCEND TO HAVE THE DENSITY INCREASE BY A SIGNIFICANT FRACTION OF ITSELF DUE TO PRESSURE.

(1) IMPLIES ρ_{POT} IS CONSERVED $\boxed{\frac{D\rho_{POT}}{Dt} = 0}$, ASSUMING NO HEAT OR SALINITY INPUT

SUPPOSING THE SPECIFIC HEAT AND THERMAL EXPANSION COEFFICIENT VARY LITTLE, GET A SIMILAR EXPRESSION FOR THE POTENTIAL TEMPERATURE:

$$\boxed{\theta \equiv T + T_0 \frac{z}{H_T}} \quad \text{SO THAT} \quad \boxed{\frac{D\theta}{Dt} = 0}$$

IN WHICH $H_T = \frac{C_p}{\beta_T g} \approx \frac{4\times 10^3\ J/(kg\,K)}{(2\times10^{-4}K^{-1})(9.81\ m/s^2)} \approx 2000\,km$ IS TEMPERATURE SCALE HEIGHT

EXAMPLE

HOW MUCH DOES THE DENSITY AND TEMPERATURE OF SURFACE WATER INCREASE IF TAKEN FROM SURFACE TO 5km DEPTH?

SOLN/ $\rho_{POT} = \rho = \rho_0$ AT SURFACE $\Rightarrow \rho = \rho_{POT} - \rho_0 \frac{z}{H_\rho} = \rho_0\left(1 - \frac{(-5km)}{200\,km}\right) = \rho_0(1.025)$

$\theta = T = T_0$ AT SURFACE $\Rightarrow T = \theta - T_0 \frac{z}{H_T} = T_0\left(1 - \frac{(-5km)}{2000\,km}\right) = T_0(1.0025)$

SO, IF $\rho_0 = 1030\,kg/m^3 \Rightarrow \rho = 1056\,kg/m^3$. IF $T_0 = 300K \Rightarrow T \simeq 300.8\,K$ //

4 (cont'd)

c) Consequences of Thermodynamics

- **Density Scale Height of Atmosphere.**

Suppose the temperature is approximately constant $(T = T_0)$.
Then background hydrostatic balance and equation of state $\Rightarrow$

$$\frac{d\bar{p}}{dz} = -g\bar{\rho} \simeq -g(\bar{p}/(RT_0)) = -\frac{1}{H_\rho}\bar{p} \qquad \text{with } H_\rho = \frac{RT_0}{g} \text{ constant}$$

Solution is $\bar{p} = P_0 e^{-z/H_\rho}$ taking $\bar{p}(z=0) = P_0$

Hence $\boxed{\bar{\rho}(z) = \rho_0 e^{-z/H_\rho}}$ with $\rho_0 = P_0/(RT_0)$.

For $T_0 \approx 250K$, $H_\rho \approx \frac{(287 \, J/kgK)(250K)}{9.81 \, m/s^2} \approx 7.3 \, km$

$\Rightarrow$ atmosphere density decreases by factor $e \simeq 2.71$ going up $7.3 \, km$

- **Adiabatic Lapse Rate**

This is the rate of decrease in temperature with height in a "well-mixed" atmosphere, for which $\bar{\theta}(z)$ is constant with z.

$$0 = \frac{d\bar{\theta}}{dz} = \frac{d}{dz}\left[\bar{T}(z)\left(\frac{\bar{p}(z)}{p_R}\right)^{-\kappa}\right] = \frac{d\bar{T}}{dz}\left[\left(\frac{\bar{p}(z)}{p_r}\right)^{-\kappa}\right] - \kappa\bar{T}\left(\frac{\bar{p}}{p_R}\right)^{-\kappa-1}\frac{1}{p_R}\frac{d\bar{p}}{dz}$$

$$= \left(\frac{\bar{p}}{p_R}\right)^{-\kappa}\left[\frac{d\bar{T}}{dz} - \kappa\bar{T}\frac{1}{\bar{p}}\frac{d\bar{p}}{dz}\right]$$

hydrostatic balance

$$= \left(\frac{\bar{p}}{p_R}\right)^{-\kappa}\left[\frac{d\bar{T}}{dz} - \kappa\bar{T}\frac{1}{\bar{p}}(-\bar{\rho}g)\right]$$

equation of state

$$= \left(\frac{\bar{p}}{p_R}\right)^{-\kappa}\left[\frac{d\bar{T}}{dz} + \frac{\kappa g}{R}\right]$$

But $\frac{\kappa}{R} = \frac{1}{R}\left(\frac{c_v}{R}+1\right)^{-1} = \frac{1}{c_v+R} = \frac{1}{c_p}$ (see p. 16)

So $\boxed{\frac{d\bar{T}}{dz} = -\frac{g}{c_p} \equiv -\Gamma_d}$ where Γ_d is (dry) <u>adiabatic lapse rate</u>

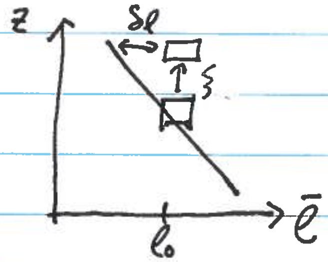
In atmosphere $\Gamma_d \approx \frac{g}{c_p} = \frac{9.81 \, m/s^2}{1004 \, J/(kgK)} \simeq 0.01 \, K/m = 10 \, K/km$

So temperature of atmosphere decreases by $10°C$ for every km going up in "well-mixed" troposphere

4 c) (cont'd)

- Buoyancy Frequency

A fluid is stratified if its (effective) density decreases with height (e.g. $\bar{\rho}_{pot}$ decreases with z in ocean, $\bar{\theta}$ increases in atmos.) Consequently, fluid displaced upward (downward) finds itself heavier (lighter) than surrounding and wants to return to its original height. The resulting motion is oscillatory



Newton's law for displaced parcel:

$$\rho_0 \frac{d^2}{dt^2}\xi = -g\,\delta\rho = -g\left(-\frac{d\bar{\rho}}{dz}\xi\right)$$

$$\Rightarrow \frac{d^2\xi}{dt^2} + \left(-\frac{g}{\rho_0}\frac{d\bar{\rho}}{dz}\right)\xi = 0 \qquad ①$$

Define $\boxed{N \equiv \left(-\frac{g}{\rho_0}\frac{d\bar{\rho}}{dz}\right)^{1/2}}$, which is real and constant if $\bar{\rho}$ decreases linearly

Solution of ①: $\xi'' + N^2\xi = 0$ is $\xi(t) = A\cos Nt + B\sin Nt$

So N is the (angular) frequency of oscillation.

N is called the "Buoyancy Frequency". The buoyancy period is $T_B = \frac{2\pi}{N}$

Example:

The density in the ocean decreases from $1030\ kg/m^3$ to $1025\ kg/m^3$ linearly going up $500m$. What is N and T_B?

Sol^n

$$N^2 = -\frac{g}{\rho_0}\frac{d\bar{\rho}}{dz} \simeq -\frac{(9.8\ m/s^2)}{1025}\ \frac{(1030-1025\ kg/m^3)}{500\ m} \approx 10^{-4}\ s^{-2}$$

So $N \simeq 10^{-2}\ s^{-1}$ (typical of ocean)

$T_B = \frac{2\pi}{N} \simeq 628\ s \approx 10.5\ min.$ //

For the atmosphere find oscillation frequency in terms of potential temperature: $\boxed{N = \left[+\frac{g}{\theta}\frac{d\theta}{dz}\right]^{1/2}}$

Typically $N \approx 10^{-2}\ s^{-1}$ for atmosphere. Note $N=0$ in "well-mixed" atmosphere

(5) EQUATIONS OF MOTION I SUMMARY AND SIMPLE APPROXIMATIONS

SO FAR WE HAVE DERIVED THE FOLLOWING:

MASS CONSERVATION : $\frac{D\rho_T}{Dt} + \rho_T \nabla\cdot\underline{u} = 0$ (a) / $\frac{\partial \rho_T}{\partial t} + \nabla\cdot(\rho_T\underline{u}) = 0$ (b) (1)

MOMENTUM CONSERVATION: $\bar{\rho}\left(\frac{D\underline{u}}{Dt} + \underline{f}\times\underline{u}_H\right) = -\nabla p - \rho g\hat{z} + \mu\nabla^2\underline{u}$ (2)

INTENAL ENERGY CONSERVATION: $\frac{D\theta_T}{Dt} = 0$ (ATMOS) (a) $\frac{D\rho_{pot,T}}{Dt} = 0$ (OCEAN) (b) (3)

IGNORES DIFFUSION

THESE CAN BE SIMPLIED FURTHER WITH A FEW APPROXIMATIONS

A) INCOMPRESSIBILITY

ASSUME $\frac{D\rho_T}{Dt} = 0$ IN (1). EFFECTIVELY THIS FILTERS SOUND WAVES. SO (1a) $\Rightarrow$ $\boxed{\nabla\cdot\underline{u} = 0}$

THIS IS GOOD FOR OCEAN AND FOR OCEAN OVER SMALL VERTICAL SCALES

B) ANELASTIC APPROXIMATION

OVER LARGE VERTICAL SCALES IN ATMOSPHERE, BETTER TO MAKE A WEAKER APPROXIMATION IN (1) THAT $\rho_T \approx \bar{\rho}(z)$

So (1b) $\Rightarrow$ $\boxed{\nabla\cdot(\bar{\rho}\underline{u}) = 0}$

C) BOUSSINESQ APPROXIMATION

IN OCEAN AND OVER SMALL VERTICAL SCALES IN ATMOSPHERE SUPPOSE $\bar{\rho} \approx \rho_0$ CONSTANT ON LEFT-HAND-SIDE OF MOM$^{\underline{M}}$ EQUATION

So (3) $\Rightarrow$ $\boxed{\rho_0\left(\frac{D\underline{u}}{Dt} + \underline{f}\times\underline{u}_H\right) = -\nabla p - \rho g\hat{z} + \mu\nabla^2\underline{u}}$

(HENCE DENSITY VARIATIONS MATTER ONLY IN BUOYANCY TERM)

D) f-PLANE APPROXIMATION

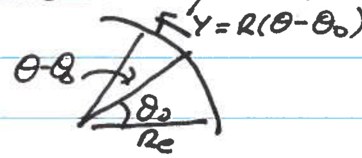
GENERALLY $f = 2\Omega_e \sin\theta$ WITH θ LATITUDE.

FOR PHENOMENA WITH SMALL NORTH-SOUTH SCALE CAN TAKE $\boxed{f \approx f_0 = 2\Omega_e \sin\theta_0}$ ABOUT SOME LATITUDE θ_0

5 (cont'd)

E) β-PLANE APPROXIMATION

FOR PHENOMENA WITH SOMEWHAT LARGER NORTH-SOUTH EXTENT ABOUT θ_0, TAYLOR EXPAND $\sin\theta \approx \sin\theta_0 + (\theta-\theta_0)\cos\theta_0$ AND NOTE THAT NORTH-SOUTH DISTANCE IS $y = R_e(\theta-\theta_0)$ WITH R_e THE EARTH'S RADIUS



So $f \approx 2\Omega_e \sin\theta_0 + 2\Omega_e \frac{y}{R_e}\cos\theta_0$

$\Rightarrow$ $\boxed{f \approx f_0 + \beta y}$ WITH $\beta \equiv \frac{\cos\theta_0}{R_e}$

AT EQUATOR ($\theta_0 = 0$) $\quad \beta \simeq 2\left(\frac{2\pi}{60\times60\times24\,s}\right)\frac{1}{6\times10^6\,m}\cos 0° \approx 2\times10^{-11}\frac{1}{m\cdot s}$

F) BUOYANCY

IN TERMS OF DENSITY (USEFUL FOR OCEAN), BUOYANCY IS DEFINED BY $\boxed{b \equiv -g\rho/\rho_0}$ IN WHICH ρ IS THE FLUCTATION DENSITY AND ρ_0 THE CHARACTERISTIC DENSITY

FOR THE ATMOSPHERE, WE USE THE RELATION $\rho \approx -\frac{\bar\rho}{\bar\theta}\theta$ AND SO DEFINE BUOYANCY AS $\boxed{b \equiv +g\,\theta/\bar\theta}$

THESE EXPRESSIONS REPLACE ρ AND θ IN THE MOMENTUM AND INTERNAL ENERGY EQUATIONS: E.g. $0 = \frac{D\theta_T}{Dt} = \frac{D(\bar\theta+\theta)}{Dt} = \frac{D\theta}{Dt} + w\frac{d\bar\theta}{dz}$

$\times\frac{g}{\bar\theta} \Rightarrow \frac{Db}{Dt} = -w\frac{g}{\bar\theta}\frac{d\bar\theta}{dz} = -N^2 w$

AND SO WE HAVE THE FOLLOWING EQUATIONS

ATMOSPHERE (ANELASTIC)

$$\nabla\cdot(\bar\rho\,\underline{u}) = 0$$
$$\frac{Du}{Dt} - fv = -\frac{\partial}{\partial x}\left(\frac{p}{\bar\rho}\right) + \mu\nabla^2 u$$
$$\frac{Dv}{Dt} + fu = -\frac{\partial}{\partial y}\left(\frac{p}{\bar\rho}\right) + \mu\nabla^2 v$$
$$\frac{Dw}{Dt} = -\frac{\partial}{\partial z}\left(\frac{p}{\bar\rho}\right) + b$$
$$\frac{D}{Dt} = -N^2 w$$

OCEAN & SHALLOW ATMOSPHERE

$$\nabla\cdot\underline{u} = 0$$
$$\frac{Du}{Dt} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu\nabla^2 u$$
$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \nu\nabla^2 v$$
$$\frac{Dw}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + b$$
$$\frac{Db}{Dt} = -N^2 w$$

WITH $f = f_0$ OR $f_0 + \beta y$, $N^2 = \frac{g}{\bar\theta}\frac{d\bar\theta}{dz}$ OR $-\frac{g}{\rho_0}\frac{d\bar\rho}{dz}$, $\nu = \frac{\mu}{\rho_0}$ THE KINEMATIC VISCOSITY