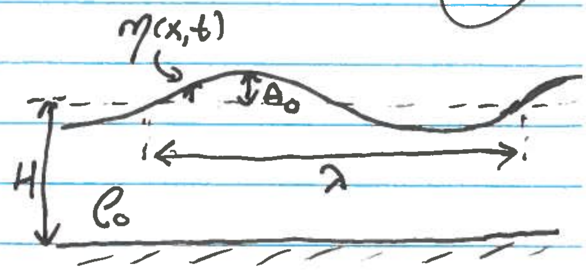
4] SURFACE WAVES



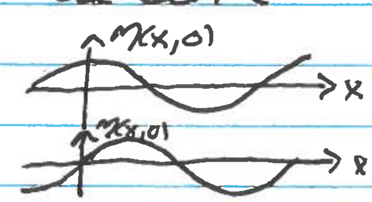
HERE WE CONSIDER THE MOTION OF WAVES ON THE SURFACE OF UNIFORM-DENSITY FLUID OF DEPTH H. THE SURFACE DISPLACEMENT FROM EQUILIBRIUM IS DENOTED BY $\eta(x,t)$, ASSUMING MOTION IN X-DIRECTION ALONE. WE WILL SEE MANY DIFFERENT CLASSES OF WAVES EXIST DEPENDING UPON THE RELATIVE DEPTH H AND THE INFLUENCE OF CORIOLIS FORCES.

1) WAVE THEORY

A] REPRESENTATION OF PLANE WAVES IN 1 DIMENSION

A PLANE WAVE IS INFINITELY PERIODIC IS SPACE AND TIME WITH FIXED WAVELENGTH λ AND PERIOD T, AND HENCE FIXED WAVENUMBER $k \equiv \frac{2\pi}{\lambda}$ AND FREQUENCY $\omega \equiv \frac{2\pi}{T}$.

DENOTING THE MAXIMUM UPWARD DISPLACEMENT OF THE SURFACE BY A_0 (THE HALF PEAK-TO-PEAK AMPLITUDE), CAN REPRESENT THE SURFACE DISPLACEMENT OF THE WHOLE WAVE BY



a) $\eta(x,t) = A_0 \cos(kx - \omega t)$

b) $\eta(x,t) = A_0 \sin(kx - \omega t)$

c) $\eta(x,t) = A_0 \cos(kx - \omega t + \phi_0)$

IN WHICH THE LAST EXPRESSION MAKES NO ASSUMPTION ABOUT THE PHASE OF THE WAVE: GET a) IF $\phi_0 = 0$; GET b) IF $\phi_0 = -\frac{\pi}{2}$.

THOUGH LESS INTUITIVE, THE EASIEST REPRESENTATION IS IN TERMS OF COMPLEX EXPONENTIALS, USING THE IDENTITY

$$\boxed{e^{i\phi} = \cos\phi + i\sin\phi}$$

① A] (CONT'd)

THUS WE REPRESENT THE SURFACE DISPLACEMENT BY

$$\boxed{\eta(x,t) = \mathcal{A}_0 e^{i(kx-\omega t)}} \quad (*)$$

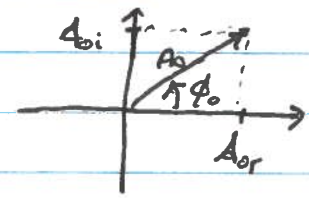


FIG 1

IN WHICH $\mathcal{A}_0$ IS A COMPLEX NUMBER GIVING BOTH THE AMPLITUDE AND PHASE, AND IT IS UNDERSTOOD IN (*) THAT η IS THE REAL PART OF THE RIGHT-HAND SIDE

INTERPRETATION OF $\mathcal{A}_0 = \mathcal{A}_{0r} + i\mathcal{A}_{0i}$:

CONVERT FROM CARTESIAN TO POLAR CO-ORDS AS SHOWN IN FIG 1

$$\mathcal{A}_{0r} + i\mathcal{A}_{0i} = A_0 \cos\phi_0 + iA_0 \sin\phi_0 = A_0 e^{i\phi_0}$$

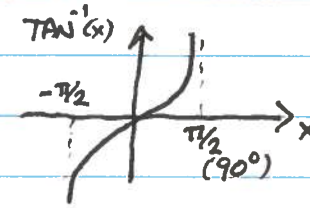
So $(*) \Rightarrow \eta(x,t) = A_0 e^{i\phi_0} e^{i(kx-\omega t)} = A_0 e^{i(kx-\omega t+\phi_0)}$

TAKING REAL PART GIVES $\eta = A_0 \cos(kx-\omega t+\phi_0)$ AS IN c) on p 49.

GIVEN $\mathcal{A}_0$, EXPLICITLY FIND AMPLITUDE AND PHASE FROM

$$A_0 = |\mathcal{A}_0| = \sqrt{\mathcal{A}_{0r}^2 + \mathcal{A}_{0i}^2}$$

$$\phi_0 = \tan^{-1}(\mathcal{A}_{0i}/\mathcal{A}_{0r})$$



EXAMPLES

① SUPPOSE $\mathcal{A}_0 = 10\text{ cm}$

$\Rightarrow A_0 = 10\text{cm}$ AND $\phi_0 = \tan^{-1}\left(\frac{0}{10}\right) = \tan^{-1} 0 = 0$

$\Rightarrow \eta = A_0 \cos(kx-\omega t)$ WITH $A_0 = 10\text{cm}$ //

② SUPPOSE $\mathcal{A}_0 = -i\,2\text{ cm}$

$\Rightarrow A_0 = \sqrt{0^2+(-2)^2} = 2\text{cm}$ AND $\phi_0 = \tan^{-1}\left(\frac{-2}{0}\right) = -\frac{\pi}{2}$

$\Rightarrow \eta = \mathrm{Re}\{-2i\,e^{i(kx-\omega t)}\} = \mathrm{Re}\{-2i\cos(kx-\omega t) - 2i[i\sin(kx-\omega t)]\}$

[TAKE REAL PART]

$= 2\sin(kx-\omega t)$ //

(ALTERNATELY $\eta = \mathrm{Re}\{2e^{i(kx-\omega t-\pi/2)}\} = 2\cos(kx-\omega t-\pi/2)$

$= 2\sin(kx-\omega t)$)

① (cont'd)

B] DISPERSION RELATION AND PHASE & GROUP SPEEDS

FOR ALL WAVES THE FREQUENCY DEPENDS UPON THE WAVENUMBER:

$\omega = \omega(k)$ ← THE "<u>DISPERSION RELATION</u>"

THIS FUNCTIONAL DEPENDENCE COMES FROM PHYSICS DRIVING THE WAVE: e.g.

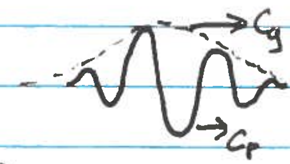
SHALLOW WATER WAVES	$\omega = \sqrt{gH}\, k$	H - WATER DEPTH
DEEP WATER WAVES	$\omega = \pm[g\lvert k\rvert]^{1/2}$	
CAPILLARY WAVES	$\omega = \pm\left[\frac{\sigma}{\rho_0}\lvert k\rvert^3\right]^{1/2}$	σ - SURFACE TENSION

(IN THE SECOND TWO EXPRESSIONS THE SIGN OF ω GIVES PROPAGATION DIRECTION)

GIVEN THE DISPERSION RELATION, CAN FIND

① THE PHASE SPEED $C_p \equiv \frac{\omega}{k}$

(THE SPEED AT WHICH THE CRESTS MOVE)



② THE GROUP SPEED $C_g \equiv \frac{d\omega}{dk}$

(THE SPEED AT WHICH THE WAVEPACKET MOVES)

EXAMPLES

① FOR 10m WAVELENGTH WAVES IN FLUID OF DEPTH 10cm,

(SHALLOW WAVES) ⇒ $c \equiv \sqrt{gH} \simeq \sqrt{(10m/s^2)(0.1m)} = 1m/s \Rightarrow \omega = c\overbrace{\left(\frac{2\pi}{\lambda}\right)}^{k} \simeq 0.63s^{-1}$

$\Rightarrow T = \frac{2\pi}{\omega} = 10s$

So $C_p = \frac{\omega}{k} = c = 1m/s$, $C_g = \frac{d\omega}{dk} = c = 1m/s$

[NOTE: SOLUTIONS INDEPENDENT OF $k = \frac{2\pi}{\lambda}$, PROVIDED $\lambda \gg 0.1m$]

② FOR 1m WAVELENGTH WAVES IN FLUID OF DEPTH 100m

(DEEP WAVES) ⇒ $\omega = \sqrt{gk} = \left[(10m/s^2)\left(\frac{2\pi}{1m}\right)\right]^{1/2} \simeq 7.9s^{-1}$

$\Rightarrow T = \frac{2\pi}{\omega} \simeq 0.79s$

AND $C_p = \frac{\omega}{k} = \sqrt{g/k} = \left[(10m/s^2)/\left(\frac{2\pi}{1m}\right)\right]^{1/2} \simeq 1.3m/s$

$C_g = \frac{d\omega}{dk} = \frac{1}{2}\sqrt{g/k} \simeq 0.63\ m/s$

NOTE: $C_g = \frac{1}{2}C_p$ FOR ALL DEEP WATER WAVES

So SEE WAVE CREST APPEAR FROM REAR AND ADVANCE TO THE FRONT OF THE WAVEPACKET.

① (cont'd)

c] Cross-correlations

To assess energy & momentum transport by waves, need to compute the average of the product of two fields, η and ξ, say. (These could be e.g. displacement and horizontal velocity.)

For average over a wavelength, this cross-correlation is

$$\langle \eta \xi \rangle = \frac{1}{\lambda}\int_0^\lambda \eta(x)\,\xi(x)\,dx$$

Examples

① $\eta = A_0 \cos(kx)$, $\xi = B_0 \cos(kx)$

$$\Rightarrow \langle \eta\xi\rangle = \frac{1}{\lambda}\int_0^\lambda A_0\cos(kx)\,B_0\cos(kx)\,dx$$
$$= A_0B_0\frac{1}{\lambda}\int_0^\lambda \cos^2(kx)\,dx$$
$$= A_0B_0\frac{1}{\lambda}\int_0^\lambda \frac{1}{2}(1+\cos(2kx))\,dx$$
$$= A_0B_0\frac{1}{\lambda}\left[\frac{1}{2}x\Big|_0^\lambda + \frac{1}{4k}\sin(2kx)\Big|_0^\lambda\right]$$
$$= A_0B_0\frac{1}{\lambda}\left[\frac{1}{2}\lambda + \frac{1}{4k}\left(\sin(4\pi)-\sin(0)\right)\right] \quad k=\frac{2\pi}{\lambda}$$
$$\Rightarrow \boxed{\langle\eta\xi\rangle = \frac{1}{2}A_0B_0}$$

② Alternately write $\eta = \frac{1}{2}A_0(e^{ikx}+e^{-ikx})$, $\xi = \frac{1}{2}B_0(e^{ikx}+e^{-ikx})$

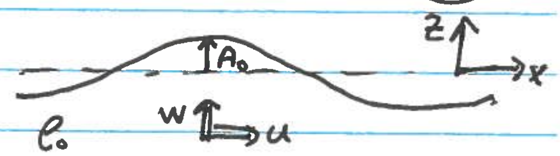
$$\Rightarrow \langle\eta\xi\rangle = \frac{1}{4}A_0B_0\frac{1}{\lambda}\int_0^\lambda (e^{ikx}+e^{-ikx})(e^{ikx}+e^{-ikx})\,dx$$
$$= \frac{1}{4}A_0B_0\frac{1}{\lambda}\int_0^\lambda e^{2ikx}+2+e^{-2ikx}\,dx$$
$$= \frac{1}{4}A_0B_0\frac{1}{\lambda}\left[\frac{1}{2ik}e^{2ikx}\Big|_0^\lambda + 2x\Big|_0^\lambda - \frac{1}{2ik}e^{-2ikx}\Big|_0^\lambda\right]$$
$$= \frac{1}{4}A_0B_0\frac{1}{\lambda}\left[\frac{1}{2ik}\underbrace{(e^{4\pi}-e^{0})}_{\to 0} + 2\lambda - \frac{1}{2ik}\underbrace{(e^{-4\pi}-e^{0})}_{\to 0}\right]$$
$$= \frac{1}{2}A_0B_0$$

Likewise find if $\eta = A_0\sin kx$, $\xi = B_0\sin kx \Rightarrow \langle\eta\xi\rangle = \frac{1}{2}A_0B_0$

& if $\eta = A_0\cos kx$, $\xi = B_0 \sin kx \Rightarrow \langle\eta\xi\rangle = 0$.

Averaging over a period $T=\frac{2\pi}{\omega}$ instead of λ gives same results.

② DEEP WATER WAVES



IGNORE ROTATION AND ASSUME SMALL-AMPLITUDE

⇒ EQUATIONS OF MOTION ARE

① $\nabla \cdot \underline{u} = 0$ (MASS CONS FOR INCOMPRESSIBLE FLUID)

② $\rho_0 \frac{\partial u}{\partial t} = -\frac{\partial P_T}{\partial x}$ (x-MOM$^{\underline{t}}$ EQ$^{\underline{t}}$)

③ $\rho_0 \frac{\partial w}{\partial t} = -\frac{\partial P_T}{\partial z} - \rho_0 g$ (z-MOM$^{\underline{t}}$ EQ$^{\underline{t}}$)

[DON'T NEED INTERNAL ENERGY EQUATION $\frac{D\rho}{Dt} = 0$ SINCE DENSITY IS UNIFORM]

[$\underline{u} \cdot \nabla$ (ADVECTION TERMS) NEGLIGIBLY SMALL FOR SMALL-AMP WAVES]

DEFINE VORTICITY $\underline{\zeta} = \nabla \times \underline{u} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ u & 0 & w \end{vmatrix} = \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{y} = \zeta \hat{y}$

WITH $\zeta \equiv \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$ BEING "SPANWISE VORTICITY"

SO TAKING $\frac{\partial}{\partial z}$② $- \frac{\partial}{\partial x}$③ GIVES THE VORTICITY EQ$^{\underline{N}}$

$$\frac{\partial}{\partial z}\left[\rho_0 \frac{\partial u}{\partial t} = -\frac{\partial P_T}{\partial x}\right] - \frac{\partial}{\partial x}\left[\rho_0 \frac{\partial w}{\partial t} = -\frac{\partial P_T}{\partial z} - \rho_0 g\right] \Rightarrow \rho_0 \frac{\partial \zeta}{\partial t} = -\frac{\partial^2 P_T}{\partial z \partial x} + \frac{\partial^2 P_T}{\partial x \partial z} = 0$$

$$\Rightarrow \boxed{\frac{\partial \zeta}{\partial t} = 0}$$

IF THERE IS NO VORTICITY IN FLUID BEFORE WAVES ARRIVE, THERE WILL BE NO VORTICITY DURING PASSAGE AS WELL: THE FLUID IS "IRROTATIONAL" : $\zeta = 0$ EVERYWHERE

BECAUSE $\zeta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0$ ④ WE CAN DEFINE THE "VELOCITY POTENTIAL" $\phi(x,z,t)$ SUCH THAT $u = \frac{\partial \phi}{\partial x}$, $w = \frac{\partial \phi}{\partial z}$

(HENCE IN ④ $\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial z}\right) = 0$ IS INDEED SATISFIED)

NOW WRITE ① IN TERMS OF ϕ

$$0 = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = \nabla^2 \phi \quad ⑤$$

THE SOLUTION OF ⑤ REQUIRES BOUNDARY CONDITIONS:

a) $\phi|_{z \to -\infty}$ IS BOUNDED

b) $B \equiv \frac{\partial \phi}{\partial t} + \frac{1}{2}|\underline{u}|^2 + \frac{1}{\rho_0} P + gz$ IS CONSTANT ON SURFACE (BERNOULLI'S PRINCIPLE)

$\Rightarrow \frac{\partial \phi}{\partial t}\Big|_{z=0} + g\eta \approx 0$ (NEGLECTING $\frac{1}{2}|\underline{u}|^2$ AND SETTING $P=0$ ON SURFACE)

c) $\frac{\partial \eta}{\partial t} \simeq w|_{z=0} = \frac{\partial \phi}{\partial z}\Big|_{z=0}$ (FLUID AT SURFACE REMAINS THERE)

2 (cont'd)

NOW SEEK SOLUTIONS FOR PLANE WAVES SUCH THAT

$$\eta(x,t) = A_0 e^{i(kx-\omega t)} \quad (6a) \quad \text{(ARBITRARILY SUPPOSE } A_0>0 \text{ REAL)}$$

$$\phi(x,z,t) = \hat{\phi}(z) e^{i(kx-\omega t)} \quad (6b) \quad (\hat{\phi} \text{ POSSIBLY COMPLEX F}^{\underline{n}} \text{ OF DEPTH } z)$$

(5): $\nabla^2\phi = 0 \Rightarrow (ik)^2 \hat{\phi}(z) e^{i(kx-\omega t)} + \frac{d^2\hat{\phi}}{dz^2} e^{i(kx-\omega t)} = 0$

$$\Rightarrow \hat{\phi}''(z) - k^2\hat{\phi} = 0$$

$$\Rightarrow \hat{\phi}(z) = C_1 e^{kz} + C_2 e^{-kz}$$

BC a): ϕ BOUNDED AS $z \to -\infty \Rightarrow$ MUST HAVE $C_2 = 0$ SUPPOSING $k>0$

So $\phi(x,z,t) = C_1 e^{kz} e^{i(kx-\omega t)}$

BC b): $\frac{\partial \phi}{\partial t}\Big|_{z=0} = -g\eta \Rightarrow -i\omega C_1 e^{i(kx-\omega t)} = -g A_0 e^{i(kx-\omega t)}$

So $C_1 = \frac{1}{-i\omega}(-gA_0) = -i\frac{g}{\omega}A_0$

So $\phi(x,z,t) = -i\frac{g}{\omega}A_0 e^{kz} e^{i(kx-\omega t)}$ (7)

BC c): $\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z}\Big|_{z=0} \Rightarrow -i\omega A_0 e^{i(kx-\omega t)} = -i\frac{g}{\omega}kA_0 e^{i(kx-\omega t)}$

$$\Rightarrow \boxed{\omega^2 = gk} \quad (8)$$

THIS IS THE DISPERSION RELATION FOR DEEP WATER WAVES

GET EXPLICIT WAVE STRUCTURE BY TAKING REAL PARTS OF (6a) AND (7)

$\Rightarrow$ $\eta(x,t) = A_0 \cos(kx-\omega t)$

$$\phi(x,z,t) = \frac{g}{\omega}A_0 e^{kz}\sin(kx-\omega t)$$

HENCE $u = \frac{\partial \phi}{\partial x} = \frac{gk}{\omega}A_0 e^{kz}\cos(kx-\omega t) \overset{\omega^2=gk}{=} \omega A_0 e^{kz}\cos(kx-\omega t)$

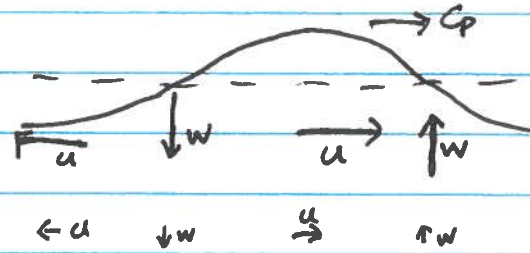
$$w = \frac{\partial \phi}{\partial z} = \frac{gk}{\omega}A_0 e^{kz}\sin(kx-\omega t) = \omega A_0 e^{kz}\sin(kx-\omega t)$$

DISPLACEMENTS IN x: $\frac{\partial \xi_x}{\partial t} = u \Rightarrow \xi_x = -A_0 e^{kz}\sin(kx-\omega t)$

IN z: $\frac{\partial \xi_z}{\partial t} = w \Rightarrow \xi_z = A_0 e^{kz}\cos(kx-\omega t)$

[NOTE $\xi_z\big|_{z=0} = A_0\cos(kx-\omega t) = \eta$]

2 (CONT'D)



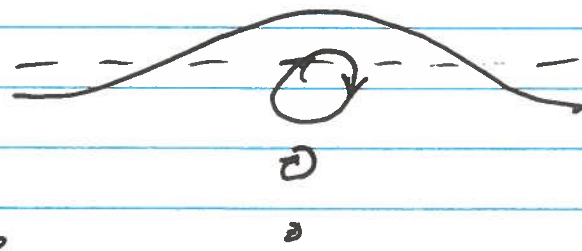
THE RELATIONSHIPS BETWEEN THE FIELDS $\eta, \phi, u, w, \xi_x, \xi_z$ etc ARE CALLED THE "POLARIZATION RELATIONS"

CAN WORK OUT THE PATH OF A FLUID PARCEL LOCATED AT SOME POSITION (x_0, z_0) BEFORE WAVE ARRIVES. POSITION (x, z) DURING PASSAGE OF WAVE IS

$$(x, z) = (x_0 + \xi_x, z_0 + \xi_z) = (x_0 - A_0 e^{kz_0} \sin(kx_0 - \omega t), z_0 + A_0 e^{kz_0} \cos(kx_0 - \omega t))$$

$$\Rightarrow (x - x_0)^2 + (z - z_0)^2 = (A_0 e^{kz_0})^2 [\sin^2(kx_0 - \omega t) + \cos^2(kx_0 - \omega t)]$$
$$= (A_0 e^{kz_0})^2$$

I.e. FLUID FOLLOWS A CIRCULAR ORBIT OF RADIUS $A_0 e^{kz_0}$, WHICH GETS SMALLER WITH DEPTH



THIS DEPTH-DEPENDENCE OF DISPLACEMENT RESULTS IN A SMALL DRIFT OF ~~NEAR~~-SURFACE WATER IN DIRECTION OF WAVE:

$$u(x, z, t) \simeq u(x_0, z_0, t) + (x - x_0) \frac{\partial u}{\partial x}\Big|_{x_0} + (z - z_0) \frac{\partial u}{\partial z}\Big|_{x_0}$$
$$= u(x_0, z_0, t) + \xi_x \frac{\partial u}{\partial x}\Big|_{x_0} + \xi_z \frac{\partial u}{\partial z}\Big|_{x_0}$$

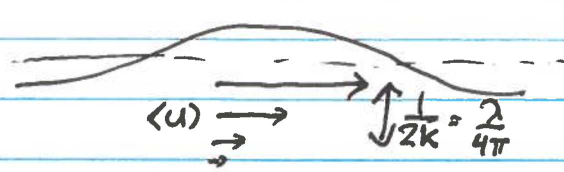
AVERAGE OVER 1 PERIOD IN TIME $\langle \cdot \rangle = \frac{1}{T}\int_0^T \cdot \, dt$

$$\Rightarrow \langle u \rangle = \langle u(x_0, z_0, t) \rangle + \langle \xi_x \frac{\partial u}{\partial x} \rangle\Big|_{x_0} + \langle \xi_z \frac{\partial u}{\partial z} \rangle\Big|_{x_0}$$
$$= \langle \omega A_0 e^{kz_0} \cos(kx_0 - \omega t) \rangle$$
$$+ \langle (-A_0 e^{kz_0} \sin(kx_0 - \omega t))(-\omega k A_0 e^{kz_0} \sin(kx_0 - \omega t)) \rangle$$
$$+ \langle (A_0 e^{kz_0} \cos(kx_0 - \omega t))(\omega k A_0 e^{kz_0} \cos(kx_0 - \omega t)) \rangle$$

$\langle \sin^2 \rangle = \langle \cos^2 \rangle = \frac{1}{2}$

$$= 0 + \frac{1}{2}(\omega k A_0^2 e^{2kz_0}) + \frac{1}{2}(\omega k A_0^2 e^{2kz_0})$$

$$\Rightarrow \boxed{\langle u \rangle = \omega k A_0^2 e^{2kz_0}}$$



THIS IS THE "STOKES DRIFT"

USING $c_p = \frac{\omega}{k} \Rightarrow \langle u \rangle = c_p (A_0 k)^2 e^{2kz_0} \ll c_p$ IF SMALL AMPL. $(A_0 k) \ll 1$

② (cont'd)

EXAMPLE

CONSIDER A $\lambda = 10\,m$ WAVELENGTH RIGHTWARD-PROPAGATING WAVE AT THE SURFACE OF AN EFFECTIVELY INFINITELY DEEP OCEAN AND TAKE ITS (HALF PEAK-TO-PEAK) AMPLITUDE TO BE $A_0 = 10\,cm$.

FROM THIS INFORMATION ALONE CAN FIND SEVERAL PROPERTIES:

$k = \frac{2\pi}{\lambda} \simeq 0.63\ m^{-1}$

$\omega = \sqrt{gk} \simeq 2.5\ s^{-1}$

$C_p = 4.0\,m/s$

MAXIMUM SPEED AT SURFACE: $\max u|_{z=0} = \omega A_0 = 0.25\,m/s$ (UNDER CREST)

$\max w|_{z=0} = \omega A_0 = 0.25\,m/s$ (AT LEADING NODE)

SPEED 1m BELOW CREST (e.g. AT $x=0$, FOR $t=0$): $u(0,-1m,0) = \omega A_0 e^{k(-1m)}$

$\simeq (0.25\,m/s)\, e^{-0.63} \simeq 0.13\,m/s$

STOKES DRIFT AT SURFACE: $\langle u \rangle = C_p (kA_0)^2 \simeq (4.0\,m/s)(0.063)^2 \simeq 1.6\times10^{-2}\,m/s$

STOKES DRIFT AT 1m DEPTH: $\langle u \rangle = C_p (kA_0)^2 e^{2kz_0} \simeq (0.016\,m/s)\, e^{-1.26} \simeq 4.5\times10^{-3}\,m/s$

WE HAVE SEEN THAT THE AMPLITUDE OF MOTION DECREASES EXPONENTIALLY WITH DEPTH AS $e^{kz} = e^{-k|z|} = e^{-|z|/\sigma}$ IN WHICH $\sigma \equiv 1/k$ IS THE "e-FOLDING" DEPTH (THE VERTICAL DISTANCE OVER WHICH THE MOTION DECREASES BY A FACTOR $e \approx 2.7$.

FOR THE EXAMPLE ABOVE, THE e-FOLDING DEPTH IS $\sigma = \frac{1}{k} = \frac{\lambda}{2\pi} \simeq 1.6\,m$

NOTE THAT THE STOKES DRIFT REDUCES AT A FASTER RATE WITH DEPTH; AS e^{2kz}. THE e-FOLDING DEPTH IS $\frac{1}{2k} = \frac{\lambda}{4\pi} \approx 0.8\,m$.

3 ENERGETICS

ONE REASON IT IS IMPORTANT TO STUDY WAVES IS THAT THEY TRANSPORT ENERGY, NAMELY KINETIC & POTENTIAL ENERGY.

DERIVE K.E. EQ^{N} FROM MOMENTUM EQUATIONS:

$$u \times \left[\rho_0 \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} \right] + w \times \left[\rho_0 \frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} - \rho g \right]$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho_0 (u^2+w^2) \right) = -u\frac{\partial p}{\partial x} - w\frac{\partial p}{\partial z} - \rho g w$$

$$\underset{\nabla\cdot \underline{u}=0}{\big(} = -\frac{\partial}{\partial x}(up) - \frac{\partial}{\partial z}(wp) - \rho g w \big)$$

TOTAL K. ENERGY/HORIZONTAL AREA IS GIVEN BY VERTICALLY INTEGRATING $\int_{-\infty}^{0} \cdot \, dz$

$$\Rightarrow \frac{\partial}{\partial t}\left[\int_{-\infty}^{0} \tfrac{1}{2}\rho_0 (u^2+w^2)\, dz \right] = -\frac{\partial}{\partial x}\left[\int_{-\infty}^{0} up \right] - [wp]\Big|_{-\infty}^{0} - \rho g \int_{-\infty}^{0} w\, dz \quad (*)$$

DEFINE $E_K \equiv \tfrac{1}{2}\rho_0 \int_{-\infty}^{0} u^2+w^2\, dz$ KINETIC ENERGY

$F_K \equiv \int_{-\infty}^{0} up\, dz$ KINETIC ENERGY FLUX

NOTE $wp|_{z=0} = wp|_{z\to-\infty} = 0$ (SINCE $p=0$ AT SURFACE AND w, p VANISH AS $z\to-\infty$)

IF WE NOW CONSIDER AVERAGES (OVER 1 WAVELENGTH OR PERIOD), FIND $\langle w \rangle = 0$. SO $(*)$ REDUCES TO

$$\boxed{\frac{\partial \langle E_K \rangle}{\partial t} = -\frac{\partial}{\partial x}\langle F_K \rangle} \qquad (1)$$

DEFINE THE "AVAILABLE POTENTIAL ENERGY" TO BE DIFFERENCE OF POTENTIAL ENERGY WITH WAVES AND WITHOUT:

$$E_p = \int_{-\infty}^{\eta} \rho_0 g z\, dz - \int_{-\infty}^{0} \rho_0 g z\, dz = \int_{0}^{\eta} \rho_0 g z\, dz = \tfrac{1}{2}\rho_0 g \eta^2$$

(THIS IS ACTUALLY ENERGY PER UNIT HORIZONTAL AREA)

SO MEAN AVAILABLE POTENTIAL ENERGY IS $\langle E_p \rangle = \tfrac{1}{2}\rho_0 g \langle \eta^2 \rangle$.

BECAUSE THERE IS NEGLIGIBLE MASS TRANSPORT BY WAVES

$$\frac{\partial}{\partial t}\langle E_p \rangle = 0$$

SO $\langle E \rangle = \langle E_K \rangle + \langle E_p \rangle$ IS TOTAL ENERGY, $\langle F \rangle = \langle F_K \rangle$ IS ENERGY FLUX AND $\frac{\partial}{\partial t}\langle E \rangle = -\frac{\partial}{\partial x}\langle F \rangle$

3 (CONT'D)

FROM POLARIZATION RELATIONS, CAN FIND $\langle E_K\rangle, \langle E_P\rangle, \langle E\rangle$ AND $\langle F\rangle$ IN TERMS OF AMPLITUDE AND WAVENUMBER OF WAVE

$$\langle E_K\rangle = \frac{1}{2}\rho_0\int_{-\infty}^{0}\langle u^2\rangle + \langle w^2\rangle\, dz \qquad \text{WITH } u = \omega A_0 e^{kz}\cos(kx-\omega t),\ w = \omega A_0 e^{kz}\sin(kx-\omega t)$$

$$= \frac{1}{2}\rho_0\int_{-\infty}^{0}\tfrac{1}{2}(\omega A_0)^2 e^{2kz} + \tfrac{1}{2}(\omega A_0)^2 e^{2kz}\, dz$$

$$= \frac{1}{2}\rho_0(\omega A_0)^2\int_{-\infty}^{0} e^{2kz}\, dz = \frac{1}{2}\rho_0(\omega A_0)^2\, \frac{1}{2k}e^{2kz}\Big|_{-\infty}^{0}$$

$$\Rightarrow \langle E_K\rangle = \frac{1}{4k}\rho_0\omega^2 A_0^2 = \underline{\frac{1}{4}\rho_0 g A_0^2} \qquad (\text{USING } \omega^2 = gk)$$

$$\langle E_P\rangle = \frac{1}{2}\rho_0 g\langle \eta^2\rangle = \frac{1}{2}\rho_0 g\left(\frac{1}{2}A_0^2\right) = \underline{\frac{1}{4}\rho_0 g A_0^2}$$

[NOTE! $\langle E_K\rangle = \langle E_P\rangle$ - ENERGY IS "EQUIPARTITIONED" ALWAYS TRUE OF WAVES IN A CONSERVATIVE (e.g. NON-ROTATING) SYSTEM]

TOTAL ENERGY IS $\langle E\rangle = \langle E_K\rangle + \langle E_P\rangle = \underline{\frac{1}{2}\rho_0 g A_0^2}$

$$\langle F\rangle = \int_{-\infty}^{0}\langle up\rangle\, dz$$

IN WHICH THE POLARIZATION REL^{N} FOR p FOUND FROM $\rho_0\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x}$

$$\Rightarrow\ p = \rho_0\left(\frac{\omega}{k}\right)\omega A_0 e^{kz}\cos(kx-\omega t)$$

$$\text{So}\quad \langle F\rangle = \int_{-\infty}^{0}\left(\omega A_0 e^{kz}\cos(kx-\omega t)\right)\left(\rho_0\frac{\omega^2}{k}A_0 e^{kz}\cos(kx-\omega t)\right)dz$$

$$= \frac{1}{2}\left(\rho_0\frac{\omega^3}{k}A_0^2\right)\int_{-\infty}^{0} e^{2kz}\, dz$$

$$= \frac{1}{4k}\left(\rho_0\frac{\omega^3}{k}A_0^2\right) = \frac{1}{4}\rho_0\frac{g\omega}{k}A_0^2$$

RECALL THE GROUP VELOCITY IS $c_g = \frac{d\omega}{dk} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}c_p = \frac{1}{2}\frac{\omega}{k}$

$$\Rightarrow \boxed{\langle F\rangle = c_g\left(\frac{1}{2}\rho_0 g A_0^2\right) = c_g\langle E\rangle}$$

SO ENERGY FLUX IS ENERGY BEING TRANSPORTED AT GROUP VELOCITY. THIS IS TRUE FOR ALL WAVES.

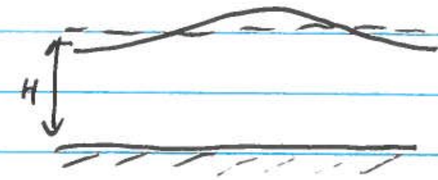
EXAMPLE: FOR WAVE CONSIDERED AT END OF LAST SECTION

$$\langle E\rangle = \frac{1}{2}\rho_0 g A_0^2 \simeq \frac{1}{2}(1000\ \text{kg/m}^3)(10\ \text{m/s}^2)(0.1\ \text{m})^2 = 50\ \text{kg/s}^2$$

$$= 50\ \text{J/m}^2 \qquad (\text{SO EVERY m}^2 \text{ HAS } 50\text{J ENERGY BELOW})$$

$\langle F\rangle = c_g\langle E\rangle = \frac{1}{2}c_p\langle E\rangle \simeq 100\ \frac{\text{J}}{\text{m}^2}\cdot\frac{\text{m}}{\text{s}} = 100\ \text{W/m}$ IS POWER CROSSING 1m SPANWISE EXTENT.

4 Surface (Gravity) Waves in Finite Depth



Here we proceed as in 2 for deep water waves but now assume the fluid has finite depth H.

The equation describing the interior motion is the same:

① $\nabla^2 \phi = 0$ (with $\underline{u} = \nabla \phi$)

The surface boundary conditions are the same

② $\left.\dfrac{\partial \phi}{\partial t}\right|_{z=0} = -g\eta$

③ $\dfrac{\partial \eta}{\partial t} = \left.\dfrac{\partial \phi}{\partial z}\right|_{z=0}$

But the lower boundary condition ($\phi|_{z \to -\infty}$ bounded) is replaced by $w|_{z=-H} = \left.\dfrac{\partial \phi}{\partial z}\right|_{z=-H} = 0$ ④

Writing $\phi(x,z,t) = \hat{\phi}(z)\, e^{i(kx-\omega t)}$ as before, ① gives

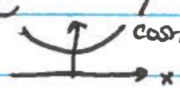
$$\hat{\phi}'' - k^2 \hat{\phi} = 0 \Rightarrow \hat{\phi}(z) = C_1 e^{kz} + C_2 e^{-kz} \quad ⑤$$

④ $\Rightarrow kC_1 e^{-kH} - kC_2 e^{kH} = 0$

$\Rightarrow C_2 = C_1 e^{-2kH}$

So ⑤ $\Rightarrow \hat{\phi}(z) = C_1 \left(e^{kz} + e^{-kz-2kH}\right) = C_1 e^{-kH}\left(e^{k(z+H)} + e^{-k(z+H)}\right)$

$\Rightarrow \hat{\phi}(z) = C \cosh k(z+H)$ ⑥



where $\cosh x \equiv \frac{1}{2}(e^x + e^{-x})$ and $C = 2C_1 e^{-kH}$ a constant

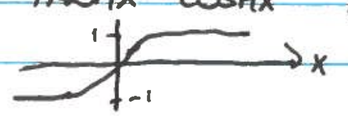
Now use ⑥ in surface boundary conditions ② & ③:

$$\begin{aligned} -i\omega C \cosh kH &= -g A_0 \\ -i\omega A_0 &= kC \sinh kH \end{aligned} \Rightarrow \underbrace{\begin{pmatrix} -i\omega & k\sinh kH \\ g & -i\omega \cosh kH \end{pmatrix}}_{\underline{\underline{M}}} \begin{pmatrix} A_0 \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For non-trivial solutions, must have $\det \underline{\underline{M}} = 0$

$\Rightarrow (-i\omega)(-i\omega \cosh kH) - gk \sinh kH = 0$

$\Rightarrow \boxed{\omega^2 = gk \tanh kH}$ where $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$



4 (CONT'D)

THIS APPROACH REVEALS THAT THE DISPERSION RELATION IS EFFECTIVELY AN EIGENVALUE. THE CORRESPONDING EIGENVECTOR GIVES THE POLARIZATION RELATIONS. I.e.

$$C = -i\,\frac{\omega}{k}\,\frac{1}{\sinh kH}\,A_0$$

$$\Rightarrow \quad \phi(x,z,t) = \mathcal{R}e\left\{-i\,\frac{\omega}{k}\,\frac{1}{\sinh kH}\,A_0 \cosh k(z+H)\,e^{i(kx-\omega t)}\right\}$$

$$= \frac{\omega}{k}\,\frac{1}{\sinh kH}\,A_0 \cosh k(z+H)\,\sin(kx-\omega t)$$

GO ON TO FIND

$$u = \frac{A_0\omega}{\sinh kH}\cosh k(z+H)\cos(kx-\omega t)$$
$$w = \frac{A_0\omega}{\sinh kH}\sinh k(z+H)\sin(kx-\omega t)$$
$$\xi_x = -\frac{A_0}{\sinh kH}\cosh k(z+H)\sin(kx-\omega t)$$
$$\xi_z = \frac{A_0}{\sinh kH}\sinh k(z+H)\cos(kx-\omega t)$$

DEEP WATER LIMIT $(kH \gg 1)$

$\Rightarrow \sinh kH = \frac{1}{2}(e^{kH}-e^{-kH}) \to \frac{1}{2}e^{kH}$, $\cosh k(z+H) \to \frac{1}{2}e^{k(z+H)}$

AND $\tanh kH \to 1$

SO $\omega^2 = gk\tanh kH \to \omega^2 = gk$ AS FOUND BEFORE

AND $\phi = \frac{\omega}{k}A_0\frac{\cosh k(z+H)}{\sinh kH}\sin(kx-\omega t) \to \phi = \frac{\omega}{k}A_0 e^{kz}\sin(kx-\omega t)$ AS BEFORE

SHALLOW WATER LIMIT $(kH \ll 1)$

$\Rightarrow \sinh kH \approx kH$, $\cosh k(z+H) \approx 1$, $\tanh kH \approx kH$

SO $\omega^2 = gk\tanh kH \to \boxed{\omega^2 = gHk^2}$ SHALLOW WATER DISP$^{\underline{n}}$ REL$^{\underline{n}}$

(THIS IS SOMETIMES WRITTEN $\omega^2 = c^2k^2$ OR $\omega = \pm ck$, WITH $c \equiv \sqrt{gH}$, THE SHALLOW WATER WAVE SPEED)

ALSO $u = A_0\frac{\omega}{kH}\cos(kx-\omega t)$ INDEPENDENT OF z!

$w = A_0\,\omega\left(1+\frac{z}{H}\right)\sin(kx-\omega t)$ MUCH SMALLER THAN u!

4 (CONT'd)

EXAMPLE:

CONSIDER A 10 m WAVELENGTH WAVE IN 10 cm DEEP WATER.
THE WAVE HAS 1 cm AMPLITUDE.

$$\text{So } kH = \left(\frac{2\pi}{10\,m}\right)(0.1\,m) \simeq 0.063 \ll 1 \Rightarrow \text{SHALLOW}$$

$$c = \sqrt{gH} \approx 1\,m/s$$

$$\omega = ck = 0.63\,s^{-1} \qquad \left(\Rightarrow \text{PERIOD } T = \frac{2\pi}{\omega} = \frac{2\pi}{\omega/k}\frac{1}{k} = \frac{1}{c}\frac{2\pi}{k} = \frac{\lambda}{c} = 10s\right)$$

$$\|u\| = A_0 \frac{\omega}{kH} = (0.01\,m)\frac{1\,m/s}{0.1\,m} = 0.1\,m/s$$

$$\|w\| = A_0\omega = (0.01\,m)(0.63\,s^{-1}) \simeq 0.0063\,m/s \ll \|u\|$$

AS IN SECTION 3 WE CAN GO ON TO FIND THE ENERGETICS OF FINITE DEPTH WAVES. IN PARTICULAR, FOR SHALLOW WATER WAVES:

$$\langle E_K \rangle = \frac{1}{2}\rho_0 \int_{-H}^{0} \langle u^2 + w^2 \rangle dz \overset{\|w\| \ll \|u\|}{\approx} \frac{1}{2}\rho_0 \int_{-H}^{0} \langle u^2 \rangle dz = \frac{1}{2}\rho_0 H \left(\frac{1}{2}\left(\frac{A_0\omega}{kH}\right)^2\right)$$

$$\Rightarrow \langle E_K \rangle = \frac{1}{4}\rho_0 \left(\frac{\omega}{k}\right)^2 \frac{1}{H} A_0^2 = \frac{1}{4}\rho_0 g A_0^2 \qquad \left(\text{USING } \omega/k = c = \sqrt{gH}\right)$$

$$\text{AS BEFORE } \langle E_P \rangle = \frac{1}{4}\rho_0 g A_0^2 = \langle E_K \rangle$$

$$\Rightarrow \langle E \rangle = \langle E_K \rangle + \langle E_P \rangle = \frac{1}{2}\rho_0 g A_0^2 \qquad \text{(SAME AS DEEP WATER RESULT)}$$

$$\text{ENERGY FLUX: } \langle F \rangle = \int_{-H}^{0} \langle up \rangle dz = H \langle up \rangle$$

$$\text{WITH } p = \rho_0 \frac{\omega^2}{k} u = \rho_0 \left(\frac{\omega}{k}\right)^2 \frac{1}{H} A_0 \cos(kx-\omega t) = \rho_0 g A_0 \cos(kx-\omega t)$$

$$\Rightarrow \langle F \rangle = H\left(\frac{1}{2}\rho_0 g \frac{\omega}{kH} A_0^2\right) = \frac{\omega}{k}\left(\frac{1}{2}\rho_0 g A_0^2\right) = c\langle E \rangle = c_g \langle E \rangle$$

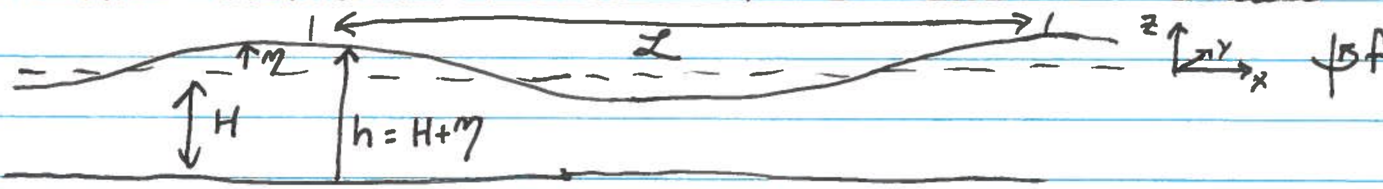
So, AS IN EXAMPLE ABOVE

$$\langle E \rangle \simeq \frac{1}{2}(1000\,kg/m^3)(10\,m/s^2)(0.01\,m)^2 = 0.5\,J/m^2$$

$$\langle F \rangle = c\langle E \rangle = 0.5\,W/m.$$

5 SHALLOW WATER THEORY

THE LARGE-SCALE MOTIONS IN THE ATMOSPHERE AND OCEAN ARE MUCH WIDER THAN THEY ARE DEEP. AND SO, RATHER THAN SOLVING THE FULL EQUATIONS OF MOTION, IT IS TYPICAL TO APPROXIMATE THEM ASSUMING THE MOTION IS "SHALLOW"



ASSUME $\alpha = \frac{H}{L} \ll 1$

HENCE VERTICAL MOTION, W IS MUCH SMALLER THAN HORIZONTAL, U:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \Rightarrow \frac{U}{L}\left(\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}}\right) + \frac{W}{H}\frac{\partial \tilde{w}}{\partial \tilde{z}} = 0 \Rightarrow \frac{U}{L} \sim \frac{W}{H} \Rightarrow W = \alpha U$$

BECAUSE W IS SMALL, THE FLOW IS HYDROSTATIC:

$$\frac{\partial p}{\partial z} = -\rho_0 g \Rightarrow p = p_0 - \rho_0 g z = \rho_0 g(\eta - z)$$

(TAKING $p = 0$ AT $z = \eta$)

SO LINEARIZED (SMALL AMPLITUDE) HORIZONTAL MOM^{E} EQ^{NS} ARE

$$\frac{\partial u}{\partial t} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} = -g\frac{\partial \eta}{\partial x} \quad ①$$

$$\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} = -g\frac{\partial \eta}{\partial y} \quad ②$$

BECAUSE FLUID IS INCOMPRESSIBLE $\quad \frac{\partial w}{\partial z} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -\nabla_h \cdot \underset{\text{HORIZONTAL VELOCITY}}{\underline{u}_h}$

ALSO RECALL THAT U_h IS INDEPENDENT OF DEPTH FOR SHALLOW FLUID

$$\Rightarrow \int_{-H}^{\eta} \frac{\partial w}{\partial z}\, dz = \int_{-H}^{\eta} -\nabla \cdot \underline{u}_h\, dz = -(\eta + H)\nabla_h \cdot \underline{u}_h = -h\,\nabla_h \cdot \underline{u}_h$$

$$= w|_{\eta} - w|_{-H} = w|_{z=\eta} = \frac{D\eta}{Dt}$$

LINEARIZING GIVES

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \quad ③$$

①, ②, ③ ARE 3 EQUATIONS IN 3 UNKNOWNS u, v, η.

TOGETHER, THESE ARE THE LINEARIZED "SHALLOW WATER EQ^{NS}"

5 (CONT'D)

WORKING WITH THE FULLY NONLINEAR EQUATIONS CAN LIKEWISE FIND THE FULLY NONLINEAR SHALLOW WATER EQUATIONS:

$$① \quad \frac{Du}{Dt} - fv = -g\frac{\partial \eta}{\partial x}$$

$$② \quad \frac{Dv}{Dt} + fu = -g\frac{\partial \eta}{\partial y}$$

$$③ \quad \frac{Dh}{Dt} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \qquad \text{WITH } h = H + \eta$$

IN WHICH $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$ (NO $w\frac{\partial}{\partial z}$ TERM)

① - ③ GIVE TWO USEFUL DIAGNOSTIC EQUATIONS

A] POTENTIAL VORTICITY

LET $\zeta \equiv \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ BE THE VERTICAL COMPONENT OF VORTICITY

SO $\frac{\partial}{\partial x}② - \frac{\partial}{\partial y}① \Rightarrow \frac{\partial}{\partial x}\left[\frac{\partial v}{\partial t} + \underline{u}_h\cdot\nabla_h v + fu = -g\frac{\partial \eta}{\partial y}\right] - \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial t} + \underline{u}_h\cdot\nabla u - fv = -g\frac{\partial \eta}{\partial x}\right]$

$$\Rightarrow \underbrace{\frac{\partial}{\partial t}\zeta + \underline{u}_h\cdot\nabla\zeta}_{\frac{D\zeta}{Dt}} + \underbrace{\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial y}\right)}_{\zeta\,\nabla_h\cdot\underline{u}_h} + f\underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}_{\nabla_h\cdot\underline{u}_h} + \underbrace{\frac{df}{dy}}_{\beta}v = 0$$

$$\Rightarrow \frac{D\zeta}{Dt} + \zeta\,\nabla_h\cdot\underline{u}_h + f\,\nabla_h\cdot\underline{u}_h + \beta v = 0$$

USING ③ $\Rightarrow \frac{D\zeta}{Dt} + (\zeta + f)\left(-\frac{1}{h}\frac{Dh}{Dt}\right) + \beta v = 0$

$$\Rightarrow \frac{D(\zeta+f)}{Dt} - \frac{\zeta+f}{h}\frac{Dh}{Dt} = 0 \qquad \left(\text{SINCE } \frac{Df}{Dt} = v\frac{\partial f}{\partial y} = \beta v\right)$$

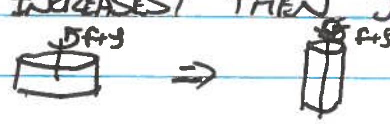
$$\Rightarrow \frac{D}{Dt}\left(\frac{\zeta+f}{h}\right) = 0 \qquad (*) \quad \text{(FROM RECIPROCAL RULE OF DERIVATIVES)}$$

SO DEFINE $Q \equiv \frac{\zeta+f}{h} = \frac{\zeta+f}{\eta+H}$ TO BE "POTENTIAL VORTICITY"

(*) SHOWS THAT Q IS CONSERVED: $\frac{DQ}{Dt} = 0$

INTERPRETATION: $\zeta + f$ IS $\overbrace{\text{RELATIVE + PLANETARY}}^{\text{ABSOLUTE}}$ VORTICITY.

IF THE VORTEX IS STRETCHED (h INCREASES) THEN ζ INCREASES TO KEEP $\frac{\zeta+f}{h}$ CONSTANT



FASTER SPIN AS DRAWN INWARDS

LINEAR APPROXIMATION ($\|\eta\| \ll H$)

$$Q \simeq \frac{\zeta+f}{H}\frac{1}{1+\eta/H} \simeq \frac{1}{H}(\zeta+f)\left(1-\frac{\eta}{H}\right) \approx \frac{1}{H}\left(\zeta + f - \frac{f}{H}\eta\right)$$

ON f-PLANE $\Rightarrow \zeta - \frac{f}{H}\eta$ CONSTANT IN TIME

ON β-PLANE $\Rightarrow \zeta + \beta y$ CONSTANT (IF η VERY SMALL)

ON SLOPING TOPOGRAPHY ($H = H_0 - sy$) $\Rightarrow \zeta + s\frac{f_0}{H_0}y$ CONSTANT $\Rightarrow \beta \approx f_0 s/H$ ← TOPOGRAPHIC 'β'

5 (CONT'D)

B] ENERGETICS

WORK WITH LINEARIZED SHALLOW WATER EQUATIONS

$$u\left(\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}\right) + v\left(\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}\right)$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\tfrac{1}{2}(u^2+v^2)\right) = -g\left(u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y}\right) = -g\left[\frac{\partial(u\eta)}{\partial x} + \frac{\partial(v\eta)}{\partial y} - \eta\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right]$$

$$= -g\,\nabla_h\cdot(\underline{u}_h\eta) - g\,\eta\frac{\partial w}{\partial z}$$

INTEGRATE VERTICALLY, RECALLING THAT $\underline{u}_h$ IS INDEPENDENT OF z

$$\Rightarrow \frac{\partial}{\partial t}\left(\tfrac{1}{2}H(u^2+v^2)\right) = -gH\,\nabla_h\cdot(\underline{u}_h\eta) - g\,\eta\left[\frac{\partial \eta}{\partial t} - \underbrace{0}_{w|_{z=0}}\right]$$

USE $\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$

$$\Rightarrow \frac{\partial}{\partial t}\left(\tfrac{1}{2}H(u^2+v^2)\right) = -gH\left[\nabla_h\cdot(\underline{u}_h\eta)\right] + gH\eta\,\nabla_h\cdot\underline{u}_h \qquad (1)$$

GET AVAILABLE POTENTIAL ENERGY FROM $\left[\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0\right] \times g\eta$

$$\Rightarrow \frac{\partial}{\partial t}\left(\tfrac{1}{2}g\eta^2\right) = -gH\eta\,\nabla_h\cdot\underline{u}_h \qquad (2)$$

SO DEFINE

$E_K \equiv \tfrac{1}{2}H(u^2+v^2)$ K.E. /MASS /AREA

$E_P \equiv \tfrac{1}{2}g\eta^2$ A.P.E

$\underline{F} = gH\,\underline{u}_h\,\eta$ ENERGY FLUX.

THEN (1) + (2) $\Rightarrow$ $\frac{\partial}{\partial t}\big(\underbrace{E_K + E_P}_{E}\big) = -\nabla_h\cdot\underline{F}$

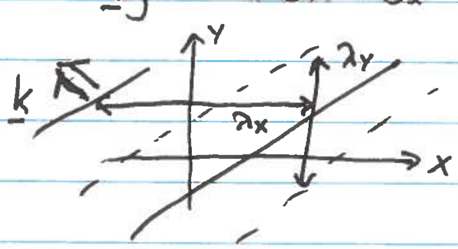
WHEN DEALING WITH WAVES OUR EXPECTATION IS THAT

$$\langle \underline{F} \rangle = \underline{c}_g \langle E \rangle$$

IN WHICH

$\underline{c}_g = \left(\frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial l}\right)$ WITH $\underline{k} = (k, l)$ THE WAVENUMBER w2D

i.e.

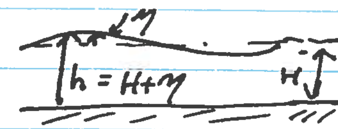


$$(k, l) = \left(\frac{2\pi}{\lambda_x}, \frac{2\pi}{\lambda_y}\right)$$

6 NON-ROTATING SHALLOW WATER WAVES

ASSUMING $f=0$, THE SHALLOW WATER EQUATIONS (1-3 ON p.63)

ARE:
1. $\frac{Du}{Dt} = -g\frac{\partial \eta}{\partial x}$
2. $\frac{Dv}{Dt} = -g\frac{\partial \eta}{\partial y}$
3. $\frac{Dh}{Dt} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$ WITH $h = H + \eta$



FOR SMALL AMPLITUDE WAVES, $\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u}\cdot\nabla \approx \frac{\partial}{\partial t}$.

ALSO, ARBITRARILY SUPPOSE MOTION IS IN X-DIRECTION ALONE.

$\Rightarrow u = A_u e^{i(kx-\omega t)}$ AND $\eta = A_0 e^{i(kx-\omega t)}$ (ACTUALLY REAL PARTS)

(SO 2 $\Rightarrow v = 0$)

FINALLY, WRITING $h = H + \eta$, WITH H CONSTANT, GIVES

1a $\quad \frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x}$

3a $\quad \frac{\partial \eta}{\partial t} + H\frac{\partial u}{\partial x} = 0$

SOLVE BY ELIMINATING u IN 1a + 3a:

$H\frac{\partial}{\partial x}$ 1a $- \frac{\partial}{\partial t}$ 3a $\Rightarrow \frac{\partial^2 \eta}{\partial t} = gH\frac{\partial^2 \eta}{\partial x^2}$

DEFINING $c = \sqrt{gH}$ GIVES THE "WAVE EQUATION"

$$\boxed{\frac{\partial^2 \eta}{\partial t^2} = c^2\frac{\partial^2 \eta}{\partial x^2}} \qquad (*)$$

WITH $\eta = Ae^{i(kx-\omega t)}$ $(*) \Rightarrow (-i\omega)^2 A_0 e^{i(kx-\omega t)} = c^2(ik)^2 A_0 e^{i(kx-\omega t)}$

$\Rightarrow \boxed{\omega^2 = c^2k^2}$

THIS IS THE SHALLOW WATER DISPERSION RELATION FOUND EARLIER (SEE p.60 OF NOTES).

FROM 1a $\quad -i\omega A_u e^{i(kx-\omega t)} = -g(ik)A_0 e^{i(kx-\omega t)}$

$\Rightarrow A_u = \frac{gk}{\omega}A_0 = \frac{g}{c}A_0 \overset{c=\sqrt{gH}}{\Longleftarrow} \frac{1}{H}cA_0 = \frac{\omega}{kH}A_0$

SO, TAKING REAL PARTS ASSUMING A_0 IS REAL

$\eta = A_0 e^{i(kx-\omega t)} \longrightarrow A_0\cos(kx-\omega t)$

$u = A_u e^{i(kx-\omega t)} \longrightarrow A_0\frac{\omega}{kH}\cos(kx-\omega t)$ (AS ON p.60)

7 Rotating shallow water waves on f-plane

(These are called "Poincaré waves" or "inertial waves")

In this case, the shallow water eq^{n}'s for small amplitude waves are:

① $\frac{\partial u}{\partial t} - f_0 v = -g\frac{\partial \eta}{\partial x}$

② $\frac{\partial v}{\partial t} + f_0 u = -g\frac{\partial \eta}{\partial y}$

③ $\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$

Even supposing propagation in x-alone, v cannot be ignored because of ②. Suppose $u = A_u e^{i(kx-\omega t)}$, $v = A_v e^{i(kx-\omega t)}$, $\eta = A_0 e^{i(kx-\omega t)}$

Then ①-③ $\Rightarrow$

$$-i\omega A_u - f_0 A_v = -g(ik)A_0$$
$$-i\omega A_v + f_0 A_u =$$
$$-i\omega A_0 + H(ikA_u) = 0$$

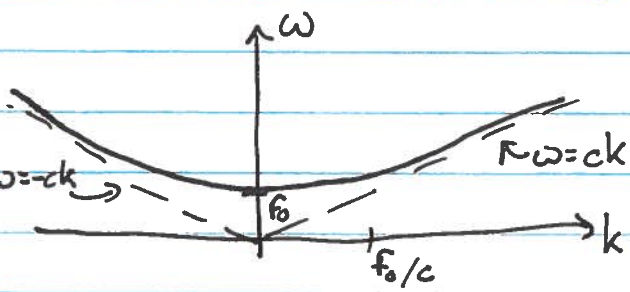
In matrix form

$$\Rightarrow \begin{pmatrix} -i\omega & -f_0 & gik \\ f_0 & -i\omega & 0 \\ Hik & 0 & -i\omega \end{pmatrix}\begin{pmatrix} A_u \\ A_v \\ A_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad (*)$$

For non-trivial solutions, the determinant of the matrix must be zero: $(-i\omega)(-i\omega)(-i\omega) - [(-f_0)(f_0)(-i\omega) + (gik)(-i\omega)(Hik)] = 0$

$$\Rightarrow \boxed{\omega^2 = c^2k^2 + f_0^2} \qquad \text{with} \quad c = \sqrt{gH}$$

This is the dispersion relation of Poincaré/inertial waves



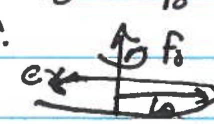
For $|k|$ 'large' $\Rightarrow$ $\omega \approx \pm ck$ shallow water gravity waves

For $|k|$ 'small' $\Rightarrow$ $\omega \approx f_0$

The cross-over from 'large' (rotation negligible) to 'small' (rotation dominant) occurs for $c|k| \sim f_0$.

This leads us to define the length scale $L_D = |k|^{-1} = \frac{c}{f_0} = \frac{\sqrt{gH}}{f_0}$

$\boxed{L_D \equiv \sqrt{gH}/f_0}$ is the "Rossby deformation radius".



It is the distance travelled by an object at speed c in time f_0^{-1}.

7 (cont'd)

EXAMPLES:

① FOR $H = 2.5\,cm$ IN A TANK OF WATER $\Rightarrow c = \sqrt{gH} \approx 50\,cm/s$

IF TANK ROTATES AT $\Omega = 0.5$ RAD/S $\Rightarrow f_0 = 1\,s^{-1}$

$\Rightarrow L_D = c/f_0 = 50\,cm$

② FOR $H = 4\,km$ DEEP OCEAN $\Rightarrow c = \sqrt{gH} \approx 200\,m/s$

AT MIDLATITUDES $L_D = c/f_0 \approx \frac{200\,m/s}{10^{-4}\,s^{-1}} \sim 2\times10^6 \sim 2000\,km$

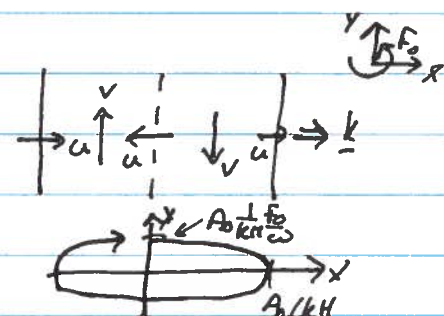
SO ROTATION IS SIGNIFICANT FOR FULL-DEPTH WAVES ONLY IF THEIR EXTENT IS COMPARABLE TO SCALE OF OCEAN BASIN.

POLARIZATION RELATIONS

FROM (*) ON p66 CAN RELATE A_u AND A_v TO A_0

$\Rightarrow A_u = \frac{\omega}{kH} A_0$ AND $A_v = \frac{-f_0}{-i\omega} A_u = -i\frac{f_0}{\omega}\left(\frac{\omega}{kH}\right)A_0 = -i\frac{f_0}{kH} A_0$

HENCE
$$\eta = A_0 e^{i(kx-\omega t)} \longrightarrow A_0 \cos(kx-\omega t)$$
$$u = A_u e^{i(kx-\omega t)} \longrightarrow A_0 \frac{\omega}{kH}\cos(kx-\omega t)$$
$$v = A_v e^{i(kx-\omega t)} \longrightarrow A_0 \frac{f_0}{kH}\sin(kx-\omega t)$$



SO FLUID PARCELS UNDERGO ELLIPTICAL ORBITS AS WAVE PASSES BY.

X-DISPLACEMENT: $\frac{\partial\xi_x}{\partial t} = u \Rightarrow \xi_x = -A_0\frac{1}{kH}\sin(kx-\omega t)$

Y-DISPLACEMENT: $\frac{\partial\xi_y}{\partial t} = v \Rightarrow \xi_y = A_0\frac{1}{kH}\frac{f_0}{\omega}\cos(kx-\omega t)$

(MOTION IS CIRCULAR AS $\omega \to f_0$, FOR VERY LONG WAVES WITH $|k| \ll 1/L_D$)

POTENTIAL VORTICITY

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = A_0\frac{f_0}{H}\cos(kx-\omega t)$$

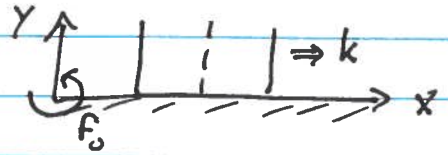
SO POTENTIAL VORTICITY IS $Q \equiv \frac{\zeta+f}{\eta+H} \approx \frac{f_0}{H}\left(1+\frac{\zeta}{f_0}-\frac{\eta}{H}\right) = \frac{f_0}{H}\left(1+\frac{A_0}{H}\cos() - \frac{A_0}{H}\cos()\right) = \frac{f_0}{H}$

$\Rightarrow Q$ IS CONSTANT, AS EXPECTED.

ENERGY: $\langle E\rangle = \langle KE\rangle + \langle APE\rangle = \left(\frac14 gA_0^2 + \frac12\frac{f_0^2}{k^2H}A_0^2\right) + \left(\frac14 gA_0^2\right) = \frac12\left(g+\frac{f_0^2}{k^2H}\right)A_0^2 = \frac{\omega^2}{2k^2H}A_0^2$

8 COASTAL KELVIN WAVES

NOW CONSIDER WAVES OF THE f-PLANE IN PRESENCE OF A SIDE BOUNDARY



MUST HAVE $v|_{y=0} = 0$. SO CANNOT ASSUME OSCILLATORY SOLUTIONS IN Y.

INSTEAD SUPPOSE $u = \hat{u}(y)e^{i(kx-\omega t)}$, $v = \hat{v}(y)e^{i(kx-\omega t)}$, $\eta = \hat{\eta}(y)e^{i(kx-\omega t)}$

SO LINEAR SHALLOW WATER EQUATIONS BECOME:

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} - f_0 v + g\frac{\partial \eta}{\partial x} = 0 \\ &\frac{\partial v}{\partial t} + f_0 u + g\frac{\partial \eta}{\partial y} = 0 \\ &\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} &-i\omega\hat{u} - f_0\hat{v} + ikg\hat{\eta} = 0 \quad (1) \\ &-i\omega\hat{v} + f_0\hat{u} + g\frac{d\hat{\eta}}{dy} = 0 \quad (2) \\ &-i\omega\hat{\eta} + ikH\hat{u} + H\frac{d\hat{v}}{dy} = 0 \quad (3) \end{aligned}$$

CAN COMBINE THESE TO GET 1 EQUATION IN $\hat{v}$ ALONE:

$$\frac{d^2\hat{v}}{dy^2} - \gamma^2\hat{v} = 0 \qquad \text{IN WHICH } \gamma^2 \equiv \frac{1}{c^2}(c^2k^2 + f_0^2 - \omega^2), \ (*) \quad c = \sqrt{gH}$$

IF $\gamma^2 < 0 \Rightarrow \gamma = il$ WITH l REAL $\Rightarrow \hat{v}(y) = e^{ily}$, SO $(*) \Rightarrow \omega^2 = c^2(k^2+l^2) - f_0^2$

WHICH IS DISPERSION RELATION FOR INERTIA WAVES IN X-Y PLANE. THE CONDITION $v|_{y=0}$ MEANS THIS CASE DESCRIBES INCIDENT WAVES REFLECTING FROM COAST.

IF $\gamma^2 > 0 \Rightarrow \hat{v}(y) = c_1 e^{\gamma y} + c_2 e^{-\gamma y}$.

SUPPOSING $\gamma > 0 \Rightarrow c_1 = 0$ SO WAVES ARE BOUNDED AS $y \to \infty$

BUT $\hat{v}(0) = 0 \Rightarrow c_2 = 0$. SO $v \equiv 0$ EVERYWHERE.

IN THIS CASE (1) → (3) ⇒

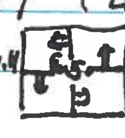
$$\begin{aligned} -i\omega\hat{u} + ikg\hat{\eta} &= 0 \quad (1)' \\ f_0\hat{u} + g\frac{d\hat{\eta}}{dy} &= 0 \quad (2)' \\ -i\omega\hat{\eta} + ikH\hat{u} &= 0 \quad (3)' \end{aligned}$$

(1)' & (3)' $\Rightarrow \begin{pmatrix} -i\omega & ikH \\ ikg & -i\omega \end{pmatrix}\begin{pmatrix} \hat{\eta} \\ \hat{u} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \boxed{\omega^2 = gHk^2 = c^2k^2}$ (AS FOR SHALLOW WATER WAVES)

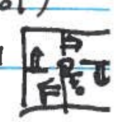
TAKE $\omega = ck$ FOR $k > 0$ (EASTWARD) AND $\omega = -ck$ FOR $k < 0$ (WESTWARD)

THEN (1)' & (2)' $\Rightarrow \frac{d\hat{\eta}}{dy} = -\frac{f_0}{g}\hat{u} = -\frac{f_0}{g}\left(\frac{kg}{\omega}\hat{\eta}\right) = -\frac{f_0 k}{\omega}\hat{\eta} = -\mathrm{SIGN}(f_0 k)\frac{1}{L_0}\hat{\eta}$ $\left(L_0 = \left|\frac{c}{f_0}\right|\right)$

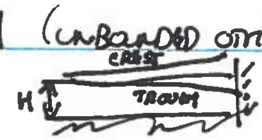
SOLUTION IS $\hat{\eta} = A_0 e^{-y/L_0}$ IF $\mathrm{SIGN}(kf_0) = +1$ (UNBOUNDED OTHERWISE) N.H.



S.H.



$\Rightarrow \eta(x,y,t) = A_0 e^{-y/L_0}\cos(kx - \omega t)$



9 ROSSBY WAVES

THESE ARE WAVES THAT ARE INFLUENCED BY THE β - EFFECT.

RECALL THAT POTENTIAL VORTICITY $q = \zeta + \beta y - \frac{f_0}{H}\eta$ IS CONSERVED

$\Rightarrow \frac{Dq}{Dt} = 0 \Rightarrow \frac{\partial q}{\partial t} \approx 0 \Rightarrow \frac{\partial \zeta}{\partial t} + \beta v - \frac{f_0}{H}\frac{\partial \eta}{\partial t} = 0$ (*) (USING $\frac{\partial y}{\partial t} = v$)

CAN RELATE ζ AND η THROUGH GEOSTROPHIC BALANCE:

$$-f_0 v = -g\frac{\partial \eta}{\partial x}$$
$$f_0 u = -g\frac{\partial \eta}{\partial y}$$

SO $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left(\frac{g}{f_0}\frac{\partial \eta}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{g}{f_0}\frac{\partial \eta}{\partial y}\right) = \frac{g}{f_0}\nabla^2\eta$ (WITH $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$)

SO (*) $\Rightarrow \frac{\partial}{\partial t}\left[\frac{g}{f_0}\nabla^2\eta - \frac{f_0}{H}\eta\right] + \beta\left(\frac{g}{f_0}\frac{\partial \eta}{\partial x}\right) = 0$

$\times\frac{f_0}{g} \Rightarrow \boxed{\frac{\partial}{\partial t}\left[\nabla^2\eta - \frac{1}{L_D^2}\eta\right] + \beta\frac{\partial \eta}{\partial x} = 0}$ (1) WITH $L_D = \frac{\sqrt{gH}}{f_0} = \frac{c}{f_0}$

THIS IS THE <u>QUASIGEOSTROPHIC POTENTIAL VORTICITY</u> (QGPV) EQUATION

SOLVE FOR WAVES IN X-Y PLANE: $\eta = A_0 e^{i(kx+ly-\omega t)}$

(1) $\Rightarrow -i\omega\left[\left((ik)^2 + (il)^2\right) - \frac{1}{L_D^2}\right] + \beta(ik) = 0$

$\Rightarrow \boxed{\omega = -\beta k / \left[k^2 + l^2 + 1/L_D^2\right]}$

THIS IS <u>THE DISPERSION RELATION FOR ROSSBY WAVES</u>

PHASE VELOCITY: $\underline{c}_p = \frac{\omega}{|\underline{k}|}\frac{\underline{k}}{|\underline{k}|} = \frac{\beta}{|\underline{k}|^2 + 1/L_D^2}\frac{1}{|\underline{k}|^2}\left(-k^2, kl\right)$

NOTE $c_{px} = -\beta k^2 / \left[|\underline{k}|^2\left(|\underline{k}|^2 + 1/L_D^2\right)\right] < 0 \Rightarrow$ CRESTS ALWAYS GO WESTWARD!

GROUP VELOCITY: $\underline{c}_g = \nabla_k \omega = \frac{\beta}{[|\underline{k}|^2 + 1/L_D^2]^2}\left(k^2 - l^2 - 1/L_D^2,\ 2kl\right)$

SO $c_{gx} < 0$ FOR LONG WAVES WITH $k < \sqrt{l^2 + 1/L_D^2}$

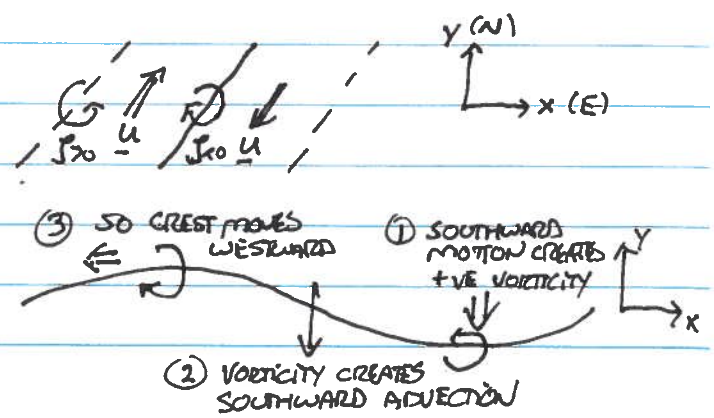
POLARIZATION RELATIONS:

$\eta = A_0\cos(kx+ly-\omega t)$

$u = A_0\frac{gl}{f_0}\sin(kx+ly-\omega t)$

$v = A_0\left(-\frac{gk}{f_0}\right)\sin(kx+ly-\omega t)$

$\zeta = A_0\left[-\frac{g}{f_0}(k^2+l^2)\right]\cos(kx+ly-\omega t)$



10 EQUATORIAL WAVES

THE β-EFFECT AT EQUATOR LEADS TO A DISTINCT SET OF WAVES 'TRAPPED' AT EQUATOR. THESE ARE ANALOGOUS TO INERTIA, KELVIN AND ROSSBY WAVES AT MIDLATITUDES.

THE SHALLOW WATER EQUATIONS ON EQUATORIAL β PLANE ARE

$$\frac{\partial u}{\partial t} - (\beta y)v = -g\frac{\partial \eta}{\partial x} \quad (1)$$
$$\frac{\partial v}{\partial t} + (\beta y)u = -g\frac{\partial \eta}{\partial y} \quad (2)$$
$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \quad (3)$$

(WHERE $\frac{D}{Dt} \approx \frac{\partial}{\partial t}$ FOR SMALL AMPLITUDE WAVES)

A] EQUATORIAL KELVIN WAVES

LET $v \equiv 0$ AND WRITE $u = \hat{u}(y)e^{i(kx-\omega t)}$, $\eta = \hat{\eta}(y)e^{i(kx-\omega t)}$

$(1) \Rightarrow -i\omega\hat{u} = -g(ik)\hat{\eta} \quad (1)'$

$(2) \Rightarrow \beta y\hat{u} = -g\frac{d\hat{\eta}}{dy} \quad (2)'$

$(3) \Rightarrow -i\omega\hat{\eta} + H(ik)\hat{u} = 0 \quad (3)'$

$(1)' \,\&\, (3)' \Rightarrow \omega = ck$ ($k>0$, EASTWARD) OR $\omega = -ck$ ($k<0$, WESTWARD)

$(1)' \,\&\, (2)' \Rightarrow \frac{d\hat{\eta}}{dy} = -\frac{\beta y}{g}\left(\frac{kg}{\omega}\right)\hat{\eta} = -\mathrm{SIGN}(k)\frac{\beta}{c}y\hat{\eta} \quad (4)$

DEFINE $\boxed{L_\beta \equiv \sqrt{c/\beta} = \left(gH/\beta^2\right)^{1/4}}$ THE EQUATORIAL DEFORMATION RADIUS

(e.g. FOR $H = 5$km $\Rightarrow L_\beta \approx 3000$ km $\approx 30°$N LATITUDE : LARGE)

SO $(4) \Rightarrow \frac{d\hat{\eta}}{dy} = -\mathrm{SIGN}(k)\frac{1}{L_\beta^2}y\hat{\eta}$

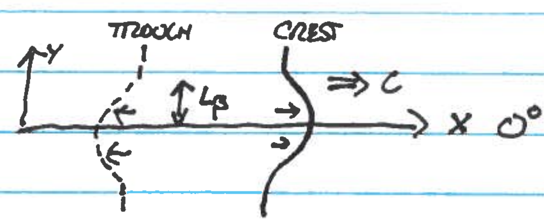
$\Rightarrow \hat{\eta}(y) = A_0 e^{-y^2/2L_\beta^2}$, $k>0$ OR $A_0 e^{+y^2/2L_\beta^2}$, $k<0$

FOR BOUNDED SOLUTION, MUST HAVE $k>0 \Rightarrow$ EASTWARD PROPAGATION

SO FOR EQUATORIAL KELVIN WAVES

$\omega = ck \quad (k>0)$

$\eta = A_0 e^{-y^2/2L_\beta^2}\cos(kx-\omega t)$



10 (cont'd)

B] EQUATORIAL INERTIAL, ROSSBY & YANAI WAVES

NOW ALLOW MOTION IN Y-DIRECTION:

$$u = \hat{u}(y)e^{i(kx-\omega t)}, \quad v = \hat{v}(y)e^{i(kx-\omega t)}, \quad \eta = \hat{\eta}(y)e^{i(kx-\omega t)}$$

SO ① - ③ ON p.70 BECOME

$$-i\omega\hat{u} - \beta y\hat{v} = -g(ik)\hat{\eta} \qquad ①'$$

$$-i\omega\hat{v} + \beta y\hat{u} = -g\frac{d\hat{\eta}}{dy} \qquad ②'$$

$$-i\omega\hat{\eta} + H\left(ik\hat{u} + \frac{d\hat{v}}{dy}\right) = 0 \qquad ③'$$

ELIMINATE $\hat{u}$ IN ①' & ②'

$$\Rightarrow \left[\omega^2 - (\beta y)^2\right]\hat{v} = -i\omega g\frac{d\hat{\eta}}{dy} - ikg(\beta y)\hat{\eta}$$

ELIMINATE $\hat{u}$ IN ①' & ③'

$$\Rightarrow \left[\omega^2 - gHk^2\right]\hat{\eta} = -i\omega H\frac{d\hat{v}}{dy} + ikH(\beta y)\hat{v}$$

NOW ELIMINATE $\hat{\eta}$ FROM THIS PAIR OF EQUATIONS

$$\Rightarrow \frac{d^2\hat{v}}{dy^2} + \frac{1}{L_\beta^2}\left[\left(\left(\frac{\omega}{\beta L_\beta}\right)^2 - (kL_\beta)^2 - (kL_\beta)\left(\frac{\beta L_\beta}{\omega}\right)\right) - \frac{1}{L_\beta^2}y^2\right]\hat{v} = 0$$

WITH $L_\beta \equiv (c/\beta)^{1/2} = (gH/\beta^2)^{1/4}$ AS BEFORE

SO NEED TO SOLVE EQUATION OF FORM $\hat{v}'' + \frac{1}{L_\beta^2}\left(\alpha^2 - \frac{y^2}{L_\beta^2}\right)\hat{v} = 0$

WITH $\alpha^2 \equiv \left(\frac{\omega}{\omega_\beta}\right)^2 - (kL_\beta) - (kL_\beta)\left(\frac{\omega_\beta}{\omega}\right)$ (*), $\omega_\beta \equiv \beta L_\beta = (\beta c)^{1/2}$

SOLUTIONS ARE "PARABOLIC CYLINDER FUNCTIONS":

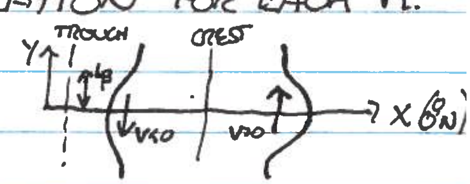
$\hat{v}_n(y) = e^{-y^2/2L_\beta^2}H_n(y/L_\beta)$, WHERE $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2, \ldots$

ARE "HERMITE POLYNOMIALS"

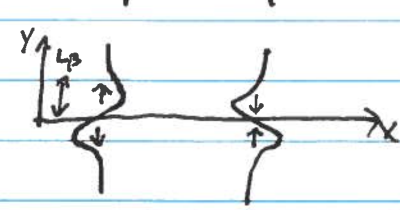
ONLY GET THESE SOLUTIONS IF $\alpha^2 = 2n+1$ FOR $n = 0, 1, 2, \ldots$

USING (*) THIS GIVES DISPERSION RELATION FOR EACH n.

E.G. $n = 0 \Rightarrow v = A_v e^{-y^2/2L_\beta^2}e^{i(kx-\omega t)}$



$n = 1 \Rightarrow v = A_v\left(2\frac{y}{L_\beta}\right)e^{-y^2/2L_\beta^2}e^{i(kx-\omega t)}$



10 B] (CONT'd)

GET MORE INSIGHT BY CONSIDERING 3 SPECIAL CASES OF THE DISPERSION RELATION: (*) $\left(\frac{\omega}{\omega_\beta}\right)^2 - (kL_\beta)^2 - (kL_\beta)\left(\frac{\omega_\beta}{\omega}\right) = 2n+1$, $n=0,1,2,\ldots$
WITH $L_\beta = \sqrt{c/\beta}$, $\omega_\beta = \beta L_\beta = \sqrt{\beta c}$

i) $n>0$ AND $\omega \gg \omega_\beta (kL_\beta) = ck$

SO (*) $\Rightarrow \left(\frac{\omega}{\omega_\beta}\right)^2 - (kL_\beta)^2 \approx 2n+1$, $n=1,2,\ldots$

$$\Rightarrow \omega^2 \approx \omega_\beta^2 (kL_\beta)^2 + (2n+1)\omega_\beta^2$$

$$\Rightarrow \boxed{\omega^2 = c^2k^2 + (2n+1)\omega_\beta^2}$$

(COMPARE THIS TO DISPERSION RELATION FOR INERTIAL (POINCARÉ) WAVES! $\omega^2 = c^2k^2 + f_0^2$)

SO THIS CASE DESCRIBES "EQUATORIAL INERTIAL WAVES"

ii) $n>0$ AND $\omega \ll \omega_\beta$

SO (*) $\Rightarrow -(kL_\beta)^2 - (kL_\beta)\left(\frac{\omega_\beta}{\omega}\right) \approx 2n+1$, $n=1,2,\ldots$

$$\Rightarrow \omega \approx -(kL_\beta\omega_\beta)/[(kL_\beta)^2 + 2n+1]$$

$$\Rightarrow \boxed{\omega \approx -kc/[(kL_\beta)^2 + (2n+1)]} = -\beta k/[k^2 + \tfrac{2n+1}{L_\beta^2}]$$

(COMPARE TO ROSSBY WAVE DISPERSION RELATION: $\omega = -\beta k/(k^2 + \frac{1}{L_D^2})$)

SO THIS CASE DESCRIBES "EQUATORIAL ROSSBY WAVES"

iii) $n=0$

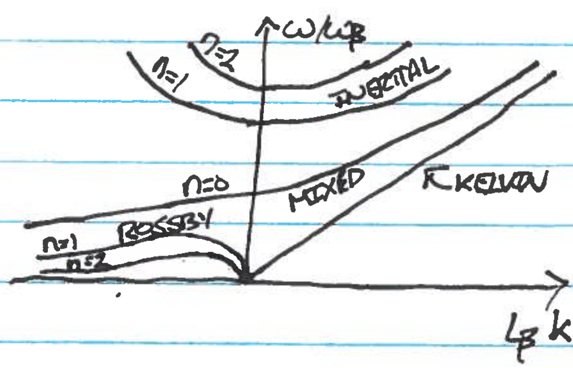
SO (*) $\Rightarrow \left(\frac{\omega}{\omega_\beta}\right)^3 - (kL_\beta)^2\left(\frac{\omega}{\omega_\beta}\right) - (kL_\beta) = \left(\frac{\omega}{\omega_\beta}\right)$

WHICH FACTORS: $\left(\left(\frac{\omega}{\omega_\beta}\right)^2 - (kL_\beta)\left(\frac{\omega}{\omega_\beta}\right) - 1\right)\left(\left(\frac{\omega}{\omega_\beta}\right) + (kL_\beta)\right) = 0$

$$\Rightarrow (\omega^2 - (kc)\omega - \beta c)(\omega + kc) = 0$$

$\omega = -kc$ SOLUTION IS UNBOUNDED SO IGNORE

$$\Rightarrow \boxed{\omega = \tfrac{1}{2}kc\left[1 \pm \left(1 + (2(kL_\beta)^2)\right)^{1/2}\right]}$$



THIS IS THE DISPERSION RELATION OF "YANAI" OR "MIXED ROSSBY-INERTIAL" WAVES