Formulae of Vector Calculus

Handy Theorems

1) Fundamental Theorem of Line Integrals

$$\int_C \nabla f \cdot d\underline{r} = f(\underline{r}(b)) - f(\underline{r}(a))$$

Notes: If $\underline{F}$ is a conservative force field, then work in moving from $\underline{r}(a)$ to $\underline{r}(b)$ is independent of path.
For closed curve $\oint_C \nabla f \cdot d\underline{r} = 0$.
Test $\underline{F} = \nabla f$ for $\underline{F}$ in 2-D with $\partial F_y/\partial x = \partial F_x/\partial y$.
If $\underline{F} = (F_x, F_y, F_z)$ perform 3 tests: $\partial F_y/\partial x = \partial F_x/\partial y$. $\partial F_z/\partial x = \partial F_x/\partial z$. $\partial F_z/\partial y = \partial F_y/\partial z$.

2) Green's Theorem

$$\oint_C \underline{F} \cdot d\underline{r} = \iint_D (\nabla \times \underline{F}) \cdot \hat{k}\; dA \equiv \iint_D \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\; dA$$

Notes: Sum of small loops (vortices) contained by (closed curve) C in x-y plane gives big loop (surrounding circulation).
Integration around C must be in counterclockwise direction.

3) Stokes' Theorem

$$\oint_C \underline{F} \cdot d\underline{r} = \iint_{\mathcal{S}} \nabla \times \underline{F} \cdot d\underline{S} \equiv \iint_{\mathcal{S}} (\nabla \times \underline{F}) \cdot \hat{n}\; dS$$

Notes: 3-D counterpart to Green's Theorem:
sum of vorticity on surface of balloon, the surface $\mathcal{S}$, gives circulation around opening in surface.
$\mathcal{S}$ is NOT a closed surface
$\hat{n}$ must point outward from "interior" of balloon.
Integration around C must be in counterclockwise direction.

4) Divergence Theorem (2-D)

$$\oint_C \underline{F} \cdot \hat{n}\, ds = \iint_D \nabla \cdot \underline{F}\; dA \equiv \iint_D \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\; dA$$

Notes: Sum of small divergences contained in (closed curve) C equals total flux of mass across C.
$\hat{n}$ must point outward from interior of C.
Integration around C must be in counterclockwise direction.

5) Divergence Theorem (3-D)

$$\iint_{\mathcal{S}} \underline{F} \cdot d\underline{S} \equiv \iint_{\mathcal{S}} \underline{F} \cdot \hat{n}\; dS = \iiint_E \nabla \cdot \underline{F}\; dV$$

Notes: 3-D counterpart to divergence theorem in 2-D:
sum of small divergences contained inside surface $\mathcal{S}$ equals total flux of mass across $\mathcal{S}$.
$\mathcal{S}$ MUST be a closed surface (even though there is no circle through the double integrals)
$\hat{n}$ must point outward from interior of $\mathcal{S}$.