Separation of Variables: Flow Chart

1. SET UP PDE AND INITIAL/BOUNDARY CONDITIONS
   - Determine appropriate partial differential equation (wave/diffusion/potential equation)
   - Determine co-ordinate system (Cartesian/polar/cylindrical/spherical)
   - If possible, exploit symmetry (azimuthal) to reduce dimensionality of problem.
     E.g. $u(r,\theta,t) \Rightarrow u(r,t)$, if initial and boundary conditions are independent of θ.
   - Explicitly rewrite resulting initial-boundary value problem.

2. SEPARATE VARIABLES
   - Write u as product of "separated" functions of each variable (*e.g.* $u(r,t) = R(r)T(t)$).
   - Substitute into partial differential equation and derive set of ordinary differential equations for each separated function.
     E.g. $T' + \lambda\kappa T = 0$; $r^2R'' + rR' + \lambda r^2 R = 0$.
   - Use given zero/periodic/boundedness boundary conditions on u to find boundary conditions on corresponding separated functions.
     E.g. $u(a,t) = 0 \Rightarrow R(a) = 0$, R - bounded
   - *Do not try to impose boundary conditions on separated functions of a variable where u is given generally as a function of that variable. (E.g. if $u(r,t) = f(t)$ is given, do not try to impose condition on T.)*

3. FIND GENERAL SOLUTION
   - Solve ordinary differential equations for separated functions with explicitly given boundary conditions.
   - *In finding your solution, you may rely on experience - otherwise check cases where the separation constant (λ) is positive, negative and zero and, together with the boundary conditions, determine when you get non-trivial solutions.*
     E.g. $X'' - \lambda X = 0; X(0) = X(\pi) = 0$ *has nontrivial solutions* ($X \neq 0$) *only if* $\lambda = -n^2$ *in which case* $X(x) \propto \sin(nx)$.
   - *your result is typically an infinite number of (eigenfunction) solutions with a corresponding infinite set of eigenvalues. (e.g.* $R_n(r) = J_0(\alpha_n r/a)$ *corresponding to* $\lambda_n = (\alpha_n/a)^2$, $n = 0,1,2,\ldots$*)*
   - Substitute these eigenvalues into the remaining ordinary differential equations and solve these equations.
     E.g. $T' + (\alpha_n/a)^2\kappa T = 0 \Rightarrow T(t) \propto \exp[-(\alpha_n/a)^2\kappa t]$.
   - Form superposition of the product of separated functions to get the general solution.
     E.g. $u(r,t) = \sum_{n=0}^{\infty} a_n J_0(\alpha_n r/a) \exp[-(\alpha_n/a)^2\kappa t]$.

4. FIND COEFFICIENTS IN GENERAL SOLUTION
   - Apply the non-zero initial/boundary condition (given in terms of some arbitrary function, f, say) and so write f as a series (*e.g.* $f(r) = u(r,0) = \sum_{n=0}^{\infty} a_n J_0(\alpha_n r/a)$).
   - Recognize that this series is an orthogonal expansion (Fourier series/Bessel series/Legendre series) of f and use property of orthogonality to pull out coefficients from the sum.
     e.g.

$$\begin{aligned}\int_0^a f(r)J_0(\alpha_m r/a)r\,dr &= \sum_{n=0}^{\infty} a_n \int_0^a J_0(\alpha_n r/a)J_0(\alpha_m r/a)r\,dr \\ &= a_m \int_0^a [J_0(\alpha_m r/a)]^2 r\,dr\end{aligned}$$

   - Evaluate integrals using tables or by direct calculation.
   - Substitute results back into general solution.
   - Explicitly evaluate first few non-zero terms if required.