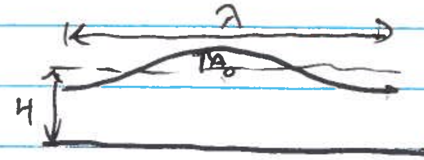
Assignment 3 Solutions



1 a) $A_0 = 0.1\,cm$, $H = 10\,cm$, $\lambda = 6\,cm$

$\Rightarrow k = \frac{2\pi}{\lambda} \simeq 1.05\,cm^{-1}$

BECAUSE $kH \approx 10 \gg 1$ CAN TREAT AS "DEEP" WATER WAVE

SO $\omega = \sqrt{gk} \simeq [(980\,cm/s^2)(1.05\,cm^{-1})]^{1/2} \simeq 32\,s^{-1}$

$\Rightarrow$ PHASE SPEED $C_p = \frac{\omega}{k} \simeq 31\,cm/s$

GROUP SPEED $C_g = \frac{d\omega}{dk} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}C_p \simeq 15\,cm/s$

AT CREST $W=0$ AND $u = \omega A_0 e^{kz}\big|_{z=0} = \omega A_0 = (32\,s^{-1})(0.1\,cm) \simeq 3.2\,cm/s$

b) $A_0 = 1\,cm$, $H = 10\,cm$, $\lambda = 60\,cm$

$\Rightarrow k \simeq 0.105\,cm^{-1}$

BECAUSE $kH \approx 1$ WAVE IS NEITHER DEEP OR SHALLOW

$\Rightarrow \omega = (gk \tanh kH)^{1/2} = (980 \times 0.105 \tanh(1.05))^{1/2} \simeq 9.0\,s^{-1}$

PHASE SPEED $C_p = \frac{\omega}{k} \simeq 85\,cm/s$

GROUP SPEED $C_g = \frac{d}{dk}(gk \tanh kH)^{1/2} = \frac{1}{2}(gk \tanh kH)^{-1/2}(g \tanh kH + gkH \operatorname{sech}^2 kH)$

$= \frac{1}{2}\frac{1}{k}(gk \tanh kH)^{1/2}\left(1 + kH \frac{1}{\sinh kH \cosh kH}\right)$

$= \frac{1}{2}\frac{\omega}{k}(1 + 2kH/\sinh(2kH))$

$\simeq 65\,cm/s$

AT CREST $W=0$ AND $u = \omega A_0 \frac{\cosh k(z+H)}{\sinh kH}\Big|_{z=0} = \frac{\omega A_0}{\tanh kH} \simeq 12\,cm/s$

c) $A_0 =$ [illegible] cm, $H = 10\,cm$, $\lambda = 600\,cm$

$\Rightarrow k = 0.0105\,cm^{-1}$ $\Rightarrow kH \simeq 0.10 \ll 1$ SO APPROXIMATELY SHALLOW

$\Rightarrow \omega = \sqrt{gH}\,k \simeq \sqrt{980 \times 10}\,k \simeq (99\,cm/s)\,k \simeq 1.04\,s^{-1}$

PHASE SPEED $C_p = \sqrt{gH} \simeq 99\,cm/s$

GROUP SPEED $C_g = \sqrt{gH} \simeq 99\,cm/s$

AT CREST $u = \omega A_0/(kH) \simeq 9.9\,cm/s$

2) THE STOKES DRIFT IS

$$\langle u \rangle = \omega k A_0^2 e^{2kz}$$

So, AT SURFACE THE SURFER DRIFTS AT SPEED

$$\langle u \rangle|_{z=0} = \omega k A_0^2$$

IN WHICH $A_0 = 0.20\,m$

$$k = \frac{2\pi}{20\,m} \simeq 0.31\,m^{-1}$$

AND $\omega = \sqrt{gk} = (9.8\,m/s^2 \cdot 0.31\,m^{-1})^{1/2} \simeq 1.8\,s^{-1}$

$$\Rightarrow \langle u \rangle|_{z=0} \simeq (1.8\,s^{-1})(0.31\,m^{-1})(0.20\,m)^2 \simeq 0.022\ m/s$$

3) a) $c = \sqrt{(9.8\,m/s^2)(4000\,m)} \simeq 2.0\times10^2\ m/s$

$\lambda = 200\,km = 2\times10^5\,m \Rightarrow k = \frac{2\pi}{\lambda} \simeq 3.1\times10^{-5}\,m^{-1}$

So $\omega = \sqrt{c^2k^2 + f_0^2} = \sqrt{[(2.0\times10^2)(3.1\times10^{-5})]^2 + (10^{-4})^2} = \sqrt{3.8\times10^{-5} + 10^{-8}} \approx 6.2\times10^{-3}\,s^{-1}$

[NOTE $kL_D = kc/f_0 = (6.2\times10^{-3}\,s^{-1})/(10^{-4}\,s^{-1}) = 62 \gg 1 \Rightarrow$ ROTATION ~ NEGLIGIBLE

so $\omega \approx kc$]

$$C_{gx} = \frac{d\omega}{dk} \overset{\omega\approx kc}{\simeq} c = \boxed{2.0\times10^2\,m/s}$$

[GENERALLY $C_{gx} = \frac{d}{dk}\sqrt{c^2k^2+f_0^2} = \frac{1}{2}(c^2k^2+f_0^2)^{-1/2}\,2c^2k = \frac{c^2k}{\omega} = \frac{c^2}{c_p} \overset{kL_D\gg1}{\approx} c$]

TRAVEL TIME IS $\frac{d}{C_{gx}} = \frac{6.2\times10^6\,m}{2.0\times10^2\,m/s} = 3.1\times10^4\,s \simeq \boxed{8.6\ \text{HOURS}}$

b) $A\ (=A_\eta) = 0.02\,m$, $\underline{k} = (k,0)$ (EASTWARD WAVE)

$\Rightarrow A_u = \frac{\omega}{kH}A \simeq cA/H \simeq (2.0\times10^2\,m/s)(0.02\,m)/(4000\,m)$

$\simeq \boxed{1.0\times10^{-3}\,m/s}$ ZONAL VELOCITY AMPL.

$A_v = \frac{f_0}{kH}A \approx \frac{10^{-4}\,s^{-1}}{(3.1\times10^{-5}\,m^{-1})(4000\,m)}(0.02\,m)$

$\simeq \boxed{1.6\times10^{-5}\,m/s}$ MERIDIONAL VELOCITY AMPL.

[NOTE $A_v \ll A_u$, CONSISTENT WITH ROTATION ~ NEGLIGIBLE]

c) "ENERGY" FLUX IS $\langle F_x\rangle = C_{gx}\langle E\rangle = c^2\langle u\eta\rangle = \frac{1}{2}c^2\frac{\omega}{kH}A^2$

SINCE $kL_D \gg 1 \Rightarrow \langle F_x\rangle \approx \frac{1}{2}c^3A^2/H = \frac{1}{2}(2.0\times10^2\,m/s)^3(0.02\,m)^2/(4000\,m)$

$\simeq 0.4\ m^4/s^3$

ACTUAL ENERGY DENSITY (ENERGY IN JOULES/VOLUME) IS $\rho_0\frac{1}{H}\langle E\rangle$

So ACTUAL ENERGY FLUX DENSITY IS $\rho_0\frac{1}{H}\langle F_x\rangle = \frac{\text{ENERGY}}{\text{AREA}\cdot\text{TIME}} = \frac{\text{POWER}}{\text{AREA}}$

IS THE ENERGY TRANSPORT ACROSS A UNIT AREA PER UNIT TIME

For $L_y = 1\,km$ SPAN OF WAVE, ENERGY TRANSPORT IS THE

POWER $P = (\rho_0\frac{1}{H}\langle F_x\rangle)\times\underbrace{(L_yH)}_{\text{AREA}} = \rho_0L_y\langle F_x\rangle \simeq (10^3\,kg/m^3)(10^3\,m)(0.4\,m^4/s^3)$

$= 4\times10^5\,(kg\,m^2/s^2)/s$

$\Rightarrow \boxed{P = 4\times10^5\,W}$

4a) $c = \sqrt{gH} \simeq (9.8\, m/s^2 \cdot 11m)^{1/2} \simeq 10.4\ m/s$

$f_0 = 2\Omega_e \sin 55° \simeq (1.45\times10^{-4} s^{-1})(0.819) \simeq 1.19\times10^{-4}\ s^{-1}$

So $L_D = \frac{c}{|f_0|} \simeq (10.4\, m/s)/(1.19\times10^{-4} s^{-1}) \simeq 8.74\times10^{4} m$

$\Rightarrow L_D \simeq \underline{87 km}$

CRITICAL WAVENUMBER IS $k_c = 1/L_D$

$\Rightarrow$ CRITICAL WAVELENGTH IS $\lambda_c = \frac{2\pi}{k_c} = 2\pi L_D$

$\Rightarrow \quad \lambda_c \simeq 2\pi\,(87 km) \simeq \underline{549\ km}$

b) $c = (9.8\, m/s^2 \cdot 100m)^{1/2} \simeq 31\ m/s$

$f_0 = 2\Omega_e \sin 60° \simeq 1.26\times10^{-4}\ s^{-1}$

So $L_D = (31\, m/s)/(1.26\times10^{-4} s^{-1}) \simeq 2.47\times10^{5} m$

$\Rightarrow L_D \simeq 247\ km$

CRITICAL WAVELENGTH IS $\lambda_c = 2\pi L_D \simeq 2\pi\,(247\ km) \simeq 1.55\times10^{3}\ km$

5a) THERE IS NO MOTION IN DIRECTION TOWARD COAST $\Rightarrow u = 0$

i.e. MAXIMUM ZONAL VELOCITY IS $\underline{A_u = 0}$

FROM $\frac{\partial v}{\partial t} + f_0 u = -g \frac{\partial \eta}{\partial y} \overset{u=0}{\Rightarrow} \frac{\partial v}{\partial t} = -g \frac{\partial \eta}{\partial y}$

$\Rightarrow -i\omega A_v = -g i l A_0$ (USING $v = A_v e^{x/L_D\, i(ly-\omega t)}$)

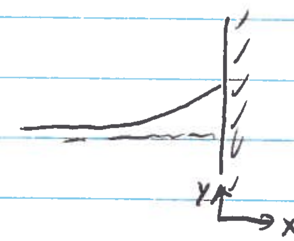
$\Rightarrow A_v = (gl/\omega) A_0 = (g/c) A_0$ (USING $\omega/l = c$)

$= \sqrt{g/H}\, A_0$ (USING $c = \sqrt{gH}$)

$\simeq (9.8\,m/s^2 / 4000\,m)^{1/2} (0.10\,m)$

$\underline{\simeq 5.0 \times 10^{-3}\ m/s}$ IS MAXIMUM MERIDIONAL VELOCITY

b) $\eta = A_0 e^{x/L_D} \cos(ly - \omega t)$ WITH $x < 0$



WHERE $L_D = \frac{c}{|f_0|} = \frac{198\,m/s}{(1.45 \times 10^{-4} s^{-1} \sin 33°)}$

$\simeq 2.5 \times 10^6\,m.$

So 1000km WEST OF COAST $(x = -10^6\,m)$, MAXIMUM DISPLACEMENT IS

$A_0 e^{x/L_D} \Big|_{x=-10^6 m} \simeq (0.10\,m)\, e^{-10^6 m/(2.5 \times 10^6 m)}$

$\simeq (0.10\,m)\, e^{-0.40}$

$\simeq 0.067\,m$

i.e. MAXIMUM DISPLACEMENT IS $\underline{\simeq 6.7\,cm}$