Special Functions: Summary

Special Function	ODE	Type	Symbol	FORMULAE
Trig. Functions	$X'' + X = 0;\ x \in (0, L)$		$\sin(x), \cos(x)$	
	$X'' + \lambda X = 0$		$X_n = \sin(n\pi \frac{x}{L})$	$\int_0^L \sin\left(n\pi \frac{x}{L}\right) \sin\left(m\pi \frac{x}{L}\right)\ dx = 0, n \neq m$
	$x \in (0, L), X(0) = X(L) = 0$		$\lambda_n = (n\pi/L)^2, n = 1, 2, \ldots$	
	$X'' + \lambda X = 0$		$X_n = \cos(n\pi \frac{x}{L})$	$\int_0^L \cos\left(n\pi \frac{x}{L}\right) \cos\left(m\pi \frac{x}{L}\right)\ dx = 0, n \neq m$
	$x \in (0, L), X'(0) = X'(L) = 0$		$\lambda_n = (n\pi/L)^2, n = 0, 1, \ldots$	
	$X'' - X = 0;\ x \in (0, L)$		$X(x) = \exp(\pm x)$	
Bessel Functions	$r^2 R'' + rR' + r^2 R = 0$	1st kind, order 0	$J_0(r)$	$J_0 = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}(k!)^2} r^{2k} \simeq 1 - \frac{1}{4} r^2, \quad r \gtrsim 0$
	$r > 0$			$J_0 \simeq \sqrt{\frac{2}{\pi}} r^{-1/2} \cos(r - \pi/4), \quad r \to \infty$
		2nd kind, order 0	$Y_0(r)$	$Y_0 = \frac{2}{\pi} \ln r, \quad r \gtrsim 0$
				$Y_0 \simeq \sqrt{\frac{2}{\pi}} r^{-1/2} \sin(r - \pi/4), \quad r \to \infty$
	$r^2 R'' + rR' + \lambda r^2 R = 0$		$J_0\left(\alpha_n \frac{r}{a}\right)$	$\int_0^a J_0\left(\alpha_n \frac{r}{a}\right) J_0\left(\alpha_m \frac{r}{a}\right) r\ dr = 0, n \neq m$
	$r \in (0, a), R(a) = 0,\ R$ bounded		$\lambda_n = (\alpha_n/a)^2, n = 0, 1, \ldots$	
	$r^2 R'' + rR' + (r^2 - n^2) R = 0$	1st kind, order n	$J_n(r)$	$J_n =\simeq \left(\frac{r}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}[(k+n)!]^2} r^{2k} \simeq \frac{1}{2^n\,(n!)^2} r^n$
	$r > 0$			
	$r^2 R'' + rR' - r^2 R = 0$	Modified	$I_0(r)$	$I_0 \simeq \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{r}{2}\right)^2, \quad r \gtrsim 0$
	$r > 0$	1st kind, order 0		$I_0 = J_0(\imath r)$
		Modified	$K_0(r)$	$K_0 \simeq -I_0(r) \ln\left(\frac{r}{2}\right), \quad r \gtrsim 0$
		2nd kind, order 0		
Legendre Polynomial	$(1 - x^2) X'' - 2xX' + \lambda X = 0$		$P_n(x) = \frac{1}{2^n n!} \left[\frac{d^n}{dx^n} (x^2 - 1)^n\right]$	$\int_{-1}^{1} P_n(x) P_m(x)\ dx = 0, n \neq m$
	$X \in (-1, 1)$ bounded		$\lambda_n = n(n+1), n = 0, 1, \ldots$	$P_0 = 1, P_1 = x, P_2 = \frac{1}{2}(3x^2 - 1), P_3 = \frac{1}{2}(5x^3 - 3x)$
	$(1 - x^2) X'' - 2xX' + \left(\lambda - \frac{m^2}{1 - x^2}\right) X = 0$	Associated	${P_n}^m = (-1)^m (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_n(x)$	$\int_{-1}^{1} {P_n}^m(x) {P_k}^m(x)\ dx = 0, n \neq k$
	$X \in (-1, 1)$ bounded		$\lambda_n = n(n+1), m \leq n = 0, 1, \ldots$	${P_n}^0 \equiv P_n, {P_1}^1 = (1 - x^2)^{1/2}$
				${P_2}^1 = 3x(1 - x^2)^{1/2}, {P_2}^2 = 3(1 - x^2)$
Hermite Polynomial	$X'' - 2xX' + (\lambda - 1) X = 0$		$H_n(x) = (-1)^n e^{x^2} \left[\frac{d^n}{dx^n} e^{-x^2}\right]$	$\int_{-\infty}^{\infty} H_n(x) H_m(x) e^{-x^2}\ dx = 0, n \neq m$
	$X \in \mathcal{R}$ bounded		$\lambda_n = 2n + 1$, n=0,1,...	$H_0 = 1, H_1 = 2x, H_2 = -2 + 4x^2, H_3 = -12x + 8x^3$