Formulae of Vector Calculus

Notation

- explicit line integral formulae are given only in 2 dimensions (x,y), but are easily extended to 3 dimensions (x,y,z).
- Only explicit parametric representations are given. For curves in Cartesian co-ordinates, let $t = x$ and $y(t) = y(x)$.

1) $\int_C f\, ds$

$$\equiv \int f(\underline{r}(t)) \left|\frac{d\underline{r}}{dt}\right| dt = \int f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Notes: If $f = 1$, integral gives line length.
If f is linear density, integral gives mass.

2) $\int_C \underline{F} \cdot \hat{n}\, ds$

$$\equiv \int (F_x, F_y) \cdot \left[\pm \left(\frac{d(-y)/dt}{|\underline{x}'|}, \frac{dx/dt}{|\underline{x}'|}\right)\right] |\underline{x}'|\, dt = \pm \int \left(F_x \frac{d(-y)}{dt} + F_y \frac{dx}{dt}\right) dt$$

Notes: If $\underline{F} \equiv (F_x, F_y)$ is velocity, gives flow across curve C.
Choose sign of $\pm$ so that $\hat{n}$ points in direction of flow that you decide is positive.
If C is a closed curve, choose sign so that $\hat{n}$ points outward (positive direction for divergent flow).

3) $\int_C \underline{F} \cdot d\underline{r} \equiv \int_C F_x\, dx + F_y\, dy$

$$\equiv \int (F_x, F_y) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}\right) dt = \int \left(F_x \frac{dx}{dt} + F_y \frac{dy}{dt}\right) dt$$

Notes: If $\underline{F}$ is force, integral gives work.

4) $\int\int_S f dS$

$$\equiv \int\int_D f(\underline{r}(u,v))\ |\underline{r}_u \times \underline{r}_v|\, dA$$

Notes: If $f = 1$, integral is surface area.
If f is density (mass per unit area), integral is mass.
$dA \equiv du\, dv$

5) $\int\int_S \underline{F} \cdot d\underline{S} \equiv \int\int_S \underline{F} \cdot \hat{n}\, dS$

$$\equiv \int\int_D \underline{F} \cdot \left(\pm \frac{\underline{r}_u \times \underline{r}_v}{|\underline{r}_u \times \underline{r}_v|}\right)\ |\underline{r}_u \times \underline{r}_v|\, dA = \pm \int\int_D \underline{F} \cdot (\underline{r}_u \times \underline{r}_v)\ dA$$

Notes: If $\underline{F}$ is velocity, integral represents flow across surface $\mathcal{S}$.
Choose sign of $\pm$ so that $\hat{n}$ points in direction of flow that you decide is positive.
If S is a closed surface, choose sign so that $\hat{n}$ points outward (positive direction for divergent flow).
$dA \equiv du\, dv$