Assignment 1: Solutions

(1) a) (H) $\xleftrightarrow{\quad 1000\text{ km} \quad}$ (L)

103 kPa   99 kPa

THE PRESSURE GRADIENT FORCE IS $-\frac{\partial P}{\partial x} \simeq -\frac{(99\times10^3\,Pa - 103\times10^3\,Pa)}{10^6\,m}$

$$\Rightarrow -\frac{\partial P}{\partial x} \simeq 4\times10^{-3}\ Pa/m = 4\times10^{-3}\ N/m^3$$

THE CORRESPONDING ACCELERATION IS $-\frac{1}{\rho}\frac{\partial P}{\partial x} \simeq \frac{4\times10^{-3}\ N/m^3}{1.2\ kg/m^3}$

$$\Rightarrow \left\|\frac{Du}{Dt}\right\|\Bigg|_{\text{Pressure}} \simeq \underline{\underline{3.3\times10^{-3}\ m/s^2 \quad \text{EASTWARD}}}$$

b) (H) $\Downarrow$ 10 m/s   (L)

EDMONTON'S LATITUDE IS $\approx 53.5°N$

$$\Rightarrow f = 2\Omega_e \sin\theta = 2\,(2\pi/((24\times60\times60)s))\ \sin 53.5$$
$$\simeq (1.45\times10^{-4}\ s^{-1})(0.804)$$
$$= 1.17\times10^{-4}\ s^{-1}$$

From $\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \mu\nabla^2 u$

SEE THAT SOUTHWARD FLOW ($v = -10\,m/s$) CONTRIBUTES TO AN ACCELERATION IN X-DIRECTION:

$$\frac{Du}{Dt}\bigg|_{\text{coriolis}} = fv = (1.17\times10^{-4}\,s^{-1})(-10\,m/s)$$
$$= -1.17\times10^{-3}\ m/s^2$$

I.e. $\left\|\frac{Du}{Dt}\right\|\Bigg|_{\text{coriolis}} = \underline{\underline{1.17\times10^{-3}\ m/s^2 \quad \text{WESTWARD}}}$

② a) $\rho = P_0/(R_aT) = (2.50\times10^4\,\mathrm{Pa})/(287\,\frac{\mathrm{J}}{\mathrm{kg\,K}})\cdot(193\,\mathrm{K})$

$\simeq \underline{\underline{4.51\times10^{-1}\ \mathrm{kg/m^3}}}$

b) $H_\rho = RT_0/g \simeq (287\,\frac{\mathrm{J}}{\mathrm{kg\,K}})(193\,\mathrm{K})/(9.81\,\mathrm{m/s^2})$

$\simeq 5.65\times10^3\,\mathrm{m} = \underline{\underline{5.65\ \mathrm{km}}}$

c) FOR ISOTHERMAL AIR $\bar{p}(z) = P_{\infty} e^{-z/H_\rho}$

GIVEN $p = P_0 = 250$ mBAR AT $z = z_0 = 10\,\mathrm{km}$, WRITE AS

$\bar{p}(z) = P_0\, e^{-(z-z_0)/H_\rho}$

SO AT $z = 20\,\mathrm{km}$ $\bar{p}(20\,\mathrm{km}) = (250\,\mathrm{mBAR})\,\mathrm{EXP}[-(20-10\,\mathrm{km})/5.65\,\mathrm{km}]$

$\simeq \underline{\underline{42.6\ \mathrm{mBAR}}}$

d) $\theta = T(P/P_R)^{-\kappa} \simeq (193\,\mathrm{K})[42.6\,\mathrm{mBAR}/1000\,\mathrm{mBAR}]^{-2/7}$

$\simeq (193\,\mathrm{K})(2.46) \simeq \underline{\underline{476\,\mathrm{K}}}$

e) θ IS CONSTANT DURING ADIABATIC DESCENT

$\Rightarrow \theta_{PARCEL} = 476\,\mathrm{K}$ AT $z = 10\,\mathrm{km}$ WHERE $p = P_0 = 250\,\mathrm{mBAR}$

$\Rightarrow T_{PARCEL} = \theta\,(P_0/P_R)^{+\kappa} \simeq (476\,\mathrm{K})[250/1000]^{2/7}$

$\simeq 320\,\mathrm{K}$

$\Rightarrow \rho_{PARCEL} = P_0/(RT) \simeq (2.50\times10^4\,\mathrm{Pa})/(287\,\frac{\mathrm{J}}{\mathrm{kg\,K}})\cdot(320\,\mathrm{K})$

$\simeq \underline{\underline{2.72\times10^{-1}\ \mathrm{kg/m^3}}}$

BUOYANCY IS $b = g[(\theta_{PARCEL} - \bar{\theta})/\bar{\theta}]$

WHERE $\bar{\theta}(10\,\mathrm{km}) = T(P(10\,\mathrm{km})/P_R)^{-\kappa} \simeq 193\,\mathrm{K}\,[250/1000]^{-2/7} \simeq 287\,\mathrm{K}$

$\Rightarrow b \simeq (9.81\,\mathrm{m/s^2})[(476-287)/287] \simeq \underline{\underline{6.47\ \mathrm{m/s^2}}}$

SINCE $\rho_{PARCEL} < \rho(10\,\mathrm{km})$ AND $b > 0 \Rightarrow$ PARCEL WILL RISE.

(3) FOR ATMOSPHERE, THE SQUARED BUOYANCY FREQUENCY IS $N^2 = \frac{g}{\bar{\theta}} \frac{d\bar{\theta}}{dz}$

WHERE $\bar{\theta}(z) = \bar{T}(z) \left[\bar{P}(z)/P_R \right]^{-\varkappa}$

IN WHICH $\bar{T} = T_0 = 193\,K$ AND $\bar{P} = P_0 e^{-(z-z_0)/H_\rho}$

$$\Rightarrow \bar{\theta}(z) = T_0 \left[(P_0/P_R)\, e^{-(z-z_0)/H_\rho} \right]^{-\varkappa}$$

$$= T_0 \,(P_0/P_R)^{-\varkappa} \; \mathrm{EXP}\left[+\varkappa(z-z_0)/H_\rho \right]$$

So $\frac{d\bar{\theta}}{dz} = T_0 (P_0/P_R)^{-\varkappa} \, \frac{\varkappa}{H_\rho} \, \mathrm{EXP}\left[+\varkappa(z-z_0)/H_\rho \right]$

$$\Rightarrow \frac{1}{\bar{\theta}} \frac{d\bar{\theta}}{dz} = \varkappa/H_\rho$$

So $N^2 = g\,\varkappa/H_\rho \simeq (9.81\,m/s^2)(2/7)/(5.65 \times 10^3\,m)$

$$\simeq 4.96 \times 10^{-4}\, s^{-2}$$

$$\Rightarrow \underline{\underline{N \simeq 2.23 \times 10^{-2}\, s^{-1}}}$$

BUOYANCY PERIOD IS $T_B = \frac{2\pi}{N} = 282\,s \simeq 4.69$ MINUTES

(4) FOR SALT WATER NEAR $T_0 = 20°C$, SALINITY $S_0 = 0$ ppt AND PRESSURE $P_0 = 100$ kPa

$$\rho \approx \rho_0 [1 - \beta_T(T - T_0) + \beta_S(S - S_0) + \beta_P(P - P_0)]$$

WITH $\rho_0 \approx 998.23$ kg/m^3

$\beta_T \approx 2.1 \times 10^{-4}$ K^{-1}, $\beta_S \approx 7.6 \times 10^{-4}$ ppt^{-1}, $\beta_P \approx 4.1 \times 10^{-10}$ Pa^{-1}

a) IF $T = 21°C$, $S = 0$ ppt, $P = 1000$ mBAR

$$\rho \simeq \rho_0 [1 - \beta_T(T - T_0) + 0 + 0]$$
$$\simeq 998.23 [1 - (2.1 \times 10^{-4} \text{ K}^{-1})(21 - 20 \text{ K})]$$
$$\simeq 998.23 - 0.21 \simeq \underline{\underline{998.02 \text{ kg/m}^3}}$$

b) WANT TO FIND S SO THAT

$$\rho_0 [1 - \beta_T(\Delta T) + \beta_S(S)] = \rho_0$$
$$\Rightarrow S = \beta_T(\Delta T)/\beta_S \simeq (2.1 \times 10^{-4} \text{ K}^{-1})(1 \text{ K})/(7.4 \times 10^{-4} \text{ ppt}^{-1})$$
$$\simeq \underline{\underline{0.28 \text{ ppt}}}$$ (i.e. 28g NaCl PER L OF SALT WATER)

c) WANT TO FIND P SO THAT

$$\rho_0 [1 + \beta_P(P - P_0)] = \rho_0 [1 + \beta_S(S)]$$
$$\Rightarrow P = P_0 + \beta_S(S)/\beta_P$$
$$= 10^5 \text{ Pa} + (7.4 \times 10^{-4} \text{ ppt}^{-1})(0.3 \text{ ppt})/(4.1 \times 10^{-10} \text{ Pa}^{-1})$$
$$= 10^5 \text{ Pa} + 5.4 \times 10^5 \text{ Pa}$$
$$\simeq 6.4 \times 10^5 \text{ Pa} = \underline{\underline{6.4 \text{ BARS}}}$$

d) ESTIMATE THE OCEAN HAS A NEAR-CONSTANT DENSITY ρ_0.

FROM HYDROSTATIC BALANCE $\frac{dP}{dz} = -\rho g \Rightarrow P(z) = P_0 - \rho_0 g z$

$$\Rightarrow z = (P_0 - P)/(\rho_0 g)$$ (1000 kg/m^3 WORKS JUST AS WELL FOR ACCURACY)

$$\simeq (10^5 \text{ Pa} - 6.4 \times 10^5 \text{ Pa})/(1030 \text{ kg/m}^3 \times 9.81 \text{ m/s}^2)$$
$$\simeq \underline{\underline{-53 \text{ m}}}$$

⑤ a) $\rho_{POT} \equiv \rho + \rho_0\, z/H_\rho$

WITH $H_\rho \equiv C_s^2/g \approx (1500\,m/s)^2/(9.81\,m/s^2) \simeq 2.29\times10^5\,m$

$\Rightarrow \rho = \rho_{POT} - \rho_0\, z/H_\rho$ WITH $\rho_0 = 1027.8\,kg/m^3 = \rho|_{z=0}$

$\simeq 1027.8\,kg/m^3\,(1 - (-1000m)/(2.29\times10^5 m))$

$\simeq \underline{\underline{1032.3\,kg/m^3}}$

b) $\theta = T + T_0\, z/H_T$

WITH $H_T \simeq 2000\,km$ AND $T_0 = 2^\circ C \simeq 275\,K$

BECAUSE DESCENT IS ADIABATIC, $\theta = 275\,K$ AT $z = -1000m$.

$\Rightarrow T = \theta - T_0\, z/H_T$

$\simeq 275\,K\,(1 - (-1000\,m)/(2\times10^6 m))$

$\simeq 275.14\,K$

SO TEMPERATURE INCREASE IS $\underline{\underline{0.14^\circ C}}$

6) a) FOR ISOTHERMAL ATMOSPHERE $\bar{\rho}(z) = \rho_0 e^{-z/H_\rho}$
WITH $H_\rho = RT_0/g \simeq (287\ \text{J/kg K})(240\text{K})/(9.81\text{m/s}^2) \simeq 7.02\times10^3\text{m}$
AND ESTIMATE $\rho_0 \simeq 1.2\ \text{kg/m}^3$

SO MASS/(AREA IN HORIZONTAL) $= \int_0^{H} \bar{\rho}\, dz = -H_\rho \rho_0 e^{-z/H_\rho}\Big|_0^H$

$= -H_\rho\rho_0[e^{-H/H_\rho} - 1] = H_\rho\rho_0[1-e^{-H/H_\rho}]$

$\simeq (7.02\times10^3\text{m})(1.2\ \text{kg/m}^3)[1 - e^{-10\text{km}/7.02\text{km}}] \simeq 6.40\times10^3\ \text{kg/m}^2$

BECAUSE TROPOSPHERE IS THIN COMPARED TO EARTH'S RADIUS, CAN SIMPLY MULTIPLY BY EARTH'S SURFACE AREA $4\pi R_e^2$ TO GET MASS:

$m = (6.40\times10^3\ \text{kg/m}^2)(4\pi\times(6.371\times10^6\text{m})^2) \simeq \underline{3.26\times10^{18}}\ \text{kg}$

b) ENERGY TO RAISE MASS, m, BY ΔT IS $E_{AIR} = c_v m \Delta T$

$\Rightarrow E_{AIR} \simeq (718\ \text{J/(kg K)})(3.26\times10^{18}\ \text{kg})(1\text{K})$

$\simeq \underline{2.34\times10^{21}\ \text{J}}$

c) INCIDENT ENERGY SHINES ON A CROSS-SECTIONAL AREA πR_e^2
SO POWER REACHING EARTH IS

$P \simeq F_s \pi R_e^2 \simeq (1370\ \text{W/m}^2)\pi(6.371\times10^6\text{m})^2 \simeq 1.75\times10^{17}\ \text{W}$

SO TIME TO INPUT ENERGY OF E_{AIR} IS

$t = E_{AIR}/P \simeq (2.34\times10^{21}\text{J})/(1.75\times10^{17}\text{J/s}) \simeq 1.34\times10^4\text{s}$

$\simeq \underline{3.72\ \text{HOURS}}$

FRACTION OF OCEAN COVER

d) MASS OF OCEAN OVER DEPTH h IS $m = \rho_0 h (0.714\pi R_e^2)$

$\Rightarrow m \simeq (1028\ \text{kg/m}^3)(10\text{m})(0.714\pi\times(6.371\times10^6\text{m})^2)$

$\simeq 3.72\times10^{18}\ \text{kg}$

SO ENERGY TO WARM BY $\Delta T = 1\text{K}$ IS $E_{OCEAN} = c_v m \Delta T$

$\Rightarrow E_{OCEAN} \simeq (4186\ \text{J/(kg K)})(3.72\times10^{18}\ \text{kg})(1\text{K})$

$\simeq \underline{1.56\times10^{22}\ \text{J}}$ ← NOTE! LARGER THAN E_{AIR}

.. THE OCEANS HOLD A LOT OF HEAT!