# Conservation Laws, Hamiltonian Structure, Modulational Instability Properties and Solitary Wave Solutions for a Higher-Order Model Describing Nonlinear Internal Waves

*By G. E. Swaters, H. V. Dosser, and B. R. Sutherland*

Recent theoretical advances in connecting the wave-induced mean flow with the conserved pseudomomentum per unit mass has permitted the first rational derivation of a model that describes the weakly nonlinear propagation of internal gravity plane waves in a continuously stratified fluid. Depending on the particular parameter regime examined the new model corresponds to an extended bright or dark derivative nonlinear Schrödinger equation or an extended complex-valued modified Korteweg-de Vries or Sasa–Satsuma equation. Mass, momentum, and energy conservation laws are derived. A noncanonical infinite-dimensional Hamiltonian formulation of the model is introduced. The modulational stability characteristics associated with the Stokes wave solution of the model are described. The bright and dark solitary wave solutions of the model are obtained.

## 1. Introduction

Notwithstanding the important role that nonlinearity plays in the evolution and propagation of internal waves of moderate amplitude in a stably stratified fluid, the development of a weakly nonlinear theory has been a difficult problem because of the well-known property that internal gravity plane waves are exact solutions to the full nonlinear equations of motion. This is a problem because,

Address for correspondence: Gordon E. Swaters, Institute for Geophysical Research, University of Alberta, Edmonton, Alberta, T6G 2G1 Canada; e-mail: gordon.swaters@ualberta.ca

DOI: 10.1111/j.1467-9590.2011.00533.x

*STUDIES IN APPLIED MATHEMATICS* 0:1–24

on the face of it, this property implies that it is not possible to determine the wave-induced mean flow in a straightforward application of perturbation theory (e.g., [1]) and thereby systematically derive a nonlinear amplitude evolution equation. An important breakthrough in this problem was made by Sutherland [2] who, exploiting the underlying Hamiltonian structure of the governing equations (Scinocca and Shepherd [3]), was able to connect the wave-induced mean flow with the conserved pseudomomentum per unit mass (McIntyre [4]) and thereby give the first rational derivation of a nonlinear Schrödinger equation (NLS) describing the evolution of weakly nonlinear but strongly dispersive internal gravity waves in a Boussinesq fluid.

Recently, Dosser and Sutherland [5] have extended this work to describe the weakly nonlinear evolution of internal wavepackets in a non-Boussinesq fluid. This model equation has the added feature of including higher-order nonlinear and dispersive effects. The principal purpose of this paper is to describe the most important dynamical characteristics and theoretical properties of this new model for the nonlinear propagation of internal gravity waves.

Versions of the Dosser and Sutherland [5] model have been derived in other physical contexts. For example, Grimshaw and Helfrich [6] have derived a similar model to describe the long time behavior of dispersive wave solutions to the Ostrovsky equation. In addition to presenting numerical simulations, they determined a bright solitary wave solution. Yang [7] found a bright solitary wave solution (and presented numerical simulations) to another similar model for an asymptotically restricted set of parameter values. Slunyaev [8] also obtained a very similar model to describe the nonlinear evolution of surface gravity waves and examined the modulational instability properties.

The outline of this paper is as follows. In Section 2, the dimensional model for weakly nonlinear internal waves in a non-Boussinesq fluid derived by Dosser and Sutherland [5] is briefly described. A nondimensionalization is introduced that maps the lowest-order terms to the canonical form of either the bright or dark form of the NLS equation. It is argued that the model can be identified as either a higher-order generalization of the derivative NLS equation or the complex-valued modified Korteweg-de Vries equation.

In Section 3, the *mass*, *momentum*, and *energy* conservation laws and the Hamiltonian formulation are derived for the Boussinesq limit of the model. It is shown that the precise form of these conservation laws and Hamiltonian formulation depends critically on the magnitude of the third-order dispersive term. When the third-order dispersive term cannot be neglected to leading order, the model equation is most appropriately understood as a variant of the complex-valued modified Korteweg-de Vries or Sasa–Satsuma equation (SSE). When the third-order dispersive term can be neglected to leading order, the model equation is most appropriately understood as a variant of the derivative NLS equation. The conservation laws and Hamiltonian formulation when the third-order dispersive term can be neglected to leading order and when it cannot be neglected to leading order are derived in both approximations. It is shown that the dependence of the Hamiltonian structure and conservation laws on the parameter measuring the order of magnitude of the third-order dispersive terms (denoted by $\alpha$) is not continuous in the $\alpha \to 0$ limit.

Numerical simulations presented by Dosser and Sutherland [5] suggest the possibility of modulational instability. In Section 4, the linear stability properties of the Stokes wave solution to the model are described. It is shown that (as expected) unstable sideband perturbations are possible only in the so-called bright limit of the model. The dependence of the modulational stability boundary (as well as growth rate of the most unstable mode) with respect to the Stokes wave amplitude and sideband wavenumber is described. Unlike the stability boundary associated with the classical NLS equation, the Dosser and Sutherland model predicts that modulational destabilization is independently limited by both the Stokes wave amplitude and the sideband wavenumber. Indeed, for sufficiently large but finite Stokes wave amplitude modulational instability cannot occur. In the parameter regime where the Slunyaev [8] and the Dosser and Sutherland [5] models overlap, the instability conditions agree with each other (see, also, Kakutani and Michihiro [9] and Parkes [10]).

In Section 5, the solitary wave solutions in both the bright and dark limits associated the model equation are explicitly constructed. Like the classical NLS equation, it is shown that the bright and dark solitary wave solutions correspond, respectively, to a one- and two-parameter family. In addition, it is shown that in the limit when the third-order dispersive and nonlinear derivative terms (the complex-valued Korteweg-de Vries part of the model so to speak) are neglected, these solitary wave solutions exactly reduce to the well-known bright and dark soliton solutions of the NLS equation, respectively. In the parameter regime where the Grimshaw and Helfrich [6] and the Dosser and Sutherland [5] models overlap the bright soliton solution reduce to each other. Grimshaw and Helfrich [6] did not construct a dark solitary wave solution. The paper is summarized in Section 6.

## 2. Governing equations

Although we will exclusively work with the Boussinesq limit of the Dosser and Sutherland [5] model, for completeness we will briefly present it in its non-Boussinesq form. The non-Boussinesq wavepacket model derived by Dosser and Sutherland [5] can be written in the dimensional form

$$i\left(\partial_t + c_g\partial_z\right)A + \frac{\omega''}{2}A_{zz} - \frac{N^2k^2e^{z/H}}{2\omega}A\left|A\right|^2$$

$$-i\left[\frac{\omega'''}{6}A_{zzz} + \frac{\omega}{4}\left(m + \frac{i}{H}\right)A\left(e^{z/H}\left|A\right|^2\right)_z\right] = 0, \qquad (1)$$

with $N > 0$ and $H > 0$ the (assumed constant and stably stratified) buoyancy frequency and scale height ($H^{-1} = N^2/g$, with $g$ the gravitational acceleration), respectively, $z > 0$ is the positively upward oriented vertical coordinate, $t$ is (laboratory) time, $A$ is the complex-valued amplitude of the underlying plane wave associated with the vertical displacement of the isopycnals and $\omega$ is the linear frequency as determined by the dispersion relationship

$$\omega^2 = \frac{N^2 k^2}{k^2 + m^2 + H^{-2}/4}, \tag{2}$$

which implies

$$c_g = \omega' \equiv \frac{\partial \omega}{\partial m} = -\frac{\omega m}{k^2 + m^2 + H^{-2}/4}, \tag{3}$$

$$\omega'' \equiv \frac{\partial^2 \omega}{\partial m^2} = \frac{\omega \left(2m^2 - k^2 - H^{-2}/4\right)}{\left(k^2 + m^2 + H^{-2}/4\right)^2}, \tag{4}$$

and

$$\omega''' \equiv \frac{\partial^3 \omega}{\partial m^3} = -\frac{3\omega m \left(2m^2 - 3k^2 - 3H^{-2}/4\right)}{\left(k^2 + m^2 + H^{-2}/4\right)^3}. \tag{5}$$

where (without loss of generality) $k > 0$ and $m > 0$ are the (real valued) horizontal and vertical wavenumbers, respectively. In what follows we will take, for convenience, $\omega > 0$. This implies that the phase propagates vertically upward (i.e., $c_p \equiv \omega m/(k^2 + m^2) > 0$) but the group velocity propagates vertically downward (i.e., $c_g < 0$ ). The Boussinesq limit in (1) is to let $H \to \infty$ (i.e., $N^2/g \to 0$) but retain $N$ (LeBlond and Mysak [11]).

It is convenient to introduce the scalings given by

$$t = \left(L/\left|c_g\right|\right)\tilde{t},\ z = L\tilde{x},\ (k, m) = \left(\tilde{k}, \tilde{m}\right)/L,\ A = 2\sqrt{\left|c_g\right|/(\omega m)}\,\tilde{u}, \tag{6}$$

where $L \equiv \omega''/(2|c_g|)$ and the variables with tildes are nondimensional. These scalings will map the lower-order terms in (1) to the canonical form of NLS.

Note that $L$ is not definite in sign. Nevertheless this choice ensures that coefficients associated with the co-moving time derivative and the leading-order dispersion term are both $+1$ (which is convenient for our discussion) and that the propagation direction is the same with respect to both the dimensional and nondimensional variables.

Substitution of (6) into (1) yields, after dropping the tildes,

$$\begin{aligned} i\,(\partial_t - \partial_x)\,u + u_{xx} - 2\delta\beta e^{\varepsilon\delta x} u\,|u|^2 \\ + i\,\left[\alpha u_{xxx} - (1 + i\varepsilon\delta/m)\,u\left(e^{\varepsilon\delta x}\,|u|^2\right)_x\right] = 0, \end{aligned} \tag{7}$$

where $\delta \equiv \text{sgn}(\omega'') = \text{sgn}(\alpha) = \pm 1$, $\varepsilon \equiv |L|/H > 0$ and where

$$\alpha \equiv \frac{2m^2\left(2m^2 - 3k^2 - 3\varepsilon^2/4\right)}{\left(2m^2 - k^2 - \varepsilon^2/4\right)^2}, \tag{8}$$

$$\beta \equiv \frac{\left|2m^2 - k^2 - \varepsilon^2/4\right|}{2m^2} \geq 0. \tag{9}$$

The Boussinesq limit in (7) corresponds to $\varepsilon = 0$.

As written, (7) maybe considered a variant of the SSE with variable coefficients (see [12, 13]). First derived in the context of solitary wave dynamics in optical fibers (e.g., Kodama [14]), the SSE may be thought of as either a rational extension of NLS with the next highest- (amplitude) order dynamical effects included or a complex-valued generalization of the modified Korteweg-de Vries (mKdV) equation. Unfortunately, even in the constant coefficient case it is well known that the SSE is not integrable via an inverse scattering transform except for a very small set of parameter values; see [13]. None of these possibilities seem to be realized in (7) even in the Boussinesq limit. That is, even when $\varepsilon = 0$ there appears to be no known choice for $\alpha$, $\beta$, and $\delta$ for which (7) is integrable (except in *ad hoc* approximations where it reduces to either the NLS or mKdV equations).

For example, in the Boussinesq limit and where the fourth squared-bracketed term is additionally neglected, (7) corresponds to the classical *bright* or *dark* NLS equation when $\delta = -1$ or 1, respectively. It follows from (4) and (8) that

$$\delta = \begin{cases} +1 \Longleftrightarrow 2m^2 > k^2 + \varepsilon^2/4 \Longleftrightarrow \alpha > 0, \\ -1 \Longleftrightarrow 2m^2 < k^2 + \varepsilon^2/4 \Longleftrightarrow \alpha < 0. \end{cases} \tag{10}$$

Thus, qualitatively, (7) in this approximation suggests that internal gravity waves with *short* vertical wavelengths (as compared to the horizontal wavelength, i.e., $|m| > |k|$) are governed by dark NLS dynamics while those with *long* vertical wavelengths (as compared to the horizontal wavelength, i.e., $|m| < |k|$) are governed by bright NLS dynamics in this limit. Both the bright and dark NLS equations are integrable and have, of course, soliton solutions (see, e.g., [15, 16]). The dark solitons correspond to envelope solitary waves with oscillatory cores that decay exponentially to zero at infinity. The bright solitons principally differ in that they decay to a nonzero value away from the core.

On the other hand, in the Boussinesq limit and where the cubically nonlinear term $u|u|^2$ is additionally neglected, (7) corresponds to a mKdV equation (the $u_{xx}$ term can always removed via a suitable transformation— as shown below). Thus, depending on the particular parameter values and initial conditions assumed, it is expected that the solution to (7) for compactly supported initial conditions will evolve, at least for some period of time, either like modulated envelope solitary waves with oscillatory cores of NLS type or nonoscillatory cores of mKdV type.

Grimshaw and Helfrich [6] have derived a model very similar to (7) to describe the long time behavior of dispersive wave solutions to the Ostrovsky equation. In addition to presenting numerical simulations, they determined a bright solitary wave solution. In the parameter regime where our model and theirs overlap the bright solitary wave solution we describe and the one found by Grimshaw and Helfrich [6] are identical. We also determine the dark solitary wave solution. We also remark that Yang [7] found a bright solitary wave solution (and presented numerical simulations) to a model similar to (7) for an asymptotically restricted set of parameter values. Slunyaev [8] also obtained an equation very similar to (7) to describe the nonlinear evolution of surface gravity waves and examined the modulational instability properties. Again, in the parameter regime where (7) and the model Slunyaev [8] examined overlap, our modulational stability results agree. For the remainder of this paper we will ignore the non-Boussinesq terms in the Dosser and Sutherland [5] model.

## 3. Conservation balances and Hamiltonian formulation

The dynamics associated with (7) depend critically on the magnitude of $\alpha$, which is the coefficient of the third-order dispersive term. When this term can be neglected (7) is, roughly speaking, NLS-like in structure and has (in the Boussinesq limit) a similar (but not necessarily an infinite number of) conservation laws and a Hamiltonian formulation. When the third-order dispersive term cannot be neglected, (7) may be considered as a complex or vector-valued mKdV equation and has (in the Boussinesq limit) a similar (but not necessarily an infinite number of) conservation laws and a Hamiltonian formulation.

Of particular note, it will be shown that the conservation laws and Hamiltonian structure associated with neglecting the third-order dispersive or $\alpha$ term *cannot* be recovered by taking the $\alpha \to 0$ limit of the conservation laws and Hamiltonian structure associated with the $\alpha \neq 0$ equations. The limit is singular and this is a consequence of that fact that the (spatial) order of the $\alpha \neq 0$ equations is three while the (spatial) order of the $\alpha = 0$ equations is only two. Accordingly, the dynamical properties of the solutions or model equations cannot be expected to necessarily depend continuously on $\alpha$ as $\alpha \to 0$.

### *3.1. Conservation laws and Hamiltonian structure when $\alpha \simeq O(\varepsilon)$*

Let us first consider the situation where $\alpha \simeq O(\varepsilon)$, which corresponds physically to an internal gravity wave beam satisfying $m \simeq k\sqrt{3/2} + O(\varepsilon)$ implying that $\omega \simeq N/\sqrt{2}$. The phases for these waves propagate upward at about $50.77^\circ$

from the horizontal. Neglecting terms of $O(\varepsilon)$ in (7) leads to

$$iu_t + u_{xx} - 2\delta\beta u\,|u|^2 - iu\left(|u|^2\right)_x = 0, \tag{11}$$

where we have introduced, without loss of generality, the co-moving coordinate system $x \to x + t$. Equation (11) is a hybrid model exhibiting features of both the classical and (generalized) derivative NLS equations. Models similar to (11) have been called a modified vector derivative NLS or MVDNLS equations (see, e.g., [17, 18]). Equation (11) does not appear to be related, via a gauge transformation, to any of the known integrable forms of the derivative NLS equation; see [19].

The mass, momentum, and energy conservation equation associated with (11) are given by, respectively,

$$\left(|u|^2\right)_t + \left[i\left(u\bar{u}_x - u_x\bar{u}\right) - \delta\beta\,|u|^4\right]_x = 0. \tag{12}$$

$$[i(u\bar{u}_x - u_x\bar{u}) - |u|^4]_t + \left[\frac{4}{3}|u|^6 + 2i|u|^2(u_x\bar{u} - u\bar{u}_x) + i(u_t\bar{u} - u\bar{u}_t) + 2|u_x|^2 - 2\delta\beta|u|^4\right]_x = 0, \tag{13}$$

$$\left[|u_x|^2 + \delta\beta\,|u|^4 + \frac{1}{3}\,|u|^6 + \frac{i}{2}\,|u|^2\left(u_x\bar{u} - u\bar{u}_x\right)\right]_t + \left[\frac{i}{2}\,|u|^2\left(u\bar{u}_t - u_t\bar{u}\right) - u_x\bar{u}_t - u_t\bar{u}_x + \frac{1}{2}\left(u_x\bar{u} - u\bar{u}_x\right)^2 + i\,|u|^4\left(u\bar{u}_x - u_x\bar{u}\right) - \frac{1}{2}\,|u|^8\right]_x = 0, \tag{14}$$

where $\bar{u}$ is the complex-conjugate of $u$. The details of the derivation of these conservations laws is contained in the Appendix.

The reason we call (13) *momentum conservation* is that its existence, via Noether's theorem, is due to the invariance with respect to arbitrary translations with respect to $x$ of the yet-to-be described Hamiltonian formulation (as shown below). The *density* associated with the conservation law (14) will form the Hamiltonian (as shown below) and that is why we have called (14) *energy conservation*. These conservation laws are critical in developing the perturbation theory assuming a weakly non-Boussinesq approximation (to be described elsewhere).

Equation (11) possesses a noncanonical nonlocal Hamiltonian formulation. If we write $u = p + iq$ where $p$ and $q$ are real-valued functions of $(x, t)$, then (11) takes the form of the $2 \times 2$ system

$$p_t = -q_{xx} + 2\delta\beta q\left(p^2 + q^2\right) + p\left(p^2 + q^2\right)_x\,, \tag{15}$$

$$q_t = p_{xx} - 2\delta\beta p \left(p^2 + q^2\right) + q \left(p^2 + q^2\right)_x . \tag{16}$$

The energy density in (14) forms the integrand for the Hamiltonian, given by

$$H = \frac{1}{2}\int_{-\infty}^{\infty} p_x^2 + q_x^2 + \delta\beta \left(p^2 + q^2\right)^2 + \frac{1}{3}\left(p^2 + q^2\right)^3 + \frac{4}{3}\left(q^3 p_x - p^3 q_x\right) dx. \tag{17}$$

It is assumed that $p$ and $q$ and all their derivatives are smooth functions and vanish sufficiently rapidly at infinity so that all required integrals exist. Thus, it follows that $|H| < \infty$ and that $dH/dt = 0$.

For our purposes, it is sufficient to remark that a system of $n$ partial differential equations written abstractly in the form

$$\Phi\left(\mathbf{q}, \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) = \mathbf{0},$$

where $t$ is time and $\mathbf{q}(x, t) = (q_1(x, t), ..., q_n(x, t))^\top$ is a column vector of $n$ dependent variables defined on $\mathbb{R}\times [0, \infty)$ is said to be Hamiltonian if there exists a conserved functional $H(\mathbf{q})$ (the Hamiltonian), and a matrix $\mathcal{M}$ of differential operators (the cosymplectic form) such that the system of partial differential equations can be written in the form

$$\mathbf{q}_t = \mathcal{M}\frac{\delta H}{\delta \mathbf{q}},$$

where $\frac{\delta H}{\delta \mathbf{q}}$ is the vector variational derivative of $H$ with respect to $\mathbf{q}$ and where the associated Poisson bracket is defined by

$$[F, G] \equiv \left\langle \frac{\delta F}{\delta \mathbf{q}}, \mathcal{M}\frac{\delta G}{\delta \mathbf{q}} \right\rangle,$$

where $F$ and $G$ are arbitrary smooth functionals of $\mathbf{q}$ and $\langle *_1, *_2\rangle$ is the inner product

$$\langle *_1, *_2\rangle = \int_{-\infty}^{\infty} *_1 \cdot *_2 \, dx,$$

satisfies the algebraic properties of skew symmetry, distributive and associative laws and the Jacobi identity ([20]; for a specific fluid dynamic application see, e.g., [21]).

The system (15) and (16) is Hamiltonian for $H$ given by (17) and cosymplectic form $\mathcal{M}$ given by

$$\mathcal{M} = \begin{bmatrix} 2q\,\partial_x^{-1} \circ q & 1 - 2q\,\partial_x^{-1} \circ p \\ -1 - 2p\,\partial_x^{-1} \circ q & 2p\,\partial_x^{-1} \circ p \end{bmatrix}, \tag{18}$$

where $\partial_x$ denotes partial differentiation with respect to $x$ and $\partial_x^{-1}$ is its inverse so that $\partial_x^{-1}\partial_x = \partial_x\partial_x^{-1} = 1$ and it is understood that $\partial_x^{-1}\circ *_1 *_2 = \partial_x^{-1}(*_1 *_2)$.

It follows that

$$\begin{pmatrix} p_t \\ q_t \end{pmatrix} = \mathcal{M} \begin{pmatrix} \delta H/\delta p \\ \delta H/\delta q \end{pmatrix}, \tag{19}$$

where

$$\begin{pmatrix} \delta H/\delta p \\ \delta H/\delta q \end{pmatrix} = \begin{pmatrix} -p_{xx} + p\left(p^2+q^2\right)\left(2\delta\beta + p^2 + q^2\right) - 2\left(p^2+q^2\right)q_x \\ -q_{xx} + q\left(p^2+q^2\right)\left(2\delta\beta + p^2 + q^2\right) + 2\left(p^2+q^2\right)p_x \end{pmatrix}. \tag{20}$$

Direct substitution will verify that (19) and (20) reproduces (15) and (16). Additionally, it is also straightforward to verify the that required algebraic properties are satisfied by $\mathcal{M}$. Taken all together these details are, however, long and tedious and are not fully reproduced here. However, to give an indication of the arguments required, we show in the Appendix that $\mathcal{M}$ is skew symmetric.

Clearly, the above Hamiltonian structure is invariant with respect to arbitrary translations in $x$, i.e., the mapping $x \to x + \lambda$ leaves the Hamiltonian formulation invariant. Thus, via Noether's theorem (see, e.g., [20]), there exists an invariant functional, denoted by $L$, satisfying

$$\mathcal{M} \begin{pmatrix} \delta L/\delta p \\ \delta L/\delta q \end{pmatrix} = -\begin{pmatrix} p_x \\ q_x \end{pmatrix}.$$

It may be verified that

$$L = \frac{1}{4}\int_{-\infty}^{\infty} i\left(u\overline{u}_x - u_x\overline{u}\right) - |u|^4 \, dx = \frac{1}{4}\int_{-\infty}^{\infty} 2\left(pq_x - p_x q\right) - \left(p^2+q^2\right)^2 dx$$

$$\Longrightarrow \begin{pmatrix} \delta L/\delta p \\ \delta L/\delta q \end{pmatrix} = \begin{pmatrix} q_x - \left(p^2+q^2\right)p \\ -p_x - \left(p^2+q^2\right)q \end{pmatrix}.$$

The functional $L$ is simply 1/4 times the invariant associated with the momentum conservation law (13).

### 3.2. *Conservation laws and Hamiltonian structure when $\alpha \simeq O(1)$*

Let us now consider the case where $\alpha \neq 0$ and cannot be neglected. In the Boussinesq limit, (7) reduces to

$$i\left(\partial_t - \partial_x\right)u + u_{xx} - 2\delta\beta\,|u|^2 u + i\left[\alpha u_{xxx} - \left(|u|^2\right)_x u\right] = 0. \tag{21}$$

This equation can be further simplified by a transformation that eliminates the $u_x$ and $u_{xx}$ terms. Introducing

$$u = \widehat{u}\left(\widehat{x}, \widehat{t}\right)\exp\left[\frac{i}{3\alpha}\left(\widehat{x} + \frac{\widehat{t}}{9\alpha}\right)\right],$$

where

$$\widehat{t} = t \text{ and } \widehat{x} = x + \left(1 - \frac{1}{3\alpha}\right) t,$$

into (21) leads to (after dropping the carets)

$$u_t - u\left(|u|^2\right)_x + \alpha u_{xxx} + 2i\delta\beta u\,|u|^2 = 0. \tag{22}$$

While (22) can be identified as a version of the SSE, as written it may be profitably understood as a complex or vector-valued generalization of the mKdV equation (see, e.g., [22] or [23]). Closely related models have been examined, for example, by Yang [7], Slunyaev [8], and Grimshaw and Helfrich [6].

We have found two conservation laws for (22). The energy and momentum conservation laws are given by, respectively,

$$\left(|u|^2\right)_t + \left[\alpha\left(\overline{u}u_{xx} + u\overline{u}_{xx} - |u_x|^2\right) - |u|^4\right]_x = 0. \tag{23}$$

$$\begin{aligned}
&\left[|u_x|^2 + \frac{1}{3\alpha}\,|u|^4 + \delta\beta i\left(u\overline{u}_x - \overline{u}u_x\right)\right]_t + \left[\frac{2}{3}\,|u|^2\left(\overline{u}u_{xx} + u\overline{u}_{xx}\right)\right.\\
&\quad - \frac{4}{3\alpha}|u|^6 - \frac{1}{3}\,|u_x|^2\,|u|^2 - \alpha\,|u_{xx}|^2 - \overline{u}_t u_x - u_t\overline{u}_x\\
&\quad + \frac{1}{6}\left(\overline{u}^2 u_x^2 + u^2\overline{u}_x^2\right) + \delta\beta i\left(u_t\overline{u} - \overline{u}_t u\right) + 2\alpha\delta\beta i\left(\overline{u}_x u_{xx} - u_x\overline{u}_{xx}\right)\\
&\quad \left. - 2\left(\delta\beta\right)^2|u|^4 + \delta\beta i\,|u|^2\left(\overline{u}u_x - u\overline{u}_x\right)\right]_x = 0.
\end{aligned} \tag{24}$$

We have labeled (23) and (24) as the energy and momentum conservation laws in a desire to be consistent with the established labels associated with the mKdV equation. The details of the derivation of these conservations laws is contained in the Appendix.

Equation (22) possesses a noncanonical nonlocal Hamiltonian formulation. If we write $u = p + iq$, where $p$ and $q$ are real-valued functions of $(x, t)$, then (22) takes the form of the $2 \times 2$ system

$$p_t = -\alpha p_{xxx} + p\left(p^2 + q^2\right)_x + 2\delta\beta q\left(p^2 + q^2\right), \tag{25}$$

$$q_t = -\alpha q_{xxx} + q\left(p^2 + q^2\right)_x - 2\delta\beta p\left(p^2 + q^2\right). \tag{26}$$

The system (25) and (26) is Hamiltonian for $H$ given by

$$H = \frac{1}{2}\int_{-\infty}^{\infty} |u|^2\,dx = \frac{1}{2}\int_{-\infty}^{\infty} p^2 + q^2\,dx, \tag{27}$$

with the cosymplectic form $\mathcal{M}$ given by

$$\mathcal{M} = \begin{bmatrix} -\alpha\partial_x^3 + 2q_x\,\partial_x^{-1}\circ q_x + 4p_x\,\partial_x^{-1}\circ p_x & 2q_x\partial_x^{-1}\circ p_x + 2\delta\beta\left(p^2+q^2\right) \\ 2p_x\partial_x^{-1}\circ q_x - 2\delta\beta\left(p^2+q^2\right) & -\alpha\partial_x^3 + 2p_x\,\partial_x^{-1}\circ p_x + 4q_x\,\partial_x^{-1}\circ q_x \end{bmatrix}. \tag{28}$$

It follows that

$$\begin{pmatrix} p_t \\ q_t \end{pmatrix} = \mathcal{M}\begin{pmatrix} \delta H/\delta p \\ \delta H/\delta q \end{pmatrix}, \tag{29}$$

where $\delta H/\delta p = p$ and $\delta H/\delta q = q$.

Direct substitution will verify that (28) and (29) reproduces (25) and (26). Additionally, it is also straightforward to verify the that required algebraic properties are satisfied by $\mathcal{M}$. Taken all together these details are, however, long and tedious and are not fully reproduced here. As before, to give an indication of the arguments required, we will show in the Appendix that $\mathcal{M}$ is skew symmetric.

## 4. Stokes wave solution and modulational instability

In this section, we determine the modulational stability properties of the Stokes wave solution to (7) in the Boussinesq limit. Our results agree with those presented in Kakutani and Michihiro [9] (see, also, Parkes [10] and Slunyaev [8]) in the parameter regime where the models overlap. When $\varepsilon = 0$, (7) can be written in the form

$$u_t - iu_{xx} + 2i\delta\beta\,|u|^2 u + \alpha u_{xxx} - \left(|u|^2\right)_x u = 0, \tag{30}$$

where we have introduced, without loss of generality, the co-moving coordinate system $x \to x + t$.

Equation (30) has the Stokes wave solution

$$u = u_0 \exp\left(-2i\delta\beta u_0^2 t\right), \tag{31}$$

where, for convenience, $u_0 \in \mathbb{R}$. To examine its stability, we introduce the perturbed solution in the form

$$u\,(x,t) = [u_0 + v\,(x,t)]\exp\left(-2i\delta\beta u_0^2 t\right), \tag{32}$$

where $v(x, t)\exp(-2i\delta\beta u_0^2 t)$ corresponds to the complex-valued perturbation. Substitution of (32) into (30) leads to the linear stability problem

$$v_t - iv_{xx} + \alpha v_{xxx} + u_0^2\,(2i\delta\beta - \partial_x)\,(v + \overline{v}) = 0. \tag{33}$$

Decomposing $v$ as

$$v = \phi(x,t) + i\psi(x,t), \tag{34}$$

where $\phi(x, t)$ and $\psi(x, t)$ are both real-valued functions, leads to the pair of equations

$$\left(\partial_t - 2u_0^2\partial_x + \alpha\partial_{xxx}\right)\phi + \psi_{xx} = 0,$$
$$\left(\partial_t + \alpha\partial_{xxx}\right)\psi + \left(4\delta\beta u_0^2 - \partial_{xx}\right)\phi = 0,$$

which together imply, for example, that

$$\left(\partial_t - 2u_0^2\partial_x + \alpha\partial_{xxx}\right)\left(\partial_t + \alpha\partial_{xxx}\right)\phi + \left(\partial_{xx} - 4\delta\beta u_0^2\right)\phi_{xx} = 0. \tag{35}$$

Assuming the normal mode solution

$$\phi = \phi_0 \exp\left(i\mu x - i\Omega t\right) + c.c., \tag{36}$$

where $\mu$ and $\Omega$ are the real-valued wavenumber and the complex-valued frequency, respectively, of the sideband perturbation, leads to the dispersion relationship

$$\Omega = -\mu u_0^2 - \alpha\mu^3 \pm |\mu|\sqrt{\left(u_0^2 + 4\delta\beta\right)u_0^2 + \mu^2}. \tag{37}$$

It follows from (9) that $\beta \geq 0$. Hence, if $\delta = 1$ ($\delta \equiv \pm\text{sgn}(\omega'')$; see the discussion associated with (7)), then $\Omega \in \mathbb{R}$ and (31) is modulationally (neutrally) stable. However, if $\delta = -1$, then $\Omega = \Omega_R \pm i\Omega_I$ with $\Omega_I > 0$ and (31) is modulationally unstable, provided

$$u_0^2 < -4\delta\beta = 4\beta, \tag{38}$$

for those wavenumbers in the interval

$$0 < \mu^2 < \left(4\beta - u_0^2\right)u_0^2. \tag{39}$$

Alternatively, (39) can be rewritten in the form

$$\left(u_0^2 - 2\beta\right)^2 + \mu^2 < 4\beta^2.$$

which describes the interior of the circle with radius $2\beta$ centered at $(0, 2\beta)$ in the $(\mu, u_0^2)$-plane.

The most unstable mode has sideband wavenumber $\mu_{\max}$ given by

$$\mu_{\max} = \pm\sqrt{\left(4\beta - u_0^2\right)u_0^2/2}, \tag{40}$$

and has the growth rate

$$\Omega_{I_{\max}} = \left(4\beta - u_0^2\right)u_0^2/2 \leq 2\beta^2. \tag{41}$$

When $\delta = -1$, modulational instability is not only limited by the magnitude of the sideband wavenumber $\mu$, it is also *independently* limited by the amplitude of the Stokes wave, i.e., a sufficiently large $|u_0|$ leads to neutral stability irrespective of the value of $|\mu|$ (see, also, [8]). This is different than that seen associated with the classical NLS equation (see, e.g., [24]) and is a consequence of the additional nonlinear derivative $u(|u|^2)_x$ term in (30). In addition, we see that the presence of the third-order dispersive term (proportional to $\alpha$) plays no explicit role in determining modulational stability (other than $\delta = -1 \Longleftrightarrow \alpha < 0$; see (9)). It does, however, play a crucial role in determining the frequency of the perturbations. In the large time limit, the disintegrating wavetrains evolve into propagating solitary waves; see [24]. These solutions are presented in the next section.

## 5. Solitary wave solutions

It is convenient to work with (21) directly in the form

$$u_t = u_x + i u_{xx} - 2i\delta\beta\, |u|^2\, u - \alpha u_{xxx} + \left(|u|^2\right)_x u = 0. \qquad (42)$$

Here we want to construct solitary wave solutions to (42) when the higher-order $\alpha u_{xxx}$ and $(|u|^2)_x u$ terms are explicitly included. It is briefly noted that if these terms are not included then (42) is simply the classical NLS equation which has well-known soliton solutions (see, e.g., [15]) and are not reproduced here.

### *5.1. Bright solitary waves*

Although it is possible to obtain the bright solitary wave solution by introducing the general ansatz

$$u = \Phi\,(x + \Omega t)\exp\left[i\,(kx + \omega t)\right],$$

into (42) where $\Phi(x + \Omega t)$ is an arbitrary real-valued function and imposing appropriate smoothly vanishing boundary conditions on $\Phi$ as $x \to \pm\infty$, as has been done, for example, by Grimshaw and Helfrich [6] for a closely related model, our presentation will follow the argument as described by Lou [25] for a similar model (which, in our opinion, cleanly obtains *both* the bright and dark solitary wave solutions).

Recalling that both the NLS and the cubically nonlinear mKdV possess soliton solutions in which the envelope is proportional to a sech function suggests the solitary wave solution to (42) will be similarly structured. Moreover, because (42) is invariant under the transformation $u \to (a + ib)\, u\,/\sqrt{a^2 + b^2}$ for all nonzero real-valued $a$ and $b$ implies that we may, without loss of generality, assume that the *amplitude parameter* associated with the bright solitary wave solution is real valued and positive.

Thus, the bright solitary wave solution to (42) will be of the form

$$u = a\,\mathrm{sech}\,(\lambda x + \Omega t)\exp\left[i\,(kx + \omega t)\right], \tag{43}$$

(modulo arbitrary translations in $x$ and $t$) for suitably chosen real-valued $a$, $k$, $\omega$, $\lambda$, and $\Omega$. Substitution of (43) into (42) yields an expression that is the sum of four terms individually proportional to $\mathrm{sech}(\lambda x + \Omega t)\tanh^{n}(\lambda x + \Omega t)\exp[i(kx + \omega t)]$ where $n = 0, 1, 2$, and 3, respectively. To have a solution valid for all $(x, t)$, the coefficients of each of these terms must be zero and this leads to the respective relationships

$$\omega = k - k^2 - \lambda^2 - 2\delta\beta a^2 + \alpha k\left(k^2 + 3\lambda^2\right), \tag{44}$$

$$\Omega = \lambda - 2\lambda k + \alpha\lambda\left(5\lambda^2 + 3k^2\right) + 2\lambda a^2, \tag{45}$$

$$(1 - 3\alpha k)\,\lambda^2 + \delta\beta a^2 = 0, \tag{46}$$

$$\lambda\left(3\alpha\lambda^2 + a^2\right) = 0. \tag{47}$$

It follows from (47) that $\lambda \neq 0$ because if $\lambda = 0$, then from (45) it follows that $a = 0$ (because, in general, $\delta\beta \neq 0$; see (9)), which corresponds to the trivial solution. Hence, it follows from (47) that

$$\lambda = \frac{a}{\sqrt{-3\alpha}} \Longrightarrow \alpha < 0 \Longleftrightarrow \delta = -1, \tag{48}$$

because we are assuming $\lambda \in \mathbb{R}$ and, without loss of generality, we may take the positive root. Henceforth for the bright solitary wave solution we set $\delta = -1$. It is noted that $\delta = -1$ for these bright solitary wave solutions is precisely analogous to the situation with the classical NLS limit when both the higher-order $\alpha u_{xxx}$ and $(|u|^2)_x u$ terms are neglected in (42).

Eliminating $\lambda^2$ from (45) using (48) implies

$$k = \frac{1 + 3\alpha\beta}{3\alpha}, \tag{49}$$

which is independent of the amplitude. It therefore follows from (5.3), (45), (48), and (49) that

$$\omega = \frac{(1 + 3\alpha\beta)\left(9\alpha^2\beta^2 - 3\alpha\beta + 9\alpha - 2\right) + 27\beta\alpha^2 a^2}{27\alpha^2}, \tag{50}$$

Finally, it follows from (44), (47), (48), and (49) that

$$\Omega = \frac{a\left(9\alpha^2\beta^2 + \alpha a^2 + 3\alpha - 1\right)}{3\alpha\sqrt{-3\alpha}}. \tag{51}$$

Observe that if we set $\beta = 0$ in (49) and (50), we recover the wavenumber and the frequency associated with the transformation that yielded (22). The bright solitary wave solution (43) therefore corresponds to a *one-parameter* family of solutions in which the free parameter may be taken as the amplitude $a$. In parameter regime where (42) and the model examined by Grimshaw and Helfrich [6] overlap the solutions (49), (50), and (51) are identical.

Another important observation to make that this solution does not exist, in general, in the limit as $\alpha \to 0$ when the higher-order $\alpha u_{xxx}$ term *is not included* but the $(|u|^2)_x u$ term *is included* in (42). If $\alpha \to 0$ then it follows from (47) that either $\lambda = 0$ or $a = 0$. The latter corresponds to just the zero solution and the former possibility implies, via (45), the same trivial result. Thus *both* the higher-order $\alpha u_{xxx}$ and $(|u|^2)_x u$ terms must be explicitly included for this bright solitary wave solution to exist. However, if both the $\alpha u_{xxx}$ and $(|u|^2)_x u$ terms are neglected in (42), then it can be shown that (47) does not arise and the solutions from the appropriately modified (5.3), (44), and (45) are simply $\omega = k - k^2 + \beta a^2$, $\Omega = (a - 2ak)\sqrt{\beta}$ and $\lambda = a\sqrt{\beta}$, which correspond exactly to the two-parameter $(k, a)$ bright soliton solutions associated with the classical NLS equation (see, e.g., [15]).

Thus, in summary, the bright solitary wave solution to (42) can be written in the form

$$u = a\,\text{sech}\,[\lambda\,(x - ct - x_0)]\exp\{i\,[k\,(x - ct - x_0) + (\omega + ck)\,(t - t_0)]\}\,, \tag{52}$$

where $\lambda$, $k$, and $\omega$ are given by (48), (49), and (50), respectively, and the translation velocity is given by

$$c \equiv -\frac{\Omega}{\lambda} = \frac{1 - 9\alpha^2\beta^2 - \alpha\left(a^2 + 3\right)}{3\alpha}, \tag{53}$$

and where $x_0$ and $t_0$ are arbitrary real-valued phase shift parameters. It is a train of bright solitary waves (plus linear dispersive wave tails) that the modulationally unstable Stokes waves described in Section 4 evolve into in the large time limit.

### *5.2. Dark solitary waves*

The dark solitary wave solutions associated with (42) will be of the form

$$u = [a + ib\tanh(\lambda x + \Omega t)]\exp[i\,(kx + \omega t)]\,, \tag{54}$$

(modulo arbitrary translations in $x$ and $t$) for suitably chosen real-valued $a$, $b$, $k$, $\omega$, $\lambda$, and $\Omega$. These dark solitary wave solutions do not decay to zero as $|x| \to \infty$. The form of (54) is suggested in analogy with the dark soliton solution of the classical NLS equation.

Substitution of (54) into (42) yields an expression that is the sum of five terms individually proportional to $\tanh^n(\lambda x + \Omega t)\exp[i(kx + \omega t)]$ where

$n = 0$, 1, 2, 3, and 4, respectively. Again, to have a solution valid for all $(x, t)$, the coefficients of each of these terms must be zero and this leads to the respective relationships (assuming a nontrivial solution)

$$b\Omega + a\omega = b\lambda + ak - k(2bk + ak) - 2\delta\beta a^3 + \alpha\left[b\lambda\left(3k^2 + 2\lambda^2\right) + ak^3\right], \tag{55}$$

$$\omega = \alpha k^3 - k^2 + k - 2\delta\beta\left(a^2 + b^2\right), \tag{56}$$

$$\Omega = \lambda(1 - 2k) + 2\delta\beta ab + \alpha\lambda\left(3k^2 + 2\lambda^2\right), \tag{57}$$

$$\lambda^2(1 - 3\alpha k) = b(\delta\beta b - \lambda a), \tag{58}$$

$$b^2 = 3\alpha\lambda^2. \tag{59}$$

Equations (55) through to (59) are not independent. Direct substitution shows that (56) through to (59) will satisfy (55) for *all* parameter values. Thus, (56) through to (59) will determine the dependence of the four parameters $\omega$, $k$, $\Omega$, and $\lambda$ in terms of the two amplitude parameters $a$ and $b$.

From (59) it follows that

$$\lambda = \pm\frac{b}{\sqrt{3\alpha}} \Longrightarrow \alpha > 0 \Longleftrightarrow \delta = 1, \tag{60}$$

because we are assuming $\lambda \in \mathbb{R}$. Henceforth we assume $\delta = 1$ in these dark solitary wave solutions. It therefore follows from (58) that

$$k = \frac{1 - 3\alpha \pm a\sqrt{3\alpha}}{3\alpha}, \tag{61}$$

and subsequently from (56) and (57) that, respectively,

$$\omega = \frac{\left(1 - 3\alpha \pm a\sqrt{3\alpha}\right)\left[9\alpha^2 + 3\alpha\left(4 + a^2\right) - 2 \mp a(1 + 6\alpha)\sqrt{3\alpha}\right]}{27\alpha^2} - 2\beta\left(a^2 + b^2\right), \tag{62}$$

$$\Omega = \pm\frac{b\left[9\alpha^2 + \alpha\left(3 + 3a^2 + 2b^2\right) - 1 \mp 6\alpha a\sqrt{3\alpha}\right]}{3\alpha\sqrt{3\alpha}} + 2\beta ab. \tag{63}$$

These solutions reduce in the appropriate limit to the known dark soliton solutions of classical NLS. For example, in the limit where both the $\alpha u_{xxx}$ and

$(|u|^2)_x u$ terms are neglected and we set $\beta = 1/2$ in (42), then it can be shown that (59) does not arise and the appropriately modified (56), (57), and (58) (again, the appropriately modified (55) is trivially satisfied for all parameter values) are solved by $k = 0$, $b^2 = 2\lambda^2$, $\omega = -2\lambda^2 - a^2$, and $\Omega = \pm\sqrt{2}a\lambda$, which corresponds exactly to the classical two-parameter $(a, b)$ dark soliton solution associated with the classical NLS equation (see, e.g., [15]).

Unlike the dark soliton solution associated with classical NLS, one is not free to set $k = 0$ in the dark solitary wave solution to (42). Indeed, this situation only arises if $a = \pm(1-3\alpha)/\sqrt{3\alpha}$. One particularly simple dark solitary wave solution to (42) occurs in the limit where $a \equiv 0$ (this situation is not permitted for the classical NLS equation). In this case, $\omega$, $k$, and $\Omega$ are given by ($\lambda$ remains unchanged) the expressions

$$k = \frac{1-3\alpha}{3\alpha}, \tag{64}$$

$$\omega = \frac{(1-3\alpha)\left(9\alpha^2 + 12\alpha - 2\right) - 54\beta\alpha^2 b^2}{27\alpha^2}, \tag{65}$$

$$\Omega = \pm\frac{b\left[9\alpha^2 + \alpha\left(3 + 2b^2\right) - 1\right]}{3\alpha\sqrt{3\alpha}}. \tag{66}$$

In summary, therefore, the dark solitary wave solution to (42) may be written in the form

$$u = \{a + ib\tanh[\lambda(x - ct - x_0)]\} \times \exp\{i[k(x - ct - x_0) + (\omega + ck)(t - t_0)]\}, \tag{67}$$

where $c \equiv -\Omega/\lambda$ and where $x_0$ and $t_0$ are arbitrary real-valued phase shift parameters.

## 6. Conclusions

The principal purpose of this paper has been to describe several important mathematical properties of the Boussinesq limit associated with the Dosser and Sutherland [5] model for the propagation of weakly nonlinear internal waves in a non-Boussinesq fluid. Our study began with the introduction of a nondimensionalization is that mapped the lowest-order terms in this new model to the canonical form of either the bright or dark form of the NLS equation. Depending on the value of the particular parameters, the Dosser and Sutherland [5] model can be identified as either a higher-order generalization of the derivative NLS equation the or complex-valued modified Korteweg-de Vries equation.

Grimshaw and Helfrich [6] have derived a model very similar to Dosser and Sutherland [5] to describe the long time behavior of dispersive wave solutions to the Ostrovsky equation. In another study, Yang [7] found a bright solitary wave solution (and presented numerical simulations) to a model similar to Dosser and Sutherland [5] for an asymptotically restricted set of parameter values. Slunyaev [8] also obtained an equation very similar to Dosser and Sutherland [5] to describe the nonlinear evolution of surface gravity waves and examined the modulational instability properties.

Detailed derivations of the mass, momentum, and energy conservation laws were given (these were not entirely trivial to obtain). In addition, the noncanonical Hamiltonian formulation are introduced for the Boussinesq limit of the model. As it turned out, the precise form of these conservation laws and Hamiltonian formulation depends critically on the magnitude of the third-order dispersive term. When the third-order dispersive term cannot be neglected, the model equation is most appropriately understood as a variant of the complex-valued modified Korteweg-de Vries or SSE. When the third-order dispersive term can be neglected to leading order, the model equation is most appropriately understood as a variant of the derivative NLS equation. The conservation laws and Hamiltonian formulation when the third-order dispersive term can be neglected to leading order and when it cannot be neglected to leading order are derived in both approximations. It was shown that the dependence of the Hamiltonian structure and conservation laws on the parameter measuring the order of magnitude of the third-order dispersive terms (denoted by $\alpha$) is not continuous in the $\alpha \to 0$ limit.

Numerical simulations presented by Dosser and Sutherland [5] suggest the possibility of modulational instability. The linear stability properties of the Stokes wave solution to the model were described. It was shown that (as expected) unstable sideband perturbations are possible only in the so-called bright limit of the model. The dependence of the modulational stability boundary with respect to the Stokes wave amplitude and sideband wavenumber was described.

The solitary wave solutions in both the bright and dark limits associated the model equation were explicitly constructed. These solutions depend continuously on all the parameters. In particular it is shown that in the limit when the third-order dispersive and nonlinear derivative terms (the complex-valued Korteweg-de Vries part of the model so to speak) are neglected, these solitary wave solutions exactly reduce to the well-known bright and dark soliton solutions of the NLS equation, respectively.

Some interesting open problems remain. We have not determined whether or not the model equations admit additional conservation laws. Also, we have not examined whether or not the model equations admit rational solitary wave solutions. Finally, the determination of the effects of the non-Boussinesq terms on the solitary wave solutions would be of genuine interest to determine.

The construction of a complete singular perturbation theory for the model equation is important from the viewpoint of applications (to some degree this is addressed in Grimshaw and Helfrich [6]). It is our intention to publish these results in another paper.

**Acknowledgments**

Preparation of this manuscript was supported in part by Discovery Research Grants awarded by the Natural Sciences and Engineering Research Council (NSERC) of Canada. Email: gordon.swaters@ualberta.ca. URL: pacific.math.ualberta.ca/gordon.

## Appendix.

### *Derivation of the conservation laws*

There is sufficient algebraic subtlety in the derivation of some of the conservation laws that we describe the details here.

We first describe the derivation of the mass, momentum and energy conservation equations associated with (11). A similar derivation is given by Hayashi and Ozawa [26] and Ozawa [27] for a related NLS equation of derivative type.

The mass conservation Equation (12) is straightforwardly obtained from the difference $\overline{u} \times (11) - c.c.$ , where $\overline{u}$ denotes the complex-conjugate of $u$ and $c.c.$ denotes the complex-conjugate of the preceding term.

The momentum conservation Equation (13) is obtained from the sum $\bar{u}_x \times (11) + c.c.$, given by

$$i\,(u_t\bar{u}_x - u_x\bar{u}_t) + i\left(|u|^2\right)_x(u_x\bar{u} - u\bar{u}_x) + \left(|u_x|^2 - \delta\beta\,|u|^4\right)_x = 0.$$

However, because

$$u_t\bar{u}_x - u_x\bar{u}_t = \frac{1}{2}(u\bar{u}_x - u_x\bar{u})_t + \frac{1}{2}(u_t\bar{u} - u\bar{u}_t)_x\,,$$

and

$$\begin{aligned}\left(|u|^2\right)_x(u_x\bar{u} - u\bar{u}_x) &= \left[|u|^2(u_x\bar{u} - u\bar{u}_x)\right]_x - |u|^2(u_x\bar{u} - u\bar{u}_x)_x\\ &= \left[|u|^2(u_x\bar{u} - u\bar{u}_x) - \frac{2i}{3}\left(|u|^6\right)\right]_x + \frac{i}{2}\left(|u|^4\right)_t\,,\end{aligned}$$

where (12) has been used to eliminate the $(u_x\bar{u} - u\bar{u}_x)_x$ term, (13) follows immediately.

The energy conservation Equation (14) is obtained from the sum $\bar{u}_t \times (11) + c.c.$, given by

$$u_{xx}\bar{u}_t + u_t\bar{u}_{xx} - \delta\beta\left(|u|^4\right)_t + i\left(|u|^2\right)_x (u_t\bar{u} - u\bar{u}_t) = 0.$$

However, because

$$u_{xx}\bar{u}_t + u_t\bar{u}_{xx} = (u_x\bar{u}_t + u_t\bar{u}_x)_x - \left(|u_x|^2\right)_t,$$

and

$$\begin{aligned}
\left(|u|^2\right)_x (u_t\bar{u} - u\bar{u}_t) &= \left(|u|^2\right)_t (u_x\bar{u} - u\bar{u}_x) \\
&\quad + \frac{1}{2}\left[|u|^2 (u\bar{u}_x - u_x\bar{u})\right]_t + \frac{1}{2}\left[|u|^2 (u_t\bar{u} - u\bar{u}_t)\right]_x \\
&= \frac{1}{2}\left[|u|^2 (u\bar{u}_x - u_x\bar{u})\right]_t - |u|^4 (u_x\bar{u} - u\bar{u}_x)_x \\
&\quad + \left[\frac{1}{2}|u|^2 (u_t\bar{u} - u\bar{u}_t) + \frac{i}{2}(u_x\bar{u} - u\bar{u}_x)^2 + |u|^4 (u_x\bar{u} - u\bar{u}_x)\right]_x \\
&= \left[\frac{1}{2}|u|^2 (u\bar{u}_x - u_x\bar{u}) + \frac{i}{3}|u|^6\right]_t \\
&\quad + \left[\frac{1}{2}|u|^2 (u_t\bar{u} - u\bar{u}_t) + \frac{i}{2}(u_x\bar{u} - u\bar{u}_x)^2 + |u|^4 (u_x\bar{u} - u\bar{u}_x) - \frac{i}{2}|u|^8\right]_x,
\end{aligned}$$

where (12) has been used to eliminate the $(|u|^2)_t$ and $(u_x\bar{u} - u\bar{u}_x)_x$ terms, (14) follows.

We now describe the derivation of the two conservation laws for (22). The energy conservation Equation (23) is obtained directly from the sum $\bar{u} \times (22) + c.c.$.

The momentum conservation law (24) may be obtained from the sum $\bar{u}_{xx} \times (22) + c.c.$, given by

$$\begin{aligned}
\left(|u_x|^2\right)_t + \left[|u|^2 |u_x|^2 - \alpha |u_{xx}|^2 - \bar{u}_t u_x - u_t\bar{u}_x\right]_x + 2\delta\beta i |u|^2 (\bar{u}u_{xx} - u\bar{u}_{xx}) \\
- |u_x|^2 \left(|u|^2\right)_x + \bar{u}^2 u_x u_{xx} + u^2\bar{u}_x\bar{u}_{xx} = 0.
\end{aligned} \tag{A1}$$

However, from the sum $|u|^2\,\bar{u} \times (22) + c.c.$ one obtains

$$\begin{aligned}
\frac{1}{2}\left(|u|^4\right)_t + \left[\alpha |u|^2 (\bar{u}u_{xx} + u\bar{u}_{xx}) - \frac{2}{3}|u|^6 - 2\alpha |u|^2 |u_x|^2\right]_x \\
+ 2\alpha |u_x|^2 \left(|u|^2\right)_x - \alpha\left(\bar{u}^2 u_x u_{xx} + u^2\bar{u}_x\bar{u}_{xx}\right) = 0.
\end{aligned} \tag{A2}$$

Consequently, from the sum (A1) + 2 × (A2) / (3$\alpha$) one obtains

$$\left(|u_x|^2 + \frac{1}{3\alpha}|u|^4\right)_t + \left[\frac{2}{3}|u|^2(\bar{u}u_{xx} + u\bar{u}_{xx}) - \frac{4}{9\alpha}|u|^6 - \frac{1}{3}|u_x|^2|u|^2 - \alpha|u_{xx}|^2 - \bar{u}_t u_x - u_t\bar{u}_x + \frac{1}{6}\left(\bar{u}^2u_x^2 + u^2\bar{u}_x^2\right)\right]_x + 2\delta\beta i\,|u|^2(\bar{u}u_{xx} - u\bar{u}_{xx}) = 0. \tag{A3}$$

To cast the term proportional to $\delta\beta$ in (A3) into a space-time divergence, we consider the difference $\bar{u}_x \times (22) - c.c.$, given by

$$u_t\bar{u}_x - \bar{u}_t u_x + \left[\alpha(\bar{u}_x u_{xx} - u_x\bar{u}_{xx}) + \delta\beta i\,|u|^4\right]_x + \left(|u|^2\right)_x(\bar{u}u_x - u\bar{u}_x) = 0,$$

which implies

$$|u|^2(\bar{u}u_{xx} - u\bar{u}_{xx}) = \frac{1}{2}(u\bar{u}_x - \bar{u}u_x)_t + \left[\frac{1}{2}(u_t\bar{u} - \bar{u}_t u) + \alpha(\bar{u}_x u_{xx} - u_x\bar{u}_{xx}) + \delta\beta i\,|u|^4 + |u|^2(\bar{u}u_x - u\bar{u}_x)\right]_x,$$

which results in (24).

*Skew symmetry of the cosymplectic forms*

Here, we show that cosymplectic form (18) is skew symmetric, i.e.,

$$\langle\Phi, \mathcal{M}\Psi\rangle = -\langle\Psi, \mathcal{M}\Phi\rangle\,,$$

for all suitable $\Phi = (\phi, \psi)^\top$ and $\Psi = (f, g)^\top$. It follows that

$$\begin{aligned}\langle\Phi, \mathcal{M}\Psi\rangle &= \int_{-\infty}^{\infty}(\phi, \psi)\begin{bmatrix} 2q\,\partial_x^{-1}\circ q & 1 - 2q\,\partial_x^{-1}\circ p \\ -1 - 2p\,\partial_x^{-1}\circ q & 2p\,\partial_x^{-1}\circ p\end{bmatrix}\begin{pmatrix} f \\ g\end{pmatrix}dx \\ &= \int_{-\infty}^{\infty}\phi[2q\partial_x^{-1}(qf) + g - 2q\partial_x^{-1}(pg)] \\ &\quad + \psi[-f - 2p\,\partial_x^{-1}(qf) + 2p\,\partial_x^{-1}(pg)]\,dx\end{aligned}$$

$$
\begin{aligned}
&=\int_{-\infty}^{\infty} \phi g-\psi f+2\partial_x \partial_x^{-1}(\phi q)[\partial_x^{-1}(qf)\\
&-\partial_x^{-1}(pg)]+2\partial_x \partial_x^{-1}(\psi p)[\partial_x^{-1}(pg)-\partial_x^{-1}(qf)]\,dx\\
&=-\int_{-\infty}^{\infty} \psi f-\phi g+2\partial_x^{-1}(\phi q)(qf-pg)+2\partial_x^{-1}(\psi p)(pg-qf)\,dx\\
&=-\int_{-\infty}^{\infty} f[2q\partial_x^{-1}(q\phi)+\psi-2q\partial_x^{-1}(p\psi)]\\
&+g[-\phi-2p\partial_x^{-1}(q\phi)+2p\partial_x^{-1}(p\psi)]\,dx\\
&=-\int_{-\infty}^{\infty}(f,g)\begin{bmatrix} 2q\,\partial_x^{-1}\circ q & 1-2q\,\partial_x^{-1}\circ p\\ -1-2p\,\partial_x^{-1}\circ q & 2p\,\partial_x^{-1}\circ p\end{bmatrix}\begin{pmatrix}\phi\\ \psi\end{pmatrix}dx=\\
&-\langle \Psi,\mathcal{M}\Phi\rangle\,.
\end{aligned}
$$

Here, we show that the cosymplectic form (28) is skew symmetric. It follows that

$$
\begin{aligned}
&\langle \Phi,\mathcal{M}\Psi\rangle\\
&=\int_{-\infty}^{\infty}\Phi^{\top}\begin{bmatrix} -\alpha\partial_x^3+2q_x\,\partial_x^{-1}\circ q_x+4p_x\,\partial_x^{-1}\circ p_x & 2q_{\,x}\partial_x^{-1}\circ p_x+2\delta\beta(p^2+q^2)\\ 2p_{\,x}\partial_x^{-1}\circ q_x-2\delta\beta(p^2+q^2) & -\alpha\partial_x^3+2p_x\,\partial_x^{-1}\circ p_x+4q_x\,\partial_x^{-1}\circ q_x\end{bmatrix}\Psi\,dx\\
&=\int_{-\infty}^{\infty}\{\phi[-\alpha\partial_x^3 f+2q_x\partial_x^{-1}(q_x f)+4p_x\,\partial_x^{-1}(p_x f)+2q_{\,x}\partial_x^{-1}(p_x g)+2\delta\beta(p^2+q^2)g]\\
&+\psi[2p_{\,x}\partial_x^{-1}(q_x f)-2\delta\beta(p^2+q^2)f-\alpha\partial_x^3 g+2p_x\,\partial_x^{-1}(p_x g)+4q_x\,\partial_x^{-1}(q_x g)]\}\,dx\\
&=\int_{-\infty}^{\infty}\{-\alpha(\phi\partial_x^3 f+\psi\partial_x^3 g)+2\delta\beta(p^2+q^2)(\phi g-\psi f)\\
&+2\partial_x\partial_x^{-1}(\phi q_x)[\partial_x^{-1}(q_x f)+\partial_x^{-1}(p_x g)]+2\partial_x\partial_x^{-1}(\psi p_x)[\partial_x^{-1}(q_x f)+\partial_x^{-1}(p_x g)]\\
&+4\partial_x\partial_x^{-1}(\phi p_x)\partial_x^{-1}(p_x f)+4\partial_x\partial_x^{-1}(\psi q_x)\partial_x^{-1}(q_x g)\}\,dx\\
&=-\int_{-\infty}^{\infty}\{f[-\alpha\partial_x^3\phi+2q_x\partial_x^{-1}(q_x\phi)+4p_x\,\partial_x^{-1}(p_x\phi)+2q_{\,x}\partial_x^{-1}(p_x\psi)+2\delta\beta(p^2+q^2)\psi]\\
&+g\left[2p_{\,x}\partial_x^{-1}(q_x\phi)-2\delta\beta(p^2+q^2)\phi-\alpha\partial_x^3\psi+2p_x\,\partial_x^{-1}(p_x\psi)+4q_x\,\partial_x^{-1}(q_x\psi)\right]\}\,dx\\
&=-\int_{-\infty}^{\infty}\Psi^{\top}\begin{bmatrix} -\alpha\partial_x^3+2q_x\,\partial_x^{-1}\circ q_x+4p_x\,\partial_x^{-1}\circ p_x & 2q_{\,x}\partial_x^{-1}\circ p_x+2\delta\beta(p^2+q^2)\\ 2p_{\,x}\partial_x^{-1}\circ q_x-2\delta\beta(p^2+q^2) & -\alpha\partial_x^3+2p_x\,\partial_x^{-1}\circ p_x+4q_x\,\partial_x^{-1}\circ q_x\end{bmatrix}\Phi\,dx\\
&=-\langle \Psi,\mathcal{M}\Phi\rangle.
\end{aligned}
$$

## References

1. A. D. D. Craik, *Wave Interactions and Fluid Flows*, Cambridge, Cambridge, UK, 1985.
2. B. R. Sutherland, Weakly nonlinear internal wavepackets, *J. Fluid Mech.* 569:249–258 (2006).

3. J. F. SCINOCCA and T. G. SHEPHERD, Nonlinear wave-activity conservation laws and Hamiltonian structure for the two-dimensional anelastic equations, *J. Atmos. Sciences* 49:5–27 (1992).
4. M. E. MCINTYRE, On the "wave momentum"myth, *J. Fluid Mech.* 106:331–347 (1981).
5. H. V. DOSSER and B. R. SUTHERLAND, Weakly nonlinear non-Boussinesq internal gravity wavepackets, *Physica D* 240:346–356 (2011).
6. R. GRIMSHAW and K. HELFRICH, Long-time solutions of the Ostrovsky Equation, *Stud. Appl. Math.* 121:71–88 (2008).
7. J. YANG, Stable embedded solitons *Phys. Rev. Lett.* 91:143903(1–4) (2003).
8. A. V. SLUNYAEV, A high-order nonlinear envelope equation for gravity waves in finite-depth water, *J. Exp. Theor. Phys. (JETP)* 101:926–941 (2005).
9. T. KAKUTANI and K. MICHIHIRO, Marginal state of modulational instability—Note on Benjamin-Feir instability, *J. Phys. Soc. Japan* 52:4129–4137 (1983).
10. E. J. PARKES, The modulation of weakly non-linear dispersive waves near the marginal state of instability, *J. Phys. A: Math. Gen.* 20:2025–2036 (1987).
11. P. H. LEBLOND and L. A. MYSAK, *Waves in the Ocean*, Elsevier, New York, USA, 1978.
12. S. GHOSH, A. KUNDU and S. NANDY, Soliton solutions, Liouville integrability and gauge equivalence of Sasa Satsuma equation, *J. Math. Phys.* 40:1993–2000 (1999).
13. N. SASA and J. SATSUMA, New-type of soliton solutions for a higher-order nonlinear Schrödinger equation, *J. Phys. Soc. Japan* 60:409–417 (1991).
14. Y. KODAMA, Optical solitons in a monomode fiber, *J. Stat. Phys.* 39:597–614 (1985).
15. P. G. DRAZIN and R. S. JOHNSON, *Solitons: An Introduction*, Cambridge, Cambridge, UK, 1989.
16. V. E. ZAKHAROV and A. B. SHABAT, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media, *Sov. Phys. JETP* 34:62–69 (1972).
17. D. J. KAUP and A. C. NEWELL, An exact solution for a derivative nonlinear Schrödinger equation, *J. Math. Phys.* 19:798–801 (1978).
18. R. WILLOX, W. HEREMAN and F. VERHEEST, Complete integrability of a modified vector derivative nonlinear Schrö dinger equation, *Phys. Scripta* 52:21–26 (1985).
19. A. KUNDU, Landau-Lifshitz and higher-order nonlinear systems gauge generated from nonlinear Schrödinger type systems, *J. Math. Phys.* 25:3433–3438 (1984).
20. G. E. SWATERS, *Introduction to Hamiltonian Fluid Dynamics and Stability Theory*, Chapman and Hall/CRC, Boca Raton, USA, 2000.
21. F. J. POULIN and G. E. SWATERS, Sub-inertial dynamics of density-driven flows in a continuously stratified fluid on a sloping bottom. I. Model derivation and stability conditions, *Proc. R. Soc. Lond. A* 455:2281–2304 (1999).
22. M. V. FOURSOV, Classification of certain integrable coupled potential KdV and modified KdV-type equations, *J. Math. Phys.* 41:6173–6185 (2000).
23. A. SERGYEYEV and D. DESMKOI, Sasa-Satsuma (complex modified Korteweg-de Vries II) and the complex sine-Gordon II equation revisited: Recursion operators, nonlocal symmetries, and more, *J. Math. Phys.* 48:042702(1–10) (2007).
24. A. C. NEWELL, *Solitons in Mathematics and Physics*, CBMS-NSF Regional Conference Series in Applied Mathematics, Philadelphia, 1985.
25. S.-Y. LOU, Self-steepening and third-order dispersion induced optical solitons in fiber, *Commun. Theor. Phys.* 35:589–592 (2001).
26. H. HAYASHI and T. OZAWA, Finite energy solutions of nonlinear Schrödinger equations of derivative type, *SIAM J. Math. Anal.* 25:1488–1503 (1994).

27. T. Ozawa, On the nonlinear Schrödinger equations of derivative type, *Indiana Univ. Math. J.* 45:137–163 (1996).

Institute for Geophysical Research
Department of Mathematical & Statistical Sciences
Department of Physics, University of Alberta

(Received May 5, 2011)