

Integration by differentiating through the integral sign

Examples

1. Solve $\int_{-\infty}^{\infty} x^2 e^{-x^2} dx$.

SOLUTION: Define

$$I(a) = \int_{-\infty}^{\infty} e^{-ax^2} dx. \quad (1)$$

Then, by differentiating both sides and bringing the derivative inside the integral on the right-hand side gives

$$I'(a) = \int_{-\infty}^{\infty} (-x^2) e^{-ax^2} dx. \quad (2)$$

Recall, however, that (1) has the explicit solution

$$I(a) = \sqrt{\pi/a}.$$

From which it follows that

$$I'(a) = -\frac{1}{2} a^{-3/2} \sqrt{\pi}. \quad (3)$$

Comparing (2) and (3) gives

$$\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{1}{2} a^{-3/2} \sqrt{\pi}$$

In particular, for $a = 1$,

$$\int_{-\infty}^{\infty} x^2 e^{-x^2} dx = \frac{1}{2} \sqrt{\pi}.$$

2. Similarly, one can find the following integrals:

- $\int_{-\infty}^{\infty} x^n e^{-x^2} dx$.
- $\int_{-\infty}^{\infty} e^{-x^2} \cos(ax) dx$.