1	The Influence of Lateral Spreading upon Solitary Wave Formation by		
2	Internal Tides		
3	Kehan Li, ^a Bruce R. Sutherland, ^{a,b}		
4	^a Department of Earth and Atmospheric Sciences, University of Alberta, Edmonton, AB, T6G		
5	2E3, Canada		
6	^b Department of Physics, University of Alberta, Edmonton, AB, T6G 2E1, Canada		

7 Corresponding author: Bruce R. Sutherland, bruce.sutherland@ualberta.ca

ABSTRACT: As internal tides propagate in the ocean, they carry and dissipate energy over 8 hundreds and even thousands of kilometers. In relatively shallow seas the low vertical mode 9 internal tide can evolve to form solitary waves whose surface signature can be detected by satellites 10 as regions of high and low reflectance where the surface is roughened or smoothed respectively 11 by horizontally convergent and divergent flows induced by the waves. To gain insight into what 12 processes lead to the observation of internal solitary waves by satellites, we perform fully nonlinear 13 simulations in three dimensions to examine the evolution of horizontally propagating, vertical 14 mode-1 internal tides as it depends on wave amplitude, ocean depth, and the spanwise extent of 15 the waves. The background stratification is set up according to measurements in the South China 16 Sea. The spanwise evolution of the 3D waves is examined in terms of the lateral spreading, radius 17 of curvature, and sea surface signature corresponding to a threshold in the surface horizontal 18 convergent and divergent flow. The evolution of sea surface signature compares favorably to a 19 satellite image in the South China Sea, particularly for waves initially having spanwise extent 20 comparable to their horizontal wavelength. 21

22 1. Introduction

Internal tides are generated when barotropic tides periodically move stratified ocean water across 23 submarine topography (Balmforth et al. 2002). If the topography is sufficiently steep, vertically and 24 horizontally propagating beams of internal tides are launched near the topography (e.g. Pétrélis 25 et al. 2006). These beams can cause dramatic vertical displacements, leading to strong local 26 turbulent mixing (e.g. Rudnick et al. 2003; Vic et al. 2019). The beams are composed of a 27 superposition of high vertical modes that are observed to transform into low vertical mode internal 28 tides (with vertical structure on the scale of the ocean depth) after interacting with the near-surface 29 stratification (Martin et al. 2006). Though partially dissipated by local mixing, most of the energy 30 in these beams is transported away from the topography by the low-mode waves, mostly by mode-1 31 waves (Echeverri et al. 2009). It is estimated that 74% of the internal tide energy near the Hawaiian 32 ridge is radiated away from the generation site as low-mode waves (Klymak et al. 2006; Carter et al. 33 2008). In the Luzon Strait, east of the South China Sea, only 40% of the energy is radiated away, 34 possibly due to the more complex bottom topography in the Luzon Strait (Garrett and Kunze 2007; 35 Buijsman et al. 2010) and the westward propagating branch of the Kuroshio (Buijsman et al. 2010). 36 Their radiated energy propagates westward across the South China Sea towards the continental shelf 37 of China. These waves are observed to be dominated by a mode-1 signal having a combination of 38 semi-diurnal and diurnal frequencies (Farmer et al. 2009; Johnston et al. 2013). When the forcing 39 is stronger, the internal tides tend to steepen during their propagation to form solitary waves which 40 are visible by the sea-surface signature in satellite images as shown, for example, in Fig. 1. 41

Given the challenging operating conditions in the Luzon Strait (Alford et al. 2015), in-situ 47 observation data from the wave generation site has been notably scarce. Hence, the amplitude and 48 spanwise width of the low-mode internal tide wave near the generation site around 122°E are not 49 well established. Some progress has been made through numerical simulations (Simmons et al. 50 2011; Zhang et al. 2011) that included the complex topography between the Luzon Strait and the 51 South China Sea. These correctly predicted the propagation speed of the internal tides, and their 52 associated horizontally convergent and divergent flows compared favorably with satellite imagery. 53 Such simulations are expensive however, in that they require high resolution over large horizontal 54 domains in order correctly to capture the amplitude of the waves at the generation site and the 55 processes of nonlinear steepening as they shoal. 56

3



FIG. 1. MODIS image showing three nonlinear internal wave trains (labelled by A, B, and C) crossing the deep basin west of Luzon Strait. Ocean measurements were collected at the P1 and P2 locations indicated by red stars. Inset: Corresponding inverted echo-sounder time series for these waves at P2 from which path-averaged wave speeds are determined. 122°E is the estimated generation site of internal tides in the numerical models. (Image reproduced from Farmer et al. (2009).)

In general, because the internal tide has large horizontal spatial extent, on the order of a hundred 57 kilometers, weakly nonlinear shallow water theory has been useful to diagnose circumstances 58 leading the formation of solitary waves as it depends upon their amplitude and the strength of 59 background rotation (Ostrovsky and Stepanyants 1989; Helfrich and Melville 2006; Helfrich and 60 Grimshaw 2008; Grimshaw and Helfrich 2012). A different theoretical perspective on the formation 61 of solitary waves noted that the self-interaction of the internal tide initially excites superharmonic 62 waves with double the horizontal wavenumber (half the wavelength) (Sutherland 2016; Baker 63 and Sutherland 2020). For sufficiently large amplitude waves with sufficiently small background 64 rotation, the excited superharmonics may successively excite higher order superharmonics that 65 superimpose to form solitary waves through what has been called the superharmonic cascade 66

⁶⁷ (Sutherland and Dhaliwal 2022). Although the shallow water and superharmonic cascade equations
 ⁶⁸ are easily solved numerically, they were restricted to two dimensional internal tides having infinite
 ⁶⁹ spanwise extent. Thus they neglected the potentially important influence upon solitary wave
 ⁷⁰ formation of the spanwise spreading of laterally confined internal waves.

Extensions to shallow water theory to include the influence of lateral spreading have been explored (Karl Helfrich, pers. comm.). The approach taken here is to use numerical simulations to examine the three-dimensional evolution of spanwise-localized internal tides that disperse laterally. In this idealized study, the domain is horizontally periodic with uniform depth. Thus our results generally set criteria for the appearance of the sea-surface signature of the waves as it depends upon their initial amplitude and spanwise extent, as well as the ocean depth.

In section 2, we review the ocean measurements in the South China Sea and construct the 77 background stratification for use in the numerical models. In addition to our simulations in three 78 dimensions (3D), we also perform simulations in two dimensions (2D) in order to compare the 79 evolution of spanwise finite with spanwise infinite waves. The equations for the 2D and 3D 80 numerical models are described in section 3 along with methods used to analyze the evolution of 81 the waves. In section 4, we present the simulation results regarding the formation of solitary wave 82 trains. In particular, predictions for internal solitary waves manifesting a sea surface signature are 83 compared with satellite observations. Discussion and conclusions are presented in section 5 84

2. Observations and Initial Conditions

It is well-documented that solitary waves form during the evolution of westward propagating 86 internal tides generated at Heng Chun Ridge and Lan Yu Ridge in the South China Sea (Farmer 87 et al. 2009; Li et al. 2009; Simmons et al. 2011). This inspires our interest in investigating the 88 formation of solitary wave trains westward of this location. The observation data are taken from 89 two separate sets of observations by Farmer et al. (2009) and Johnston et al. (2013). The dataset 90 reported by Farmer et al. (2009) measured the full-depth stratification from 5 deployments around 91 21°N and 119°E during 2005 and 2007. This data is used to define approximate analytic profiles of 92 stratification employed in our numerical simulations. The dataset reported by Johnston et al. (2013) 93 measured the stratification around the top 300 meters of the ocean at 20.71°N and 120.45°E during

⁹⁵ UTC June 14th to July 1st 2011. This data is used to validate the structure of our approximate ⁹⁶ stratification profile near the surface.

⁹⁷ We choose to represent the stratification by a continuous profile of the squared buoyancy fre-⁹⁸ quency given by a double piecewise-exponential function of the form

$$N^{2}(z) = \begin{cases} N_{0}^{2} e^{(z-z_{0})/\sigma_{1}} & z_{*} \leq z \leq 0, \\ N_{*}^{2} e^{(z-z_{*})/\sigma_{2}} & -H \leq z < z_{*}, \end{cases}$$
(1)

⁹⁹ where $N^2(z_0) = N_0^2$ and $N^2(z_*) = N_0^2 e^{(z_*-z_0)/\sigma_1} = N_*^2$, such that the two exponential functions meet ¹⁰⁰ at $z = z_*$. Here, z_* is the depth where the stratification transitions from the upper layer to the abyssal ¹⁰¹ exponential profiles and z_0 represents the depth of the surface-mixed layer. Rather than setting N^2 ¹⁰² to zero above z_0 , for simplicity, we allow the stratification to increase exponentially to the surface. ¹⁰³ Previous studies have shown that the evolution of the mode-1 internal tide is insensitive to the ¹⁰⁴ details of the stratification in the surface mixed layer (Sutherland and Dhaliwal 2022).

¹⁰⁵ Due to noise in the observation data, it is easier to find an analytic fit to potential density profiles, ¹⁰⁶ $\bar{\rho}(z)$, than to $N^2(z)$. Using $N^2 = -(g/\rho_0)d\bar{\rho}/dz$ and vertically integrating Eq. (1), gives the ¹⁰⁷ corresponding analytic expression for potential density:

$$\bar{\rho}(z) = \begin{cases} -\frac{\rho_0 N_0^2 \sigma_1}{g} \left[e^{-z_0/\sigma_1} \left(e^{z/\sigma_1} - e^{z_*/\sigma_1} \right) \right] + \rho_* & z_* \le z \le 0, \\ -\frac{\rho_0 N_0^2 \sigma_2}{g} \left\{ e^{-(z_*/\sigma_2) - ((z_0 + z_*)/\sigma_1)} \left[e^{z/\sigma_2} - e^{-H/\sigma_2} \right] \right\} + \rho_b & -H \le z < z_*, \end{cases}$$
(2)

in which ρ_0 is the characteristic potential density, $\rho_* \equiv \bar{\rho}(z_*)$ and $\rho_b \equiv \bar{\rho}(-H)$.

To construct the analytic density profiles, we set $z_0 = -30.5$ m and $z_* = -362$ m, and find the values of σ_1 and σ_2 that best fit the observations. The parameters used in this fit are listed in Table 1. The analytic potential density profile is compared with observations in Fig. 2a; the corresponding squared buoyancy frequency profile is shown in Fig. 2b.

In addition to the stratification, we prescribe other background conditions and wave properties based upon observations. These values are given in Table 2. The Coriolis parameter, f_0 , was taken as a constant corresponding to a latitude of 21°N. The wave frequency, $\omega = 0.000144 \text{ s}^{-1}$ was fixed, corresponding to the forcing by the M2 internal tide. Between the Luzon Strait and the observation sites, P1 and P2 (see Fig.1), of Farmer et al. (2009), the ocean depth gradually decreases from



FIG. 2. (a) Comparison of the observed potential density profile to our approximation (observation 1 from Farmer et al. (2009) and observation 2 from Johnston et al. (2013)) and (b) the corresponding piecewise exponential function of squared buoyancy frequency.

Parameters	Values
ρ_0	1022.21 kg m ⁻³
$ ho_*$	$1026.14 \text{ kg m}^{-3}$
$ ho_b$	$1027.66 \text{ kg m}^{-3}$
z_0	-30.5 m
Z*	-362 m
N_0	0.0157 s^{-1}
N_*	0.0065 s^{-1}
σ_1	186 m
σ_2	351 m

TABLE 1. Values for the parameters that give the best-fit piecewise exponential profiles for observations of stratification in the South China Sea.

 $H \simeq 3500 \text{ m}$ to 2000 m. Because the domain of our simulations has constant depth, we examine the influence of *H* on the wave evolution by running simulations at different fixed-depths in the range between 2000 and 3500 m. The maximum vertical displacement of the waves, *A*₀, varies

Parameters	Symbols	Values
Coriolis parameter	fo	0.00005181 s ⁻¹
Ocean depth	Н	[2000-3500] m
Internal tide frequency	ω	0.000144 s ⁻¹
Wavenumber	k	$[4.86-5.30] \times 10^{-5} \text{ m}^{-1}$
Wavelength	л	[119-129] km
Initial spanwise width of internal tide	σ_y	[50-125,∞] km
Initial vertical displacement amplitude	A_0	[37.5-100] m

TABLE 2. Simulation parameters governing the background values and initial structure of internal tides based
 upon observations in the South China Sea. The ocean depth and the internal tide wavenumber, wavelength,
 spanwise width and amplitude span the ranges indicated.

between the spring and neap cycles of the tide. Our simulations are initialized with horizontally 126 periodic vertical mode-1 waves having a maximum vertical displacement amplitude in the range 127 between 12.5 m and 75 m. Given the frequency of the waves, the horizontal wavelength, λ , (and 128 hence wavenumber, k) varies depending upon the ocean depth H. This is found by solving the 129 eigenvalue problem for the dispersion relation of vertical mode-1 waves in fluid of given depth, 130 H, and determining the value of k that has frequency $\omega = 0.000144 \,\mathrm{s}^{-1}$. Although there are no 131 direct observations of the spanwise extent of the internal tide where it first develops, in our three-132 dimensional simulations we examine a range of widths between $\sigma_v = 50$ and 125 km to examine 133 how the width influences lateral dispersion and the possible formation of internal solitary waves. 134

3. Numerical models

We performed numerical simulations in two dimensions (2D spanwise-infinite waves) and three 139 dimensions (3D spanwise-localized waves). Both models solve the incompressible, Boussinesq 140 equations on the f-plane in constant-depth fluid having periodic boundaries in the horizontal with 141 free-slip upper and lower boundary conditions. In practice, the models work with non-dimensional 142 variables, with length-related units scaled by the ocean depth H and time-related units scaled by 143 the characteristic buoyancy frequency N_0 . We begin in section 3a by describing the 2D model 144 equations and their initial conditions. In section 3b we describe the 3D model equations. Analysis 145 methods are described in section 3c. 146

147 a. 2D model

¹⁴⁸ We consider the evolution of 2D waves having structure in the streamwise (x) and vertical (z) ¹⁴⁹ directions. Although there can be motion in the spanwise (y) direction, the fields of interest are ¹⁵⁰ independent of y. The 2D model has been used previously in the study of the nonlinear evolution of ¹⁵¹ internal tides Sutherland (2016); Sutherland and Dhaliwal (2022). It computes the time evolution ¹⁵² of the spanwise vorticity, ζ , spanwise velocity, v, and buoyancy, b in the x-z plane:

$$\frac{\partial \zeta}{\partial t} = -u\frac{\partial \zeta}{\partial x} - w\frac{\partial \zeta}{\partial z} - \frac{\partial b}{\partial x} + f_0\frac{\partial v}{\partial z} + v\mathcal{D}\zeta,$$
(3)

$$\frac{\partial v}{\partial t} = -u\frac{\partial v}{\partial x} - w\frac{\partial v}{\partial z} - f_0 u + v\mathcal{D}v, \qquad (4)$$

$$\frac{\partial b}{\partial t} = -u\frac{\partial b}{\partial x} - w\frac{\partial b}{\partial z} - N^2 w + \kappa \mathcal{D}b,$$
(5)

¹⁵³ in which *u*, *v*, and *w* are velocities in the *x*, *y*, and *z* directions, respectively. In practice the ¹⁵⁴ equations were recast into non-dimensional variables using the length scale *H* (the domain depth) ¹⁵⁵ and time scale N_0^{-1} , in which $N_0 = 0.0157 \text{ s}^{-1}$ is the characteristic buoyancy frequency (see Table 1). ¹⁵⁶ The results presented here, however, are given in dimensional form.

The domain was discretized on an evenly spaced grid in the *z*-direction and in terms of their horizontal Fourier components in the *x*-direction (spectral representation). The spatial resolution typically consisted of 257 vertical levels and 1024 horizontal Fourier components, corresponding to 257×2049 grid points in real space. Various resolutions were tested and it was found that doubling the resolution in both directions did not quantitatively influence the results.

Although we treat the motion to be inviscid and non-diffusive, for numerical stability we include 162 effective diffusivity in the last term on the right-hand sides of Eqs. (3) - (5). Here $v = 10^{-5}H^2N_0$ 163 represents the kinematic viscosity and $\kappa = 10^{-5} H^2 N_0$ represents the diffusivity. Although these 164 numbers were much larger than realistic values for the ocean, viscous and diffusive damping was 165 only applied to small-scale disturbances and not motion on the scale of the waves. Explicitly, the 166 diffusion operator, \mathcal{D} , is a Laplacian operator acting only upon horizontal Fourier components 167 with horizontal wavenumber greater than a cut-off wavenumber, $n_c k$, where k is the prescribed 168 horizontal wavenumber of the parent internal tide. We typically used a cut-off of $n_c = 128$. 169

The background stratification was set by the piecewise exponential function described in section 2. The simulations were initialized with between $n_w = 1$ and 4 wavelengths of a horizontally periodic, vertical mode-1 "parent" internal wave with a prescribed horizontal wavenumber *k* and vertical displacement amplitude A_0 . Explicitly, the initial fields were given by

$$\zeta(x,z,0) = \frac{1}{2}\omega k \frac{N^2 - f_0^2}{\omega^2 - f^2} A_0 \hat{\psi}(z) e^{ikx} + \text{c.c.},$$
(6)

$$v(x,z,0) = \frac{1}{2} i \frac{f_0}{k} A_0 \hat{\psi}'(z) e^{ikx} + \text{c.c.},$$
(7)

$$b(x,z,0) = \frac{1}{2}N^2(z)A_0\hat{\psi}(z)e^{ikx} + \text{c.c.},$$
(8)

¹⁷⁴ in which c.c. denotes the complex conjugate, $\hat{\psi}(z)$ is the vertical structure of the streamfunction ¹⁷⁵ and $\hat{\psi}'$ is its derivative. The vertical structure function is given by the solution of the eigenvalue ¹⁷⁶ problem

$$\hat{\psi}'' + k^2 \frac{N^2 - \omega^2}{\omega^2 - f^2} \hat{\psi} = 0, \quad \hat{\psi}(-H) = \hat{\psi}(0) = 0, \tag{9}$$

where $\hat{\psi}''$ denotes the second-order derivative of $\hat{\psi}$, and N^2 is given by Eq. (1). For given k, the eigenvalue problem was solved using a Galerkin method (Sutherland 2016; Baker and Sutherland 2020). From this we extract the vertical structure and the corresponding wave frequency, $\omega(k)$, of the lowest vertical mode.

¹⁸¹ For time-stepping, an Euler forward scheme was used for diffusive terms for numerical stability, ¹⁸² and a leapfrog scheme was employed to advance in time the non-diffusive terms with an Euler ¹⁸³ backstep taken every 20 steps to avoid splitting errors. Time steps were taken as $\Delta t = 0.05N_0^{-1} \simeq$ ¹⁸⁴ 3.2 s. Simulations performed with half Δt resulted in no significant quantitative differences.

185 b. 3D model

186 1) EVOLUTION EQUATIONS

The 3D model more realistically simulated the evolution of internal tides by considering the influence of waves having finite spanwise extent. Such waves will spread laterally even as the waves possibly steepen to form solitary waves. Similar to the 2D model, the 3D model used a rectangular domain with horizontally (x and y) periodic boundary conditions and free-slip upper and lower boundary conditions. The model computed the time evolution of u and v velocities in the x and y direction, respectively, and the buoyancy field, b. The equations for horizontal momentum and internal energy, neglecting viscosity and diffusion, are

$$\frac{\partial u}{\partial t} = -\frac{\partial u^2}{\partial x} - \frac{\partial v u}{\partial y} - \frac{\partial w u}{\partial z} + f_0 v - \frac{1}{\rho_0} \frac{\partial p}{\partial x},\tag{10}$$

$$\frac{\partial v}{\partial t} = -\frac{\partial uv}{\partial x} - \frac{\partial v^2}{\partial y} - \frac{\partial wv}{\partial z} - f_0 u - \frac{1}{\rho_0} \frac{\partial p}{\partial y},\tag{11}$$

$$\frac{\partial b}{\partial t} = -\frac{\partial ub}{\partial x} - \frac{\partial vb}{\partial y} - \frac{\partial wb}{\partial z} - N^2 w.$$
(12)

A diagnostic equation for vertical velocity, w, is given using incompressibility:

$$\frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}.$$
(13)

A diagnostic equation for the dynamic pressure, p, which includes nonhydrostatic effects, is found by taking the divergence of the 3D momentum equations and using incompressibility:

$$\frac{1}{\rho_0} \nabla^2 p = -\left[\frac{\partial^2 (u^2)}{\partial x^2} + \frac{\partial^2 (v^2)}{\partial y^2} + \frac{\partial^2 (w^2)}{\partial z^2}\right] - 2\left[\frac{\partial^2 (uv)}{\partial x \partial y} + \frac{\partial^2 (uw)}{\partial x \partial z} + \frac{\partial^2 (vw)}{\partial y \partial z}\right] + f_0 \frac{\partial v}{\partial x} - f_0 \frac{\partial u}{\partial y} + \frac{\partial b}{\partial z}.$$
(14)

¹⁹⁷ A spectral representation was used for the horizontal fields, being decomposed into their Fourier ¹⁹⁸ components in the *x* and *y* directions. The vertical fields were decomposed into Fourier cosine ¹⁹⁹ series for *u* and *v* and Fourier sine series for *b*. For example, the vertical structure of *b* is represented ²⁰⁰ by the sine-series coefficients, b_n , such that the amplitude in *z* is given by

$$\hat{b}(z) = \sum_{j=1}^{n_z} b_j \sin(m_j z), \quad m_j = j(\pi/H) \text{ and } j = 1, 2, \dots n_z,$$
 (15)

²⁰¹ in which n_z is the number of vertical modes.

The spatial domain was of size $L_x \times L_y \times H$. The streamwise extent was set by $L_x = 2\pi n_w/k$, in which n_w is the initial number of horizontal wavelengths of the internal tide in the domain. The spanwise dimension was set to be $L_y = 500H$ or 1000H depending on the initial spanwise halfwidth, σ_y of the waves. L_y was chosen to be at least ten times $2\sigma_y$. Typically, $n_x = 512$ and $n_z = 256$

grid points were used in the streamwise and vertical dimensions, respectively. Depending on the 206 spanwise width of the waves, $n_y = 256$ or 512 modes were used in the spanwise direction. Although 207 the code works in nondimensional units based on the domain depth, H, the typical dimensional 208 vertical resolution was $\Delta z \sim 10$ m and the typical dimensional streamwise and spanwise resolutions 209 were $\Delta x \sim 250$ m and $\Delta y \sim 5$ km, respectively. These are sufficient to resolve observed solitary 210 waves in the South China Sea which have typical streamwise extent on the order of 3 km (taking 211 ~ 1000 s to move past a mooring site propagating at ~ 3 m/s) and spanwise extent on the order of 212 100 km (e.g. see Figure 1). 213

214 2) EXPONENTIAL FILTER

Equations (10) - (12) do not include Laplacian diffusion for numerical stability. Because numerical noise in the 3D model grew faster compared to the 2D models, we instead applied an exponential filter to Eqs. (10), (11), and (12) at every time step. In this approach, the Fourier components with wavenumber higher than a specific cut-off wavenumber, $n_{cut}k$, were damped exponentially with increasing wavenumber (Subich et al. 2013). Taking Fourier components in the *x* direction as an example, a Fourier field f_n was filtered by $f_n \rightarrow \chi f_n$, in which

$$\chi(n) = \begin{cases} 1, & n < n_{\text{cut}}, \\ \exp\left[-e_1 \left(\frac{n - n_{\text{cut}}}{n_{\text{mx}} - n_{\text{cut}}}\right)^{e_2}\right], & n \ge n_{\text{cut}}. \end{cases}$$
(16)

Here e_1 is the filter strength, e_2 is the filter order, and $n_{mx} \equiv n_x/2 = 256$ is the total number of Fourier components in *x*. We used the default values provided in Subich et al. (2013): $n_{cut} = 0.6n_{mx}$, $e_1 = 20$, and $e_2 = 2$. Using this filter, numerical noise was damped effectively without affecting the wavenumbers having non-negligible amplitude.

225 3) INITIALIZATION AND TIME-STEPPING

The background stratification profile was identical to the 2D model, constructed by the piecewise exponential function described in section 2. A "parent" vertical mode-1 internal tide with a prescribed horizontal wave number *k* and maximum vertical displacement amplitude A_0 was initialized in the domain at t = 0. The parent wave was a plane wave in the *x* direction with amplitude decaying as a Gaussian in the *y*-direction centered at y = 0. Explicitly, the three



FIG. 3. Profiles of the initial streamwise velocity, *u*, showing (a) the vertical cross-section in the *xz*-plane at y = 0 and (b) the top view in the *xy*-plane at z = 0. Here the simulation is initialized with $n_w = 4$ wavelengths of the parent wave, H = 2500 m, $A_0 = 50$ m, $k \simeq 4.8 \times 10^{-5}$ m⁻¹ and $\sigma_y = 50$ km.

evolution equations at t = 0 are given by

$$b(x, y, z, 0) = -\frac{1}{2}A_0 N^2 e^{-y^2/(2\sigma_y^2)} \hat{\psi}(z) e^{ikx} + \text{c.c.}, \qquad (17)$$

$$u(x, y, z, 0) = \frac{1}{2} A_0 \frac{\omega}{k} e^{-y^2/(2\sigma_y^2)} \hat{\psi}'(z) e^{ikx} + \text{c.c.},$$
(18)

$$v(x, y, z, 0) = \frac{1}{2} i A_0 \frac{f_0}{k} e^{-y^2/(2\sigma_y^2)} \hat{\psi}'(z) e^{ikx} + \text{c.c.},$$
(19)

in which σ_v is the standard deviation of the Gaussian.

Fig. 3 shows cross-sections of the initial streamwise velocity field (*u*) which has four wavelengths of the parent mode in the *x* direction. Here the parent mode wavelength is $\lambda_x = 50H$ and the spanwise (half-)width is $\sigma_y = 20H$. Where the ocean depth is $H \simeq 2500$ m, the wavelength corresponds to $\lambda_x \simeq 125$ km, consistent with satellite observations of the distance between successive solitary wave trains in the South China Sea.

The time scheme for the 3D model employed the same leapfrog method as used in the 2D model for the non-diffusive terms with steps of $\Delta t = 0.05 N_0^{-1}$ (see section 3a).

243 c. Analysis methods

Our focus is upon the possible development of internal solitary waves from the internal tide and how this is influenced by the ocean depth and the initial amplitude and spanwise extent of the waves. The structure of the waves was characterised in terms of the vertical displacement field, $\xi = -b/N^2$, and the along-wave velocity field, *u*, and its *x*-derivative. The evolution of the vertical



FIG. 4. Top view of *u* at the surface after 45 hours from the simulation with initial conditions shown in Fig. 3. The black-dashed line indicates the lateral extent of the waves, y_{max} , at this time.

displacement was examined at a depth $z_{\rm m}$ where the vertical structure of the streamfunction, and hence vertical displacement, was greatest ($\hat{\psi}(z_{\rm m}) = 1$). This occurred at $z_{\rm m} = -703$ m.

In 3D simulations, the spanwise extent of the waves increased due to lateral dispersion. The surface flow pattern evolved to form near-parabolic arcs and, for initial periodic waves of sufficiently large amplitude, the flow near the centerline at y = 0 evolved to form solitary waves. An example is shown in Fig. 4 which shows the surface streamwise velocity after 45 hours from the simulation with initial conditions shown in Fig. 3. The narrowing of regions where the streamwise velocity is large is indicative of solitary wave formation. We employ several diagnostics to characterise the spread and steepening of the waves, as described below.

The spanwise spreading of the waves was characterized in two ways. From snapshots of *u* at the surface, we determined the location of the wave edge, y_{max} , defined to be the spanwise distance from y = 0 where the magnitude of the peak surface streamwise velocity is no larger than is 0.1% of the peak streamwise velocity at y = 0. For example, this is indicated by the black-dashed line in Fig. 4. By applying this diagnostic at successive times, we characterised the widening of the beam in time by $y_{max}(t)$.

We also developed a diagnostic for the spanwise extent of the waves that could be compared with satellite observations of the sea-surface signature of internal solitary waves (e.g. see Fig. 1). The bright-banded surface signatures are caused by horizontally convergent flow making the surface rougher whereas darker regions are associated with horizontally divergent flow and a

smoother surface. Alpers (1985) estimated that, for surface current gradients to be visible as 269 surface roughening by satellites, their magnitudes should lie in the range $10^{-4} - 10^{-3} \text{ s}^{-1}$. In our 270 diagnostic, we compute the x-divergent flow field, $\partial u/\partial x$, and measure where its magnitude 271 exceeds an intermediate threshold $(\partial u/\partial x)_c = 5.5 \times 10^{-4} \,\mathrm{s}^{-1}$. We denote by y_s the maximum 272 spanwise distance from the centerline beyond which this threshold condition is not met. In all our 273 simulations, $y_s(t = 0) = 0$ meaning that there would be no sea-surface signature of the initially 274 periodic waves. However, a sea-surface signature can develop $(y_s > 0)$ if the initial waves have 275 sufficiently large amplitude and spanwise width to form solitary waves. 276

²⁷⁷ We likewise used measurements of $\partial u/\partial x$ at the surface to characterize the bending of phase ²⁷⁸ lines, as shown in Fig. 5. As the waves propagated in the *x* direction, the contours where $\partial u/\partial x = 0$ ²⁷⁹ bent to form a near-parabolic arc about y = 0 at late times. We characterized this deformation at ²⁸⁰ each time by fitting the contour to a parabola of the form $x = ay^2 + x_0$. Our results were then cast ²⁸¹ in terms of the radius of curvature, $R_c = 1/(2a)$. Initially R_c is infinite. And so this analysis is ²⁸² performed only for simulation times after 5 hours.

4. Results

Here we present simulation results examining the evolution of waves as it depends on H, A_0 and σ_y . Although the simulations were performed with nondimensional parameters based on depth scale, H, and time scale, N_0^{-1} , the results here are given in dimensional units relevant to observations in the South China Sea. In all cases the background stratification N^2 was set by the piecewise exponential (Eq. (1)) with the parameters listed in Table 1. Values of the Coriolis parameter and the wave frequency were fixed, as given in Table 2.

²⁹⁵ We begin by comparing the evolution of spanwise infinite waves in a 2D model with a 3D ²⁹⁶ simulation of waves having finite spanwise extent, $\sigma_y = 50$ km. The initial condition at y = 0 is the ²⁹⁷ same in both simulations, as shown in Fig. 6(a,b). In these simulations, only 1 wavelength of the ²⁹⁸ parent wave was initialized in the domain ($n_w = 1$).

We examine the wave structure after it has evolved for 29 hours. This is the time predicted for the waves moving at their predicted horizontal group velocity, $c_{gx} \simeq 2.55 \text{ ms}^{-1}$, to reach the P2 location in the South China Sea from the generation site estimated to be located at 122°E (see Fig. 1). At this time in the 2D model, one wavelength of the periodic internal tide has evolved to form a solitary



FIG. 5. Surface velocity streamwise gradient, $\partial u/\partial x$, in simulations with initially narrow and wide waves shown at three times (a,d) 10 hours, (b,e) 20 hours and (c,f) 30 hours. The simulation in (a-c) with $\sigma_y = 50$ km has initial conditions shown in Fig. 3. The simulation in (d-f) has the same initial conditions except that $\sigma_y = 150$ km. In each plot the superimposed green contour indicates the phase line associated with one of the four solitary waves where $\partial u/\partial x = 0$. The colour scale for all plots is indicated in (a).

wave train containing four waves of depression with successive amplitudes decreasing toward the
 lee of the train (Fig. 6c,d). In contrast, the corresponding 3D simulation shows the appearance of
 only two waves of depression after 29 hours.

The degree of narrowing of the wave troughs and their maximum downward displacement becomes larger if the lateral extent of the initial waves is larger. For example, Fig. 7 plots time series of the vertical displacement at $z_m = -703$ m determined from the 2D simulation (σ_y infinite) and from 3D simulations with $\sigma_y = 50$ and 70 km. As will be shown, waves having initially smaller spanwise width spread laterally more rapidly due to dispersion. This spread of energy away from y = 0 reduces the centerline amplitude of the waves, inhibiting nonlinear steepening.



FIG. 6. Comparison of 2D and 3D simulations showing (a,c,e) vertical cross-sections of streamwise velocity at y = 0 and (b,d,f) vertical displacement at y = 0 and $z_m = -703$ m. Initial conditions, shown in (a,b) are the same for the 2D and 3D simulations. The evolution at 29 hours is shown for (c,d) the 2D model and (e,f) the 3D model with $\sigma_y = 50$ km. The simulations are initialized with one wavelength of the parent wave having $A_0 = 50$ m in total depth H = 2500 m. The colour scale for all streamwise velocity plots is indicated in (a).



FIG. 7. Time series over one period of the initial wave of vertical displacement at x = y = 0 and z = -703 m from simulations of waves with initial spanwise width indicated in the legend. As in Fig. 6, $A_0 = 50$ m and H = 2500 m.



FIG. 8. Evolution of the vertical displacement over time at x = y = 0 and z = -703 m as it depends on the initial maximum vertical displacement A_0 : (a) 12.5 m, (b) 25 m, (c) 37.5 m, (d) 50 m, (e) 62.5 m, and (f) 75 m. In all cases H = 2500 m and $\sigma_y = 50$ km.

The strength of solitary waves, if they form, depends on the initial wave amplitude, A_0 , ocean depth H, as well as the spanwise width, σ_y . Next we successively examine the influence of each of these parameters on the development of solitary waves.

Figure 8 plots time series of vertical displacement determined from six 3D simulations with different initial amplitudes. In all cases the depth is fixed at H = 2500 m and the initial beam width is $\sigma_y = 50$ km. As anticipated for sufficiently large A_0 , localized solitary waves with deeper depressions become more evident as the initial amplitude increases every tidal period. For $A_0 \gtrsim 37.5$ m, solitary waves begin to appear after $\simeq 27$ hours in simulations with $A_0 \gtrsim 50$, and solitary wave trains develop at larger amplitudes. In all cases, the maximum downward displacement, $|\xi_{\min}|$, becomes larger as A_0 increases.

To investigate the dependence of solitary waves on the ocean depth in the 3D model, we ran a sequence of simulations with *H* ranging from 2000 m to 3500 m keeping $A_0 = 50$ m and $\sigma_y = 50$ km fixed. In these simulations we adjusted the initial horizontal wavenumber, *k*, of the waves to ensure the wave frequency, ω , was that of the semi-diurnal M_2 tide. The time evolution of the vertical displacement at the depth $z_m = -703$ m is shown in Fig. 9. Qualitatively, the crests of the solitary wave trains corresponding to ξ_{max} do not significantly alter with variations in ocean depth, while



FIG. 9. As in Fig. 8, but showing the evolution of the vertical displacement as it depends on the simulated ocean depth *H*: (a) H = 2000 m and $k \simeq 5.30 \times 10^{-5}$ m⁻¹, (b) H = 2500 m and and $k \simeq 4.96 \times 10^{-5}$ m⁻¹, (c) H = 3000 m and $k \simeq 4.90 \times 10^{-5}$ m⁻¹, and (d) H = 3500 m and $k \simeq 4.86 \times 10^{-5}$ m⁻¹. In all cases $A_0 = 50$ m and $\sigma_v = 50$ km.

the maximum downward displacement $|\xi_{min}|$ decreases with increasing ocean depth. In a greater depth, there are less waves contained within the solitary wave train.

To investigate the dependence of solitary waves on the initial spanwise width, we varied σ_y in the 345 3D model, keeping $A_0 = 50$ m and H = 2500 m fixed. As demonstrated in Fig. 10, increasing σ_y 346 results in larger downward vertical displacements and more waves within the solitary wave train.

Fig. 11 quantitatively summarizes the results shown in Figs. 8, 9, and 10 by plotting the depen-350 dence of the maximum descent of isopycnals, $|\xi_{min}|$, and their maximum rise, ξ_{max} , on the initial 351 wave amplitude, A_0 , ocean depth, H, and initial spanwise width of the waves, σ_v . In all cases the 352 displacements are normalized by the initial maximum displacement, A_0 . As shown in Fig. 11a, at 353 fixed H and σ_v , $|\xi_{\min}|$ changes little from the initial value if A_0 is small. However, for $A_0 \gtrsim 37.5$ m 354 the displacement increases approximately linearly with A_0 . The best-fit line passing through the 355 origin has a slope 0.0432 \pm 0.0014. This shows that $|\xi_{\min}|$ increases quadratically with A_0 if A_0 356 is sufficiently large. At fixed $A_0 = 50$ m and $\sigma_y = 50$ km, we find the relative maximum descent 357 of isopycnals moderately decreases linearly with increasing ocean depth (Fig. 11c) whereas the 358 maximum upward displacement is nearly independent of H. As the spanwise width of the beam 359 increases, so does the maximum descent and ascent of isopycnals (Figs. 11e,f). The relative dis-360



FIG. 10. As in Fig. 8, but showing vertical displacement over time as it depends on the initial spanwise width, σ_y , of the internal tide: (a) $\sigma_y = 50$ km, (b) $\sigma_y = 75$ km, (c) $\sigma_y = 100$ km, and (d) $\sigma_y = 125$ km. In all cases H = 2500 m and $A_0 = 50$ m.



FIG. 11. Dependence of the magnitude of the minimum and maximum relative vertical displacement on the initial wave amplitude (a,b), ocean depth (c,d), and spanwise width (e,f) from 3D simulations. Initial wave amplitude $A_0 = 50$ m in (c,d,e,f); ocean depth H = 2500 m in (a,b,e,f); spanwise width $\sigma_y = 50$ km in (a,b,c,d). The triangles on the right side of (e,f) indicate values determined in 2D simulations ($\sigma_y \rightarrow \infty$). The dashed lines in (a) represent the best-fit line through the origin of the points with 37.5 m $\leq A_0 \leq 75$ m.

placements asymptote to near-constant values for $\sigma_y \gtrsim 100$ km, consistent with the values predicted for spanwise infinite waves, as determined in 2D simulations.



FIG. 12. Relative lateral extent of the waves, y_{max} , over time in simulations with (a) $\sigma_y = 50$ km and different A₀ and *H* as indicated in the legend, and with (b) $A_0 = 50$ m, and H = 2500 m and different σ_y as indicated in the legend.

Next we examine the spanwise evolution of waves characterized by their lateral spreading at the surface, the evolution of the radius of curvature of the crescent-shaped waves, and the width of the sea surface signature as might be observed by satellite.

The half-width of the laterally confined waves is characterized by y_{max} , as described in Sec. 3c. 371 Figure 12 shows that the half-width, relative to σ_v , increases in time due to lateral dispersion. 372 This spreading is insensitive to the initial amplitude of the waves, A_0 , and the ocean depth, H, as 373 evident for the three cases plotted in Fig. 12a. However, the spreading depends sensitively upon the 374 initial spanwise extent of the waves, σ_{y} , increasing most rapidly if the initial width is smaller than 375 $\simeq 100$ km. In the case with $\sigma_v = 50$ km, over 35 hours y_{max} more than doubles from $\simeq 190$ km to 376 \simeq 391 km. Over the same time these waves are predicted to propagate 321 km at their group velocity, 377 $c_g \simeq 2.55 \,\mathrm{m \, s^{-1}}$. Hence, the lateral spread of such spanwise narrow internal tides is comparable 378 with their streamwise propagation distance. 379

The increase in time of y_{max} is approximately linear over the first 35 hours of evolution. Finding a best-fit line to y_{max} versus time for each of the five simulations in Fig. 12(b), gives their lateral spreading rates, which are plotted in Fig. 13. The spreading rate decreases exponentially with increasing initial spanwise width, determined empirically by $\dot{y}_{\text{max}} \simeq$ $[17(\pm 3)\text{km/hour}]\exp[-\sigma_y/(44\pm 3\text{km})].$

We have seen that phase lines at the surface bend to form arcs over time, which we quantify by measuring the radius of curvature, R_c , at y = 0 of the zero contour associated with the $\partial_x u$ field



FIG. 13. Dependence of the linear lateral spreading rate upon initial spanwise width of the waves. The error in the rates is comparable to the size of the symbols. The red line shows the best-fit exponential through the points.

between strongest peaks in that field (see Fig. 5). Initially phase lines are parallel to the y-axis, 392 and their radius of curvature is infinite. Our analysis begins after 5 hours. The time evolution of 393 R_c between 5 and 35 hours is shown in Fig. 14. As with the lateral spreading of the waves, we 394 find the evolution of the curvature is relatively insensitive to the initial wave amplitude, A_0 , and 395 ocean depth, H (Fig. 14a), but depends strongly on the initial spanwise extent of the waves, σ_v 396 (Fig. 14b). In all cases, R_c initially decreases, being most rapid of σ_v is smaller. As solitary waves 397 begin to develop around 15 hours, R_c rapidly increases then decreases again, this change being more 398 pronounced if σ_y is larger. For simulations with $\sigma_y = 50$ km, the minimum radius of curvature is 399 $R_c \simeq 440$ km occurring after 8 hours. This is only moderately larger than the lateral extent, y_{max} , 400 at this time (see Fig. 12(a)), indicating significant lateral distortion to the wavefield. 401

Satellites observe the sea surface signature of internal waves through the enhancement and 404 reduction of the surface roughness resulting from horizontally convergent and divergent flows at 405 the surface induced by the waves. We characterise whether or not this signature can be observed 406 by measuring the spanwise half-width, y_s , from y = 0 within which the surface current gradient, 407 $\partial u/\partial x$, exceeds a threshold value of 5.5×10^{-4} s⁻¹, as described in section 3c. For the horizontal 408 convergence to pass the threshold, the internal tide must steepen to form solitary waves for which 409 the horizontal flows at the surface become larger and change over smaller horizontal distances. 410 From our previous analyses, we expect a stronger, wider and longer lasting sea surface signature 411 in shallower domains for internal waves which initially have larger amplitude and wider horizontal 412



FIG. 14. Evolution of the radius of curvature, R_c , (a) with $\sigma_y = 50$ km fixed and with different A_0 and H as indicated in the legend, and (b) with $A_0 = 50$ m and H = 2500 m fixed and σ_y varying as indicated in the legend.

extent: these more rapidly form solitary waves and, because the lateral spreading is less pronounced,
the solitary waves persist for longer times.

Figure 15 plots the evolution in time of y_s for eight simulations in which a sea surface signature of solitary waves becomes evident at some point. In all these cases, the sea surface signature becomes evident after a delay ≥ 10 hours. The half-width grows in time and then decays as lateral spreading of the waves lowers their amplitude. The decay results in the sea surface signature vanishing before 50 hours if $\sigma_y \leq 75$ km and $A_0 \leq 50$ m. As anticipated, the signature vanishes more quickly if *H* is larger and the initial amplitude or spanwise width is smaller. No sea surface signature ever becomes evident in our simulations with $H \geq 2500$ m, $\sigma_y \leq 50$ km and $A_0 \leq 50$ m.

424 **5. Discussion and conclusions**

We performed fully nonlinear simulations to examine the evolution of the low-mode internal tide as it depends on wave amplitude, ocean depth, and the spanwise extent of the waves in stratification characteristic of measurements taken in the South China Sea. Solitary waves of depression were found to evolve from the initial horizontally sinusoidal internal tide provided the initial amplitude and spanwise extent of the waves were sufficiently large, with the maximum depression amplitude being more pronounced in shallower fluid. Corresponding to the deepening and narrowing of isopycnals associated with solitary waves, the horizontal flow at the surface exhibited stronger



FIG. 15. Predicted width of sea surface roughness signature, y_s , over time for simulations with different initial vertical displacement amplitudes, A_0 , spanwise widths, σ_y , and depths, H, as indicated in the legend.

⁴³² horizontal gradients which can result in enhanced sea surface roughening that can be observed by
⁴³³ satellite. However, this manifestation of internal waves at the surface was retarded if the initial
⁴³⁴ amplitude and spanwise extent of the waves were too small or the domain too deep.

For the satellite image shown in Fig. 1, internal solitary waves in the South China Sea are evident 435 by the arc-shaped pattern of sea surface roughness at locations A, B and C. In particular, the sea 436 surface pattern indicated by A (near the observation location P1) has half-width $\simeq 50$ km. We 437 predict that the mode-1 internal tide originating $\simeq 200$ km to the east at 122°E would take 22 hours 438 to propagate to site A at the group velocity $2.55 \,\mathrm{m\,s^{-1}}$, comparable with the observed speeds. The 439 waves at site A occur in an ocean depth $H \simeq 3000$ m. The sea surface signature has a half-width of 440 46 km and a radius of curvature \simeq 400 km. Thus our analyses suggest the effective initial spanwise 441 extent of the waves was $\sigma_y \simeq 75$ km with initial maximum vertical displacement amplitude ~ 63 m. 442 For waves with $\sigma_v \simeq 75$ km, the radius of curvature is predicted to increase moderately from its 443 minimum at $\simeq 12$ hours after generation. This is observed for waves that have propagated from 444 site A to sites B and C where the satellite observed sea surface signatures have $R_c \simeq 466$ km and 445 $R_c \simeq 472$ km, respectively. 446

Although our model is able to simulate the relatively realistic evolution of internal tides, it has many simplifying assumptions. Besides assuming a stationary background, we have focused on the evolution of the vertical mode-1 internal tide in uniform-depth fluid. By assuming initially sinusoidal waves in a periodic domain, we are in effect examining the temporal evolution of the waves in a frame of reference moving at their initial group velocity. Simulations with different initial amplitudes thus examine the evolution of waves originating from different phases between the spring and neap tides. Simulations with different domain depths given insights into the influence of depth upon steepening of the waves. Generally we find that the lateral spreading and radius of curvature of the waves about their centerline do not depend significantly upon amplitude and depth, but do depend strongly on the initial lateral extent, σ_y , of the waves: the rate of spread decays exponentially with increasing σ_y ; the radius of curvature first decreases, then increases when solitary waves begin to form (after \approx 15 hours) then decreases again for $\sigma_y \gtrsim$ 75 km.

The generic nature of these results suggests they may be applied to other regions in the ocean where internal solitary waves are observed by satellite. A complete categorization of the time evolution of the span and radius of curvature of the sea surface signature as it depends upon ocean depth and initial wave amplitude and spanwise extent could prove a useful tool in understanding the origins of internal solitary waves globally.

- ⁴⁶⁴ *Acknowledgments.* Funding for this research has been provided by the Natural Sciences and ⁴⁶⁵ Engineering Research Council.
- ⁴⁶⁶ *Data availability statement.* The data from this study is available from the authors upon request.

467 **References**

- Alford, M. H., and Coauthors, 2015: The formation and fate of internal waves in the South China
 Sea. *Nature*, **521**, 65–69.
- ⁴⁷⁰ Alpers, W., 1985: Theory of radar imaging of internal waves. *Nature*, **314**, 245–247.

⁴⁷¹ Baker, L., and B. R. Sutherland, 2020: The evolution of superharmonics excited by internal tides
⁴⁷² in non-uniform stratification. *J. Fluid Mech.*, **891**, R1.

Balmforth, N. J., G. R. Ierley, and W. R. Young, 2002: Tidal conversion by subcritical topography. *J. Phys. Oceanogr.*, **32** (10), 2900–2914.

- ⁴⁷⁵ Buijsman, M., J. McWilliams, and C. Jackong, 2010: East-west asymmetry in nonlinear internal
 ⁴⁷⁶ waves from Luzon Strait. *J. Geophys. Res.*, **115**, C10057.
- 477 Carter, G. S., and Coauthors, 2008: Energetics of M2 barotropic-to-baroclinic tidal conversion at
 478 the Hawaiian Islands. *J. Phys. Oceanogr.*, **38**, 2205–2223.
- Echeverri, P., M. R. Flynn, K. B. Winters, and T. Peacock, 2009: Low-mode internal tide generation
 by topography: An experimental and numerical investigation. *J. Fluid Mech.*, 636, 91–108.
- Farmer, D., Q. Li, and J.-H. Park, 2009: Internal wave observations in the South China Sea: The
 role of rotation and non-linearity. *Atmos. Ocean*, 47, 267–280.
- Garrett, C. J. R., and E. Kunze, 2007: Internal tide generation in the deep ocean. *Annu. Rev. Fluid Mech.*, **39**, 57–87.
- Grimshaw, R. H. J., and K. R. Helfrich, 2012: The effect of rotation on internal solitary waves.
 IMA J. Appl. Math., **77**, 326–339.
- Helfrich, K. R., and R. H. J. Grimshaw, 2008: Nonlinear disintegration of the internal tide. *J. Phys. Oceanogr.*, 38, 686–701.

- Helfrich, K. R., and W. K. Melville, 2006: Long nonlinear internal waves. *Annu. Rev. Fluid Mech.*,
 38, 395–425.
- Johnston, T. M. S., D. L. Rudnick, M. H. Alford, A. Pickering, and H. L. Simmons, 2013: Internal
 tidal energy fluxes in the South China Sea from density and velocity measurements by gliders.
 J. Geophys. Res., 118, 3939–3949.
- Klymak, J. M., R. Pinkel, C. T. Liu, A. K. Liu, and L. David, 2006: Prototypical solitons in the
 South China Sea. *Geophys. Res. Lett.*, 33 (11), L11 607.
- Li, Q., D. M. Farmer, T. Duda, and S. Ramp, 2009: Acoustical measurement of nonlinear internal
 waves using the Inverted Echo Sounder. *J. Atmos. Oceanic Technol.*, 26, 2228–2242.
- Martin, J. P., D. L. Rudnick, and R. Pinkel, 2006: Spatially broad observations of internal waves
 in the upper ocean at the Hawaiian Ridge. *J. Phys. Oceanogr.*, **36**, 1085–1103.
- ⁵⁰⁰ Ostrovsky, L. A., and Y. A. Stepanyants, 1989: Do internal solitons exist in the ocean? *Rev.* ⁵⁰¹ *Geophys.*, **27**, 293–310.
- Pétrélis, F., S. L. Smith, and W. R. Young, 2006: Tidal conversion at a submarine ridge. *J. Phys. Oceanogr.*, **36**, 1053–1071.
- Rudnick, D. L., and Coauthors, 2003: From tides to mixing along the Hawaiian ridge. *Science*,
 301 (5631), 355–357.
- Simmons, H., M.-H. Chang, Y.-T. Chang, S.-Y. Chao, O. Fringer, C. R. Jackson, and D. S. Ko,
 2011: Modeling and prediction of internal waves in the South China Sea. *Oceanography*, 24,
 88–99.
- Subich, C., K. Lamb, and M. Stastna, 2013: Simulation of the Navier–Stokes equations in three
 dimensions with a spectral collocation method. *Intl. J. Num. Meth. Fluids*, **73**, 103–129.
- Sutherland, B. R., 2016: Excitation of superharmonics by internal modes in non-uniformly stratified
 fluid. *J. Fluid Mech.*, **793**, 335–352, doi:10.1017/jfm.2016.108.
- Sutherland, B. R., and M. S. Dhaliwal, 2022: The nonlinear evolution of internal tides. Part 1: The
 superharmonic cascade. *J. Fluid Mech.*, 948, A21.

- ⁵¹⁵ Vic, C., and Coauthors, 2019: Deep-ocean mixing driven by small-scale internal tides. *Nature* ⁵¹⁶ *Comm.*, **10**, 2099.
- ⁵¹⁷ Zhang, Z., O. B. Fringer, and S. R. Ramp, 2011: Three-dimensional, nonhydrostatic numerical ⁵¹⁸ simulation of nonlinear internal wave generation and propagation in the South China Sea. *J.*
- ⁵¹⁹ *Geophys. Res.*, **116**, C05 022.