Geophys. J. Int. (2002) **151,** 377–392

Inner core tilt and polar motion

Mathieu Dumberry and Jeremy Bloxham

Department of Earth and Planetary Sciences, Harvard University, Cambridge, MA, 02138, *USA. E-mail: dumberry@geophysics.harvard.edu*

Accepted 2002 May 6. Received 2002 April 5; in original form 2001 October 22

SUMMARY

A tilted inner core permits exchange of angular momentum between the core and the mantle through gravitational and pressure torques and, as a result, changes in the direction of Earth's axis of rotation with respect to the mantle. We have developed a model to calculate the amplitude of the polar motion that results from an equatorial torque at the inner core boundary which tilts the inner core out of alignment with the mantle. We specifically address the issue of the role of the inner core tilt in the decade polar motion known as the Markowitz wobble. We show that a decade polar motion of the same amplitude as the observed Markowitz wobble requires a torque of 10^{20} N m which tilts the inner core by 0.07 degrees. This result critically depends on the viscosity of the inner core; for a viscosity less than 5×10^{17} Pa s, larger torques are required. We investigate the possibility that a torque of 10^{20} N m with decadal periodicity can be produced by electromagnetic coupling between the inner core and torsional oscillations of the flow in the outer core. We demonstrate that a radial magnetic field at the inner core boundary of 3 to 4 mT is required to obtain a torque of such amplitude. The resulting polar motion is eccentric and polarized, in agreement with the observations. Our model suggests that equatorial torques at the inner core boundary might also excite the Chandler wobble, provided there exists a physical mechanism that can generate a large torque at a 14 month period.

Key words: Chandler wobble, Earth's magnetic field, Earth's rotation, inner core viscosity, inner core wobble, Markowitz wobble.

1 INTRODUCTION

The Earth's rotation axis changes its orientation with respect to the mantle on timescales that range from days to millions of years (Lambeck 1980). Tidal torques from the Moon and Sun acting on the equatorial bulge are responsible for polar motions in the diurnal frequency band, while readjustments of the moment of inertia of the mantle through post-glacial rebound and convection produce a slow drift of the pole (true polar wander). Between these two ends of the spectrum, there is an annual wobble associated with mass transport in the atmosphere, the 14-month Chandler wobble which represents the free Eulerian precession of the Earth, and a decade polar motion known as the Markowitz wobble. The latter is the focus of this study.

The Markowitz wobble is a somewhat irregular polar motion with a magnitude of roughly 20 to 50 milliarcseconds (mas) (Markowitz 1960, 1968). Until recently, the existence of this oscillation was in doubt for several reasons, including changes in the star catalogues and modifications in data processing techniques (e.g. Lambeck 1980; Dickman 1981). The latest analyses seem to confirm the real nature of this signal (Wilson & Vicente 1980; Dickman 1981; Gross 1990; McCarthy & Luzum 1996). Various authors have attempted to characterize the Markowitz wobble; the general consensus is that the signal can be described as a highly eccentric motion of amplitude 20–30 mas oriented in the general direction of 35°E longitude, with an ellipticity between 0.87 and 0.93, and with a period of 28 to 31 yr (Poma *et al.* 1987). However, the mechanism responsible to generate such a polar motion is still not identified.

In the reference frame rotating with the Earth, there are no external torques with decade periodicity, which implies that the angular momentum of the whole Earth must be conserved at these timescales. The changes in the direction of rotation with respect to the mantle must then result either from exchanges of angular momentum between the mantle and its fluid envelope (the atmosphere and oceans), from exchanges of angular momentum between the mantle and the core, from changes in the moment of inertia of the Earth, or from a combination of these effects.

Coupling between the solid Earth and its external fluid envelope was first proposed by Dickman (1983), who argued that the decade polar motion could be explained in terms of a natural free wobble of the coupled ocean–solid Earth system. This hypothesis was disputed by (Wahr 1984) on the basis that the coupling between the oceans and the solid Earth was too weak to produce such a low frequency motion (although see Dickman 1985). More recently, (Celaya *et al.* 1999) investigated the excitation of the decade polar motion that results from a numerical model of a coupled climate system which includes the effects of ground water storage, ocean currents, seafloor pressure, atmospheric pressure and winds. Their study concluded that, while some of these effects may contribute to the observed signal, the amplitude of the Markowitz wobble cannot be explained completely by the surface forcing constituents currently included in their model.

Excitation of the Markowitz wobble by exchange of angular momentum between the core and the mantle through torques at the core–mantle boundary (CMB) has also been investigated. Greff-Lefftz & Legros (1995) have shown that electromagnetic coupling is too weak by two or three orders of magnitude for reasonable profiles of conductivity in the lower mantle. Topographic coupling, where horizontal pressure gradients act on the ellipticity and other possible components of CMB topography has been proposed (Hinderer *et al.* 1987, 1990) and investigated by a number of authors (Greff-Lefftz & Legros 1995; Hide *et al.* 1996; Hulot *et al.* 1996). These studies all indicate that the direction of the resulting torque critically depends on the selected topography at the CMB, but that its amplitude is too small by about an order of magnitude for topography with amplitude of a few kilometers. Furthermore, concern has been raised regarding the method for calculating the pressure coupling used in these studies (Kuang & Bloxham 1997a), and the actual topographic torque may be even smaller. In any case, CMB coupling seems inadequate to produce the observed decade polar motion.

Another possible scenario to explain the Markowitz wobble, the one which is investigated in this work, involves the participation of the inner core. Busse (1970) was the first to suggest that an inner core which is misaligned (or tilted) with respect to the mantle could influence the direction of the Earth's rotation on decade timescales, despite the fact that its moment of inertia is a very small fraction of that of the whole Earth (about 7×10^{-4}). His idea was that the Markowitz wobble could be associated with a free Eulerian precession of the inner core tilt and its rotation axis: the equivalent of the Chandler wobble but for the inner core. In the case of a rigid body, the precession frequency is determined by the body's dynamic ellipticity, and for the inner core this corresponds to a period of about 400 days. However, because the inner core is immersed in the fluid core, the frequency of this free precession is decreased by the density contrast at the inner core boundary (ICB). For a density contrast of 6 per cent, this amounts to a prograde inner core wobble of about 24 yr, a periodicity close to the observed polar motion.

In Busse's model, the precession of the inner core tilt was communicated to the mantle by surface pressure torques acting on the ICB and CMB. This torque develops as a consequence of the tilted figures of the inner core: in a tilted state, the outer surface of the inner core is no longer aligned with surfaces of constant centrifugal potential in the fluid core. This produces a pressure torque on the ICB and a secondary flow in the fluid core, which then couples to the mantle by a similar pressure torque. Busse showed that because of the weak coupling between the inner core and the mantle that arises from such a torque, the natural frequencies of the system are not very different from the free wobble frequencies of the mantle and the inner core individually. The Markowitz wobble could then represent the signature of the inner core wobble. However, the weak pressure coupling between the inner core and the mantle results in a mantle response which is too small to be observed (Kakuta *et al.* 1975).

In both of these studies, however, two important aspects of the Earth's internal dynamics were omitted. First, there was no dynamic equation governing the fluid core. Because the periodic motion of the inner core will create a flow in the outer core at the same period, it is questionable whether the role of the angular momentum of the fluid core can be neglected. Second, and most importantly, the gravitational interaction between the tilted inner core and the rest of the Earth was not taken into account. A misalignment between the oblate figures of the inner core and the mantle will cause a gravitational torque in the equatorial direction that tries to reestablish the alignment of their figures. The fluid core also participates in the torque because its ellipsoidal surfaces of constant gravitational potential, which stay mostly aligned with the mantle, will also interact with a tilted inner core. When the combined effects of the fluid core dynamics and the gravitational and pressure torques acting on the mantle and the inner core are incorporated, the free precession associated with the tilted figure of the inner core is still prograde but reduced to a period of about 6.6 yr (Mathews *et al.* 1991a; Dehant *et al.* 1993; Xu & Szeto 1998). The period of the inner core wobble is now quite different from the observed periodicity of the Markowitz wobble and this weakens the case of the free precession considerably. In addition, as pointed out by Dickman (1981), the observation of a highly eccentric (and possibly retrograde) polar motion is difficult to reconcile with a free precession mode, which would produce a circular (and prograde) polar motion.

This suggests that if the inner core is indeed responsible for the decade polar motion, a free precession of its figure axis is not the explanation, although it does not discount the possibility that this free precession could explain part of the signal. Therefore, the only way that the inner core could play a role in the Markowitz wobble is if its tilt is controlled by equatorial torques at the ICB which vary on a decadal timescale. The resulting polar motion at the surface would then be the response of this forced oscillation. The investigation of this possible explanation for the Markowitz wobble is the focus of this work.

What is the torque required on the inner core in order to create a sufficiently large tilt of its figure to generate the observed polar motion? What will be the amplitude of that tilt? What are the mechanisms that can produce torques of the required amplitude at decade periods at the ICB? These are the questions that motivate our study.

A recent study by Greiner-Mai & Barthelmes (2001) has addressed some of these questions. They assumed a simple angular momentum balance between the polar motion of the Earth's rotation axis and the changes in the moment of inertia of the Earth caused by the tilted figure of the inner core. They then inverted for the tilt angle during the past century for the polar motion that corresponds to the Markowitz wobble. They obtained tilt variations of a few tenths of a degree. From this result, they estimated an applied torque on the inner core of about 10^{22} N m, based on a calculation of the total gravitational torque exerted on a tilted inner core by Smylie *et al.* (1984). However, both of these calculations neglect the dynamic influence of the fluid core. An applied torque on the inner core implies an equal and opposite torque on the fluid core at the ICB. The angular momentum of the fluid core will then be altered and this will, in turn, change the gravitational and pressure coupling with the inner core. This suggests that consideration of the fluid core is important both for its influence on the dynamics of the inner core and for its role in the global angular momentum balance of the Earth. We therefore suspect that the values for the tilt of the inner core and the amplitude of the torque required to create such a tilt that were reported in the study of Greiner-Mai & Barthelmes (2001) may be incorrect. In addition, the presence of a resonance at the 6.6 yr period of the free inner core wobble could amplify a decade timescale forced excitation of the inner core tilt and reduce the amplitude of the torque required to produce polar motions of observed amplitudes.

The possible role of the inner core in the decade polar motion is not well established because none of the previous studies that addressed the issue have considered the complete internal coupling dynamics of the Earth. In this work, we revisit the role of the tilted figure of the inner core by developing a model that incorporates both the effects of surface pressure torque and gravitational volume torque. This model is an adaptation of the one developed by Mathews *et al.* (1991a) for the study of the forced nutations of the Earth. We adapt the model to our study of decade polar motion by incorporating the effects of viscous relaxation of the inner core surface.

In our model, the tilt of the inner core and the perturbations in Earth's rotation are caused by a prescribed torque at the ICB. Our primary goal is to determine the torque required on the inner core in order to explain the observed decade polar motion. We demonstrate that a torque of about 10^{20} N m is sufficient. A torque of that amplitude produces a tilt of about 0.07 degrees of the figure of the inner core. Both of these results are obtained when the viscosity of the inner core is larger than 5×10^{17} Pa s. For smaller viscosities, the required torque is increased because the tilted figure of the inner core relaxes towards an alignment with the symmetry axis of the mantle.

Our second objective is to determine if the dynamics of the fluid core can generate the required torque at the ICB. We consider a scenario of electromagnetic coupling involving the action of torsional oscillations in the flow and we demonstrate that a torque of 10^{20} N m at decade periods can be produced if the radial magnetic field at the ICB is of the order of 3–4 mT. This value is large but is not unreasonable considering that larger fields are expected at the ICB due to the concentrated shear in velocity. One interesting aspect of the electromagnetic torque produced according to this scenario is that the resulting polar motion is highly eccentric and polarized, two important characteristics of the Markowitz wobble.

The results of this work suggest therefore that, given our limited knowledge of the inner core viscosity and of the magnetic field near the ICB, forced variations of the inner core tilt cannot be discounted as a possible contribution to the excitation of the Markowitz wobble.

Finally, our results also show that a torque at the ICB of 10^{19} N m can also participate in the excitation of the Chandler wobble, if such a torque can be applied at the Chandler wobble period of 14 months. However, because mantle and ocean dissipation are not included in our model, the required torque that we calculate is probably underestimated.

This work is organized as follows. In the next section we present the observations of polar motion for the last century. In Section 3 we discuss some of the basic physical arguments regarding rotating bodies, precession and internal coupling of the Earth. In Sections 4 and 5 we present some of the details of our model and show some of our important results. Finally, in Section 6 we analyse the implications of these results in terms of Earth rotation, viscosity of the inner core, electromagnetic coupling at the ICB and the Chandler wobble excitation.

2 DATA

In Fig. 1, we show the two equatorial components of the mean polar motion for the past century. The data used in this figure are from the International Earth Rotation Service (IERS) combined series EOP-C01, where the mean position of the pole was obtained by filtering the Chandler and the annual wobble from the original series. The mean position can be described by a linear drift and a decade oscillation with amplitudes that vary between 50 mas in the early part of the century to 20 mas for the more recent data. This is the Markowitz wobble.

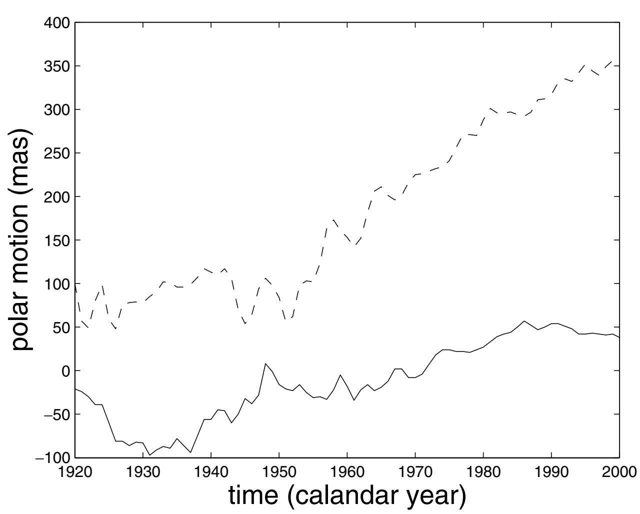

Figure 1. Components $m_1 = \omega_1/\Omega_o$ (solid line) and $m_2 = \omega_2/\Omega_o$ (dashed line) of the polar motion for the last century. The Chandler wobble and the annual wobble have been filtered from the original IERS data.

An oscillation of a roughly 6-yr period can also be seen in the data. This periodicity corresponds to the beat frequency between the Chandler wobble and annual wobble. Therefore, it is possible that this oscillation could be an artefact resulting from an incomplete elimination of the Chandler and annual wobble from the data (Dickman 1981). However, in this work we argue that if the Markowitz wobble is explained by a torque at the ICB, then the free precession of the inner core tilt should also be excited. This mode has a periodicity of 6.6 yr and hence, the periodic signal in Fig. 1 may represent this free mode.

3 BASIC DYNAMICS AND INTERNAL COUPLING OF THE EARTH

The equilibrium configuration of the Earth is one where the mantle, the fluid core and the solid inner core are all rotating at angular velocity $\mathbf{\Omega}$ around an axis which coincides with the axis of geometric symmetry of the aligned oblate figures of the mantle ($\hat{\mathbf{e}}_3$) and inner core ($\hat{\mathbf{e}}'_3$) (Fig. 2a). Because the angular momentum of the whole Earth has to be conserved, a tilt of the inner core will also produce a small polar offset between the rotation axis and the mantle (Fig. 2b). In a recent study, Greiner-Mai & Barthelmes (2001) have used this simple balance to invert for the tilt of the inner core for the past century. They concluded that a tilt of a few tenths of a degree was required in order to explain the decade polar motion.

This equilibrium balance between the tilted figures of the inner core and mantle is correct on grounds of angular momentum conservation, but is incorrect when proper considerations of the Earth's internal dynamics are allowed. Any misalignment between the figures of the mantle and inner core will cause a gravitational torque in the equatorial direction that acts to reestablish the alignment of their figures. Similarly, pressure torques on the tilted figures of the inner core and mantle will arise from the angular velocity of the fluid core pushing on its misaligned envelope. Moreover, torques acting perpendicular to the axis of a rotating body will induce precessional motions and the tilted figure of the mantle and inner core will then precess around the rotation axis.

One could argue that a steady torque imposed at the ICB would balance the combined gravitational and pressure torques and maintain an equilibrium configuration if the free precessions are rapidly damped. If this is the case, the equilibrium configuration shown in Fig. 2(b) is still incorrect because it overlooks the changes in the angular momentum of the fluid core. The imposed torque on the solid inner core at the ICB implies an equal and opposite torque on the fluid core. Similarly, the gravitational and fluid pressure torques on the misaligned boundaries imply equal and opposite torques from the boundaries on the fluid. These torques produce changes in the angular momentum of the fluid core which are characterized by variations in the direction of its rotation axis ($\mathbf{\Omega_f}$).

The correct equilibrium configuration that is established when a constant torque is applied at the ICB results not only from the tilt of the inner core and the polar offset, but also from the change in the angular momentum of the fluid core. This configuration is shown in Fig. 2(c). We defer the quantitative details of the angular momentum balance configurations of Figs 2(b) and (c) to Section 4, and for now simply state that the tilt angle in Fig. 2(c) is a factor 2.5 smaller than in Fig. 2(b), and with the reversed sign. The tilt angles derived by Greiner-Mai & Barthelmes (2001) are thus a factor 2.5 too large and with opposite direction.

Of course, we do not expect the complicated dynamics in the fluid core to apply a constant torque on the inner core. Time dependent equatorial torques at the inner core boundary will perturb the equilibrium and instigate free precessions. The polar motion at the surface will result from a superposition of the forcing due to the imposed torque at the ICB and the excitation of the free precession modes. If the periodicity of the imposed torque is close to one of the free precession periods, the amplitude of the forced wobble will be amplified.

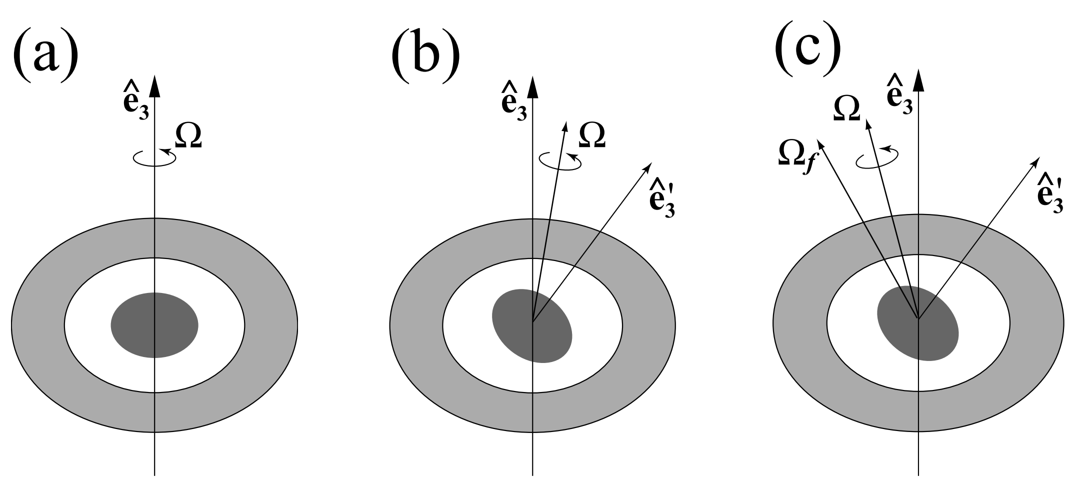

Figure 2. Equilibrium configurations of the Earth: (a) in its undisturbed state; (b) with a tilted inner core; (c) with a tilted inner core and proper considerations of the fluid core dynamics. $\hat{\mathbf{e}}_3$ is the symmetry axis of the mantle (light grey), $\hat{\mathbf{e}}'_3$ is the symmetry axis of the inner core (dark grey), $\mathbf{\Omega}$ is the rotation axis of the Earth and $\mathbf{\Omega_f}$ is the rotation axis of the fluid core. The steady applied torque is directed into the page. Not drawn to scale.

4 DESCRIPTION OF THE MODEL

Models that describe the internal gravitational and pressure coupling between the mantle, the fluid core and the inner core have been developed for the study of the Earth's forced nutations (Sasao *et al.* 1980; Wahr 1981; Mathews *et al.* 1991a; Dehant *et al.* 1993). The nutations are the small changes of the direction of the Earth's rotation axis in space that are produced by the tidal torques from the Moon, the Sun and the other planets on the oblate figure of the Earth. Because the mantle, fluid core and inner core react differently under the same external torque, exchanges of angular momentum between them result from internal gravitational and pressure torques. Therefore, the amplitude of the observed nutations ultimately depends on the coupling between Earth's interior regions. Moreover, the structure of the Earth, its elastic properties and the coupling between its different regions, determines a set of eigenmodes of rotation. If a tidal torque has a frequency close to that of a normal mode, the associated nutation amplitude will be amplified by a resonant effect. The objective of the forced nutation studies is then to construct a model for the internal coupling in the Earth that successfully reproduces the observed nutations when subjected to the known spectrum of tidal forcing. Over the years, these models have grown in sophistication in response to the increased accuracy of the observations (Mathews & Shapiro 1992; Sovers *et al.* 1998).

The polar motion generated from an imposed torque at the ICB will result from the same internal coupling dynamics that are included in these models. Therefore we can take advantage of these existing models for the purpose of our study.

4.1 Adapted model from nutation theory

We use the model developed by Mathews *et al.* (1991a). We give here only a brief description of the model and refer the interested reader to the original paper. The model is that of an axisymmetric, oceanless, rotating earth comprised of a mantle, fluid core and inner core. Surfaces of constant density are determined under the assumption of hydrostatic equilibrium between pressure and the combination of gravitational and centrifugal potential. Each region is allowed to deform elastically, but no dissipation effects are included. The basic objective of the model is to calculate the perturbations from an initial state of a uniform rotation $\mathbf{\Omega_{o=\Omega_o\hat{e}_3}}$ with respect to a reference frame fixed to the mantle. These perturbations are expressed in terms of departures in the orientation of the rotation axis of the whole earth with respect to the mantle, $\boldsymbol{\omega}$, and departures of the rotation axis of the fluid core, $\boldsymbol{\omega}_\mathbf{f}$, and of the inner core, $\boldsymbol{\omega}_\mathbf{s}$, with respect to the rotation vector of the earth $\mathbf{\Omega} = \mathbf{\Omega_o} + \boldsymbol{\omega}$. These are expressed in the conventional complex notation

$$\tilde{m} = m_1 + im_2 = (\omega_1 + i\omega_2)/\Omega_o, \tag{1}$$

$$\tilde{m}_f = (m_f)_1 + i(m_f)_2 = ((\omega_f)_1 + i(\omega_f)_2)/\Omega_o, \tag{2}$$

$$\tilde{m}_s = (m_s)_1 + i(m_s)_2 = ((\omega_s)_1 + i(\omega_s)_2/\Omega_o, \tag{3}$$

where the directions 1 and 2 refer to the two equatorial directions in the mantle reference frame. The complex scalar $\tilde{m}$ thus represents the amplitude of the polar motion. A fourth degree of freedom is added by allowing a tilted inner core with orientation $\mathbf{\hat{e}'_3}$ with respect to the orientation of the mantle $\mathbf{\hat{e}_3}$ (see Fig. 2). The difference $\hat{\mathbf{e}}'_3 - \mathbf{\hat{e}_3}$ represents the tilt angle denoted by $\mathbf{n_s}$, and is expressed in the same standard notation by

$$\tilde{n}_s = (n_s)_1 + i(n_s)_2. \tag{4}$$

The amplitudes of these perturbations are calculated by solving a system of four equations: one for the evolution of the angular momentum of the whole Earth ($\mathbf{H}$); one each for the evolution of the angular momentum of the fluid core ($\mathbf{H_f}$) and inner core ($\mathbf{H_s}$); and one kinematic equation governing the tilt of the inner core relative to the mantle. They are, respectively,

$$\frac{d}{dt}\mathbf{H} + \mathbf{\Omega} \times \mathbf{H} = 0, \tag{5}$$

$$\frac{d}{dt}\mathbf{H_f} - \boldsymbol{\omega}_\mathbf{f} \times \mathbf{H_f} = -\mathbf{\Gamma_{app}}, \tag{6}$$

$$\frac{d}{dt}\mathbf{H_s} + \mathbf{\Omega} \times \mathbf{H_s} = \mathbf{\Gamma_s} + \mathbf{\Gamma_{app}}, \tag{7}$$

$$\frac{d}{dt}\mathbf{n_s} = \Omega_o\,(\mathbf{m_s} \times \mathbf{\hat{e}_3})\,. \tag{8}$$

The last of these equations, eq. (8), will be altered in the next section in order to allow for viscous relaxation of the inner core surface. In the Mathews *et al.* (1991a) model, an external torque $\mathbf{\Gamma}$ representing the tidal forcing was applied on the whole Earth and appeared on the right hand side of (5). We are solely interested in the perturbations induced by an imposed torque at the ICB and neglect that tidal torque. This forces the angular momentum of the whole Earth to be conserved. Our prescribed equatorial forcing at the ICB is represented by the applied torque $\mathbf{\Gamma_{app}}$. The torque $\mathbf{\Gamma_s}$ represents the pressure and gravitational torques on the inner core. Following the standard notation, these torques are expressed as

$$\tilde{\Gamma}_s = (\Gamma_s)_1 + i(\Gamma_s)_2, \tag{9}$$

$$\tilde{\Gamma}_{\text{app}} = (\Gamma_{\text{app}})_1 + i(\Gamma_{\text{app}})_2. \tag{10}$$

Buffett (1992) improved the original model by incorporating the effects of magnetic stress at the CMB and the ICB. This magnetic coupling arises as a consequence of the misalignment between the rotation axes of each region, which creates small differences in velocity at the boundaries which shear the radial magnetic field. In the context of the forced nutations, this additional coupling represents a fine tuning improvement of the data fit, and on this ground, it seems safe to neglect this small effect.

The magnetic stresses will turn out to be important in our model because of the timescale at which we are exciting the system. We are interested in the response of the Earth to torques that are applied at decade timescales, as opposed to the diurnal response of the nutation studies. At daily timescales, the radial magnetic field is essentially static, but at decade timescales, the variations in the outer core flow can create sufficiently large changes in the radial magnetic field by the process of advection. Therefore a significant torque at the ICB can be generated by the action of this outer core flow. Indeed, it will become clear in Section 6.2 that the nature of the applied torque at the ICB that drives the system is most likely electromagnetic and that $\mathbf{\Gamma_{app}}$ is the prescribed form of this torque.

One can note that the prescribed torque at the ICB will evidently induce a misalignment between the rotation axis of the inner and outer core. This resulting differential velocity at the ICB will then produce an additional contribution to the magnetic stress. However, we have verified that this differential velocity is smaller by at least a factor 10 than the overriding outer core velocity that is necessary to produced the applied torque in the first place. We therefore safely neglect this additional contribution in the electromagnetic torque.

The solutions of the above system are obtained as follows. The angular momentum vectors are expanded linearly in terms of the perturbations and include the effects of elastic deformation. The torque $\mathbf{\Gamma_s}$ is similarly expanded. Thus, this system of four equations is transformed into a set of algebraic conditions on $\tilde{m}$, $\tilde{m}_f$, $\tilde{m}_s$ and $\tilde{n}_s$. The forcing and the perturbations are assumed periodic and proportional to $e^{i\Omega_o \sigma t}$ where σ is the frequency of the forcing (in cycles per day). The entire problem can be written in a matrix form as

$$\mathsf{M} \cdot \mathbf{x} = \mathbf{b}, \tag{11}$$

with the matrix M, solution vector $\mathbf{x}$ and forcing vector $\mathbf{b}$ given by

$$\mathsf{M} = \begin{pmatrix} \sigma + (1+\sigma)\kappa - e & (1+\sigma)(\xi + A_f/A) & (1+\sigma)(\zeta + A_s/A) & (1+\sigma)\alpha_3 e_s A_s/A \\ \sigma + \sigma\gamma & 1 + \sigma + \beta + e_f & \sigma\delta & -\sigma\alpha_1 e_s A_s/A_f \\ \sigma + \sigma\theta - \alpha_3 e_s & \sigma\chi + \alpha_1 e_s & 1 + \sigma + \sigma\nu & (1 + \sigma - \alpha_2)e_s \\ 0 & 0 & 1 & \sigma \end{pmatrix}, \tag{12}$$

$$\mathbf{x} = \begin{pmatrix} \tilde{m} \\ \tilde{m}_f \\ \tilde{m}_s \\ \tilde{n}_s \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 0 \\ -\tilde{\Gamma}_{\text{app}}/iA_f\Omega_o^2 \\ \tilde{\Gamma}_{\text{app}}/iA_s\Omega_o^2 \\ 0 \end{pmatrix}. \tag{13}$$

The quantities that enter the matrix M are defined as follows. A, A_f and A_s are the equatorial moments of inertia and e, e_f and e_s are the dynamic ellipticities of the whole Earth, the fluid core (subscript f) and the inner core (subscript s). The parameter α_1 essentially represents the density contrast at the ICB and is slightly less than unity. The parameter α_3 is related to α_1 by $\alpha_1 = 1 - \alpha_3$. The parameter α_2 represents the ability of the rest of the Earth to exert a torque on a tilted inner core and is defined as $\alpha_2 = \alpha_1 - \alpha_3\alpha_g$, where α_g represents the gravitational coupling alone. Finally, the parameters (κ, ξ, ζ), (γ, β, δ) and (θ, χ, ν) are three sets of compliances which characterize the effects of elastic deformations of the whole Earth, the fluid core and the inner core, respectively, which arise due to changes in centrifugal potential induced by the independent rotations of the three regions. These nine parameters are obtained by integrating the equations of elastic deformations for a specific Earth model. The values of all of the above parameters obtained from the basic Earth model PREM (Dziewonski & Anderson 1981) are presented in Mathews *et al.* (1991b) and are reproduced here for convenience in Table 1.

The periods of the free precessions of the Earth are found with the condition det $(\mathsf{M}) = 0$. Four natural modes emerge; the prograde and retrograde free core nutation which have periodicities close to 1 day; the Chandler wobble with a period of 400 days; and the inner core wobble which has a period of 6.6 yr.

In the nutation problem, the amplitude of the perturbation $\mathbf{x}$ is calculated for a given tidal forcing at a selected frequency. We proceed similarly and prescribe an applied torque at the ICB with a given frequency.

4.2 Viscous relaxation of the inner core

The inner core is not perfectly elastic and has a finite viscosity. On long timescales, when its surface does not coincide with an equipotential, the geometric figure of the inner core will viscously deform towards the imposed equipotential surface. Thus, if the inner core was a perfect fluid, its axis of symmetry would be perfectly aligned with that of the mantle. This remains true even when an equatorial torque is applied at the ICB: the applied torque will induce an equatorial rotation of the inner core, but a secondary flow within the inner core will develop in order to keep its elliptical surface aligned with that of the mantle.

Table 1. Parameters for Earth model PREM.

Moments of inertia, kg m^2	
$A = 8.0115 \times 10^{37}$	$A_f = 9.0583 \times 10^{36}$
$A_s = 5.8531 \times 10^{34}$	
Ellipticities	
$e = 3.247 \times 10^{-3}$	$e_f = 2.548 \times 10^{-3}$
$e_s = 2.422 \times 10^{-3}$	
Coupling constants	
$\alpha_g = 2.1752$	$\alpha_1 = 0.9463$
$\alpha_2 = 0.8294$	$\alpha_3 = 0.0537$
Compliances	
$\kappa = 1.039 \times 10^{-3}$	$\xi = 2.222 \times 10^{-4}$
$\zeta = 4.964 \times 10^{-9}$	$\gamma = 1.965 \times 10^{-3}$
$\beta = 6.160 \times 10^{-4}$	$\delta = -4.869 \times 10^{-7}$
$\theta = 6.794 \times 10^{-6}$	$\chi = -7.536 \times 10^{-5}$
$\nu = 7.984 \times 10^{-5}$	

We allow viscous relaxation in our model by modifying the kinematic relation between the variations in the rotation of the inner core and the tilt of its geometric figure (eq. 8). The extent of the relaxation depends on the frequency of the applied torque relative to the characteristic e-folding time of viscous relaxation, τ. We expect that for very high frequencies, the changes of the inner core tilt are essentially controlled by the variations in the rotation velocity of the inner core, as it is currently prescribed in (8). Conversely, for very low frequencies, the inner core tilt will remain essentially aligned with the mantle regardless of the variations in rotation of the inner core. These two situations can be satisfied by multiplying the left hand side of (8) by a factor $(1 + i/\sigma\tau)$. This is analogous to prescribing a time dependence which contains an oscillating and decaying part, $e^{i\Omega_o\sigma t(1+i/\sigma\tau)}$ to both $\tilde{n}_s$ and $\tilde{m}_s$ in eq. (8). The last equation that appear in matrix $\mathbf{M}$ therefore becomes

$$\tilde{m}_s + \sigma\left(1 + \frac{i}{\sigma\tau}\right)\tilde{n}_s = 0. \tag{14}$$

Since we are not interested in the details of the secondary flow that develops to maintain the alignment between the inner core and mantle, this simple prescription for the adjustment of the inner core surface is sufficient for our present purpose.

The timescale of viscous relaxation is proportional to the viscosity of the inner core, η_s, and also depends on the assumed rheology of the inner core. Here, we are considering an incompressible, self-gravitating and homogeneous inner core with a uniform viscosity. In this case, τ and η_s are related by (Buffett 1997)

$$\tau = \frac{C\eta_s}{a_s g \Delta\rho}, \tag{15}$$

where g and $\Delta\rho$ are the gravitational acceleration and the density jump at the ICB radius of a_s, and $C = 1.9$ is a numerical constant.

The viscosity of the inner core is a crucial parameter in our model because it influences directly the amplitude of the tilt of the inner core, and therefore the amplitude of the resulting polar motion at the surface. Unfortunately, it is not a well constrained quantity. Typical estimates from experiments at high-temperature on iron at ambient pressure give $\eta_s = 10^{13\pm3}$ Frost & Ashby (1982). However, the extrapolation of these values to pressure at core conditions may not be appropriate, and larger typical grain size in the inner core than those assumed in these experiments are likely to increase the value of the viscosity (Bergman 1998). A geodynamic estimate of 3×10^{16} Pa s has been suggested by Buffett (1997) based on the seismic observations of the eastward super-rotation of the inner core. This corresponds to a time relaxation of the inner core figure of $\tau \approx 0.6$ yr. The work of Greff-Lefftz *et al.* (2000) on the period of free retrograde core nutation suggests that the viscosity is larger than 10^{14} Pa s if the magnetic field at the ICB is of the order of a few milliTeslas, or alternatively, that the viscosity is smaller than 10^{14} Pa s if the magnetic field at the ICB is much larger.

Because of the uncertainty on the viscosity of the inner core at present, and because our model might serve as an additional constraint for the viscosity, we present calculations that use values of η_s ranging between 5×10^{15} Pa s and 5×10^{19} Pa s. For $\Delta\rho = 600$ kg m^{-3}, $g = 4.4$ m s^{-2} and $a_s = 1.2 \times 10^6$ m in eq. (15), this corresponds to relaxation times between 0.1 yr and 1000 yr.

The dissipation of rotational energy associated with the free modes of precession is not included in our model. This includes the effects of mantle anelasticity, dissipation in the oceans (Smith & Dahlen 1981) and electromagnetic dissipation (Buffett 1992). Since we are solving the system in the frequency domain and are mostly interested in the amplitudes that result from the forced oscillations, these omissions do not invalidate our approach. However, as a consequence, the forced polar motion amplitude that we calculate at the resonance frequency of the free modes will be overestimated.

5 RESULTS

5.1 Static case

It is instructive to first calculate the amplitude of the polar offset, the tilt of the inner core, and the changes in the rotation of the fluid core and inner core that results when a steady torque is imposed at the ICB and in the absence of free precessions. This amounts to solving eqs (5) to (8) for $d/dt = 0$ and a fixed $\Gamma_{\mathbf{app}}$. (For this discussion, we do not consider the effects of viscous deformation of the inner core, i.e. we set $\tau \to \infty$).

It is immediately clear that eq. (8) forces $\tilde{m}_s = 0$, which implies that the rotation axis of the inner core is aligned with that of the whole Earth. The equilibrium configuration is thus one where a polar offset $\tilde{m}$ is balanced by a tilted figure of the inner core and a misaligned rotation of the fluid core, as shown in Fig. 2(c). We note that since we are solving for $d/dt = 0$, we impose $\Omega \times \mathbf{H} = 0$ in (5) and therefore $\mathbf{H}$ coincides with Ω in Fig. 2. The relative balance between $\tilde{m}$, $\tilde{m}_f$ and $\tilde{n}_s$ results from solving (11) for $\tilde{m}_s = 0$ and without the last equation:

$$\begin{pmatrix} A(\kappa - e) & A\xi + A_f & \alpha_3 e_s A_s \\ 0 & 1 + e_f & 0 \\ -\alpha_3 e_s & \alpha_1 e_s & (1 - \alpha_2) e_s \end{pmatrix} \begin{pmatrix} \tilde{m} \\ \tilde{m}_f \\ \tilde{n}_s \end{pmatrix} = \begin{pmatrix} 0 \\ -\tilde{\Gamma}_{\text{app}}/i A_f \Omega_o^2 \\ \tilde{\Gamma}_{\text{app}}/i A_s \Omega_o^2 \end{pmatrix}. \quad (16)$$

Using the parameters listed in Table 1, we find that the amplitude of $\tilde{m}_f$ and $\tilde{n}_s$ relative to $\tilde{m}$ are

$$\tilde{m}_f \approx \frac{1}{35}\tilde{m}, \qquad \tilde{n}_s \approx -10^4\,\tilde{m}. \quad (17)$$

We note that if we neglect to consider the fluid core in the angular momentum balance of the whole Earth (first row of 16), we retrieve the angular momentum balance between the polar offset and the tilted inner core pictured in Fig. 2(b), which is

$$\tilde{n}_s = \frac{(e - \kappa)A}{\alpha_3 e_s A_s}\tilde{m}. \quad (18)$$

This expression is similar to the one used by Greiner-Mai & Barthelmes (2001) to invert for the tilt of the inner core. This balance implies a ratio $\tilde{n}_s \approx 2.5 \times 10^4 \tilde{m}$. The amplitude of the tilt of the inner core required to explain the decade polar motion is thus reduced by a factor 2.5 when proper consideration of the fluid core dynamics is included. Moreover, the direction of $\tilde{n}_s$ relative to $\tilde{m}$ is reversed.

According to the ratios presented in (17), a polar displacement $\tilde{m}$ of 25 mas corresponds to an opposite tilt angle of the inner core of 2.5×10^5 mas, or 0.07 degrees. The applied torque on the inner core required to produce such polar offset is 1.7×10^{20} N m. This value is two orders of magnitude smaller than the estimate of the torque presented by Greiner-Mai & Barthelmes (2001), which was based on a calculation by Smylie *et al.* (1984) which did not include the dynamic effects of the fluid core. Again, this demonstrates clearly how the fluid core plays an important role in the dynamics of the inner core tilt.

The above analysis neglects two important factors. First, the presence of the free inner core wobble at a period of 6.6 yr might enhance the response of the polar motion by resonant effects, which would decrease the required torque. Second, viscous relaxation of the inner core surface will tend to realign the oblate figures of the mantle and inner core, and result in a smaller polar offset. An increased torque is therefore required to produce a polar motion of the same magnitude. These two effects are included in the full calculation presented in the next section.

5.2 Amplitude of polar motion generated by a periodic torque

In order to quantify the ability of our mechanism to excite sufficient polar motion, we now solve eq. (11). We recall that the perturbations in rotation and the applied torque in (11) have an implied periodic time dependence $e^{i\Omega_o\sigma t}$. Therefore, the polar motion recovered from this system represents, for positive σ, a prograde circular trajectory at the surface. As was pointed out earlier, the observed decade polar motion is not circular but highly eccentric. However, our primary goal in this section is simply to establish the amplitude of the torque required to produce a polar motion amplitude of about 25 mas, not the precise details of the motion. Solving (11) is sufficient for that purpose. The case of a polarized polar motion will be addressed in Section 6.2. All the calculations presented in this section use the parameters of the basic Earth model PREM (Dziewonski & Anderson 1981) that are listed in Table 1.

In Fig. 3 we present the amplitude of the recovered circular polar motion, or 'forced wobble' (i.e. $(m_1^2 + m_2^2)^{1/2}$), as a function of frequency. We show the results for three different characteristic timescales of the viscous relaxation of the inner core surface, $\tau = 100$ yr, $\tau = 10$ yr and $\tau = 1$ yr. In all cases, the amplitude of the imposed torque at the ICB was kept constant at 10^{20} N m.

The amplitude of the forced wobble is enhanced near the resonant frequencies of the Chandler wobble ($\approx 2.5 \times 10^{-3}$ cycles day^{-1}) and the inner core wobble ($\approx 4.1 \times 10^{-4}$ cycles day^{-1}). Not surprisingly, the largest wobble amplitude is achieved when the relaxation time of the inner core shape is the longest (i.e. largest inner core viscosity). At the period of the Markowitz wobble, (≈ 30 yr, which corresponds to $\approx 9.1 \times 10^{-5}$ cycles day^{-1}), with an imposed torque of 10^{20} N m, the amplitude of the forced wobble is similar to that of the observed decade variations of 25 mas when $\tau \geq 10$ yr, while it is clear that it is not the case when $\tau = 1$ yr.

This can be seen more clearly in Fig. 4, where we have fixed the periodicity at 30 yr, and varied both the relaxation time of the inner core and the amplitude of the torque. The amplitude of the forced wobble is now represented as contour lines. The dashed contour line corresponds to an amplitude of 25 mas. The equatorial torque on the inner core required to explain the Markowitz wobble is of order 10^{20} N m for τ larger than 10 yr. Below $\tau = 10$ yr, the torque required increases proportionally to the decrease in the relaxation time. Fig. 4 also shows the inner core tilt (in degrees) from the same calculation. The inner core tilt scales proportionally to the forced wobble at the surface. The dashed contour corresponds to a tilt of 0.07 degrees. We note that for τ larger than 10 yr, we retrieve the results of the static case presented in the previous section. This illustrates that, in the case of a large inner core viscosity, the resonant effect of the inner core wobble is not very large at periods of 30 yr.

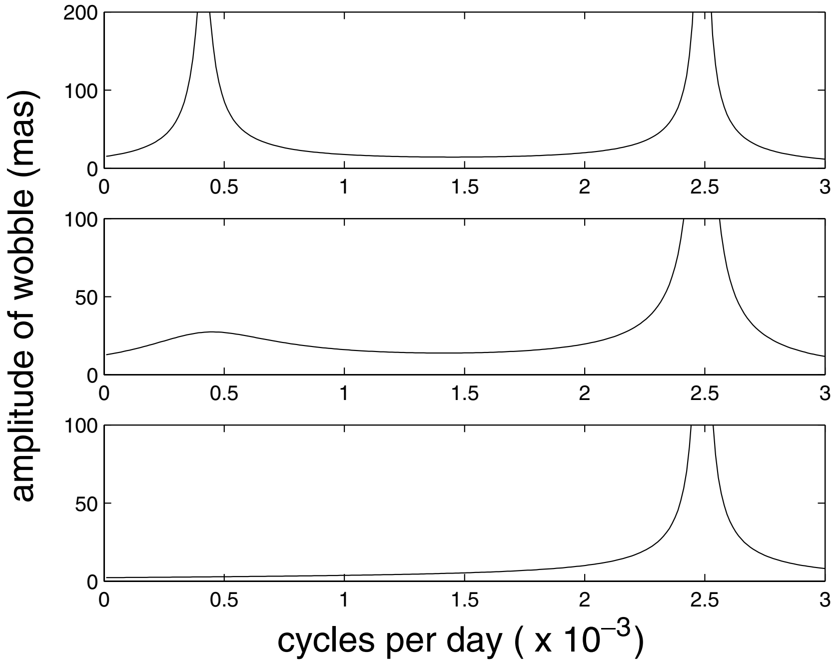

Figure 3. Amplitude of the forced wobble as a function of the frequency for an applied torque of 10^{20} N m and for a time relaxation of the inner core shape of 100 yr (top), 10 yr (middle) and 1 yr (bottom).

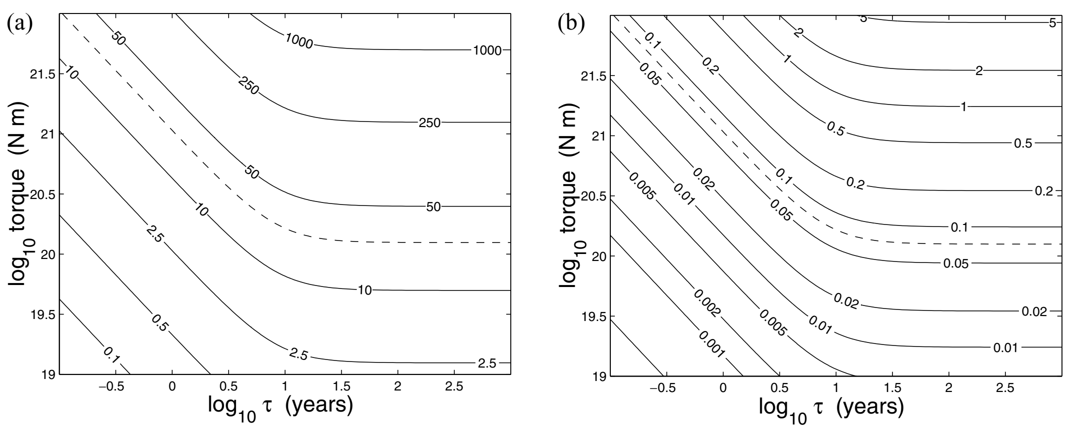

Figure 4. Contours of amplitude of the forced wobble (in mas) (a) and amplitude of inner core tilt (in degrees) (b) as a function of the applied torque and relaxation time of the inner core surface, at a fixed periodicity of 30 yr. The dashed contour corresponds to a forced wobble of 25 mas.

In contrast, the resonant effect is most important at the periodicity of the free inner core wobble (2409 days). In Fig. 5 we show the results obtained when the periodicity is fixed to that value. Compared with a periodicity of 30 yr, the torque required to generate the same forced wobble is similar for τ less than 10 yr, but decreased by a few orders of magnitude for larger τ.

Interestingly, Fig. 3 suggests that an imposed torque at the ICB is quite effective at exciting the Chandler wobble. In Fig. 6 we have set the periodicity at 400 days, the Chandler wobble period of an oceanless Earth, and again computed the amplitude of the wobble and the tilt of the inner core as a function of both the imposed torque and the relaxation time of the inner core. The observed amplitude of the Chandler wobble, roughly 150 to 200 mas (the dashed contour on the figure corresponds to 150 mas), is recovered with a torque which is about one order of magnitude smaller than that required to produce the decade polar motion.

6 DISCUSSION

6.1 Amplitude of the torque

The results of our model suggest that the required amplitude for the decade polar motion, about 25 mas, can be achieved by a torque on the inner core of the order of 1–2 $\times 10^{20}$ N m if the time relaxation of the inner core shape is larger than a decade, which corresponds to an inner core viscosity greater than 5×10^{17} Pa s. The associated tilt of the inner core is $\sim$0.07 degrees.

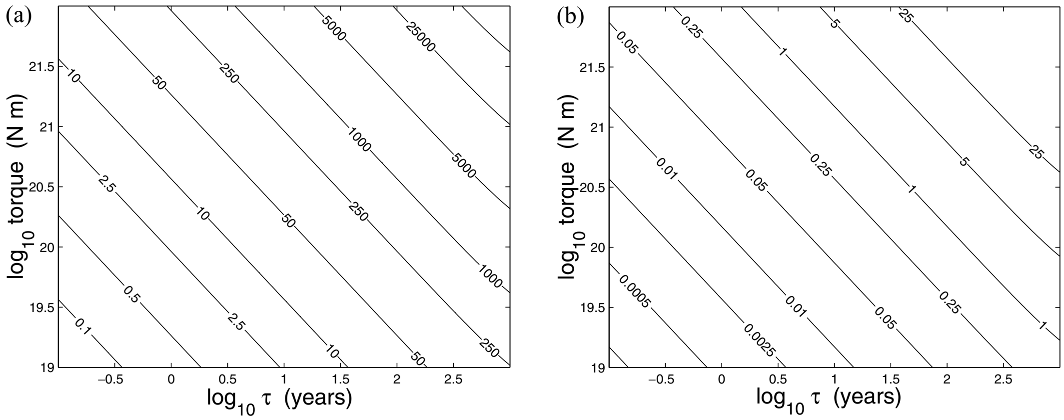

Figure 5. Contours of amplitude of the forced wobble (in mas) (a) and amplitude of inner core tilt (in degrees) (b) as a function of the applied torque and relaxation time of the inner core surface, at the fixed periodicity of the inner core free wobble (2409 days).

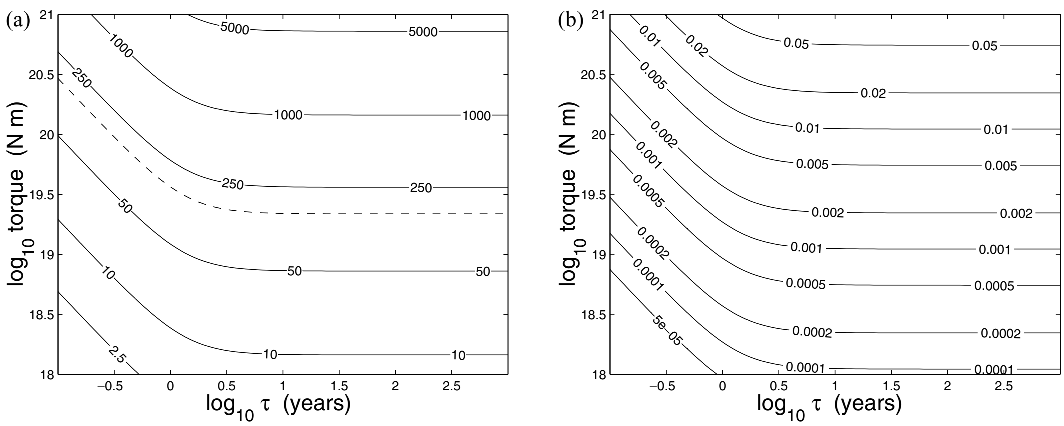

Figure 6. Contours of amplitude of the forced wobble (in mas) (a) and amplitude of inner core tilt (in degrees) (b) as a function of the applied torque and relaxation time of the inner core surface, at the fixed periodicity of the Chandler wobble (400 days). The dashed contour corresponds to a forced wobble of 150 mas.

As we will see in the next section, a torque of 10^{20} N m is at the high end of the estimated equatorial torque that can be applied on the inner core. Hence, this implies that a viscosity of 5×10^{17} Pa s is the lower limit that permits a sufficient polar motion to be generated by a forced tilting of the inner core. How does this value compare with current estimates of viscosity?

Experiments on iron at high temperature suggests $\eta_s \sim 10^{13\pm3}$ Pa s (Frost & Ashby 1982), which is considerably smaller than our lower limit. However, these estimates are uncertain due to pressure extrapolation, and larger viscosities cannot be ruled out. In addition, in these experiments, a grain size smaller than 5 mm is necessary to achieve a viscosity of less than 10^{16} Pa s and typical grain size in the core are probably much larger Bergman (1998). For instance, a growth model of the inner core suggests a grain size of about 5 m and a viscosity upper bound of order 10^{21} Pa s (Yoshida *et al.* 1996).

The geodynamic estimate of the viscosity by Buffett (1997) gives an upper bound of 3×10^{16} Pa s from reconciling axial gravitational coupling between the mantle and the inner core and the seismic observations of the eastward super-rotation of the inner core. However, the original seismic observations of a super-rotation rate of 1° per year (Song & Richards 1996; Su *et al.* 1996) have since been disputed, and the revised rate is smaller by at least a factor 10 (Laske & Masters 1999; Souriau & Poupinet 2000; Poupinet *et al.* 2000). This elevates the upper bound for the viscosity by at least one order of magnitude, and it is now consistent with our requirement of 5×10^{17} Pa s.

Another attempt to constrain the viscosity of the inner core was proposed by Greff-Lefftz *et al.* (2000). They calculated the joint effect of the magnetic dissipation at the ICB and the viscous relaxation of a viscoelastic inner core on the free modes of nutations of the Earth. They concluded that the observed periodicity of the retrograde free core nutation reflects a viscosity larger than 10^{14} Pa s if the magnetic field at the ICB is of the order of a few milliTeslas, or alternatively, that the viscosity is smaller than 10^{14} Pa s if the magnetic field at the ICB is much larger. According to their first scenario, the viscosity of the inner core could be higher than the lower limit required by our model.

The work of Smylie & McMillan (2000) focused on the viscosity in a possible 'slushy' layer in the outermost 300 to 400 km of the inner core, which would be the seat of compositional convection. They inferred a viscosity of $\eta_s \approx 10^{11}$ Pa s based on the observation of the periods of the two translational Slichter modes of the inner core (Courtier *et al.* 2000). It is clear that the presence of such a 'slushy' layer will have a profound effect on the rotational dynamics involving the inner core, as its low viscosity can readily accommodate any changes in the imposed potential. However, at present, the observation of the Slichter modes remains controversial (Hinderer & Crossley 2000). Moreover, a model of sedimentary compaction applied to the inner core growth suggest that the thickness of this 'slushy' layer is of the order of 100 m when its viscosity is 10^{18} Pa s, and that its thickness vary in proportion to the square root of the viscosity, implying even larger viscosities for a thicker layer (Sumita *et al.* 1996).

In short, the present knowledge of the viscosity of the inner core does not discount the possibility that the inner core tilt might play an observable role in the decade variations of the polar motion. Alternatively, if the inner core tilt participates in the decade polar motion, then the results of our model constrain the inner core viscosity to be larger than 5×10^{17} Pa s.

We have found that, at the free inner core wobble period of 2409 days and for a viscosity of 5×10^{17} Pa s, the torque required to generate a polar wobble of a few tens of mas is about a factor 2 smaller than that required at a 30 yr period. For larger viscosities, this factor becomes very large. Hence, our results suggest that torques on the inner core at decade periods can readily excite the inner core wobble to observable levels, if the viscosity of the inner core is sufficiently large.

This fact was noted by Greff-Lefftz *et al.* (2000). In their study of the influence of inner core viscosity on the normal modes of rotation, they calculated that the damping of the inner core wobble would be rapid if the viscosity was lower than 10^{16} Pa s. For larger viscosities, the wobble can be sustained and if torques are acting at the ICB with a sufficient magnitude, we should observe a 6.6 yr signal in the polar motion data and the gravimetric measurements. (We note that if the free inner core wobble is indeed present in the data, its excitation is most probably due to torques at the ICB since it is difficult to conceive of a different way to generate a large tilt of the inner core.)

Therefore, the detection of such a signal represents a good test for our hypothesis because if the Markowitz wobble is indeed a forced response from an imposed torque at the ICB, we expect that the inner core wobble will be excited to comparable amplitudes. The polar motion presented in Fig. 1 may indicate that a 6.6 yr signal is indeed present in the data, however a thorough analysis would be required in order to confirm that this signal is real and not an artefact due to incomplete filtering of the annual and Chandler wobble. Obviously, the detection of this signal is a requirement, but not a formal proof for our hypothesis; it is possible that the torques on the inner core are too weak to create the Markowitz wobble but large enough and at the correct periodicity to excite the free inner core wobble to an observable amplitude.

On the other hand, if it can be clearly demonstrated that the free inner core wobble is absent from the data, then it implies that the inner core viscosity is smaller than $\sim 10^{17}$ Pa s, or alternatively, that the torques at the ICB are weaker than $\sim 10^{20}$ N m. In either cases, the inner core tilt cannot be responsible for the Markowitz wobble.

6.2 Nature of the torque

So far our efforts have focused on determining the amplitude of the equatorial torque required at the ICB to produce the observed polar motion. We now turn our attention to the possible mechanisms that can generate such a torque. Mechanical coupling through viscous stresses at the ICB is probably very weak even when turbulent values of the viscosity are adopted (Aurnou & Olson 2000). Inertial coupling from the pressure imposed by the fluid flow on the elliptical ICB requires knowledge of the flow and is difficult to evaluate. Another possibility is electromagnetic coupling, which we investigate here.

Electromagnetic torques between the fluid core and its envelope are a result of the Lorentz force which is created by the interaction of the magnetic field normal to the boundary and the electrical current flowing along the boundary (Rochester 1960). In terms of the magnetic field **B**, the torque at the ICB can be expressed as

$$\boldsymbol{\Gamma} = \frac{1}{\mu} \int_{\mathrm{ICB}} (\mathbf{r} \times \mathbf{B})\, B_r \; dS, \tag{19}$$

where μ is the permeability of free space and the integral is taken on the assumed spherical inner core surface. The magnetic field in the core is maintained by complex dynamics involving the convective flow motion. One can therefore imagine a complicated morphology for the magnetic field at the ICB. At any given time, we expect that this magnetic field will produce a net equatorial torque on the inner core, unless a fortuitous cancellation occurs as a result of the surface integration. However, the real question is whether the amplitude and periodicity of this torque are within the range of values required to explain the Markowitz wobble.

In order to force a polar motion with decade periods, the torque on the inner core must have the same periodicity. This requires changes in the magnetic field on decade timescales. Since the changes in the magnetic field result from the dynamics in the fluid core, it is necessary to first identify a dynamic process in the fluid core with typical timescales of decades, and second, establish whether this process is capable of imposing equatorial torques on the inner core with the required magnitude. One possible such process is torsional oscillations. These are the rigid azimuthal oscillations of cylindrical shells aligned with the rotation axis (Taylor 1963; Braginsky 1970). The action of this torsional oscillation flow on the radial magnetic field threading the ICB can indeed produce equatorial torques on the inner core, as demonstrated in the remainder of this section.

Tangential azimuthal flow at the ICB will deform the radial magnetic field and produce an azimuthal magnetic traction on the surface of the inner core. If one pictures an axisymmetric radial magnetic field, it is obvious that the resulting torque will be entirely in the axial direction. However, the radial magnetic field will have a non-axisymmetric part which will give rise to an equatorial component of the torque.

This can most easily be demonstrated with the help of simple cartoons. Suppose, as in Fig. 7(a), that the radial magnetic field at the ICB is concentrated in two small areas at mid-latitudes, one in the southern hemisphere where the field enters the inner core and the other one in the northern hemisphere where it leaves it. An azimuthal flow acting on the inner core will generate a local azimuthal magnetic traction at

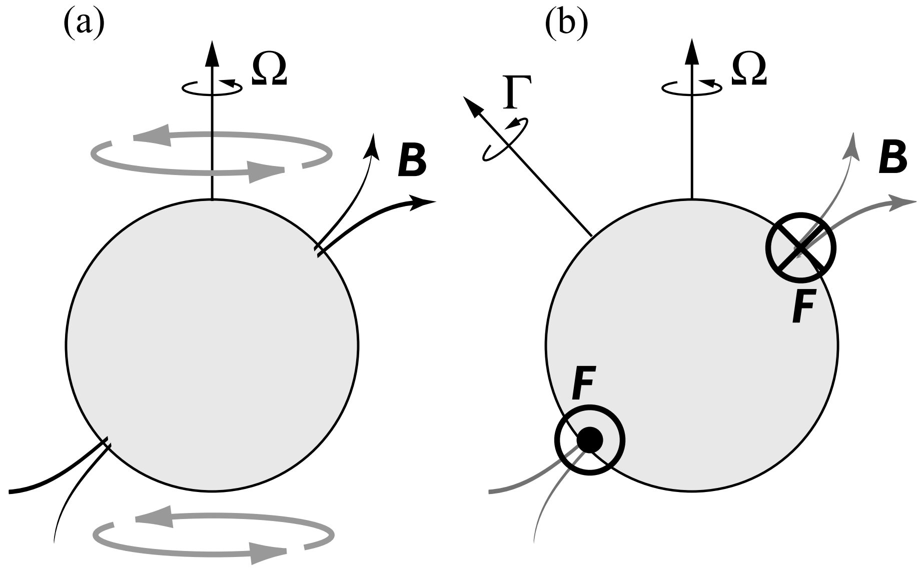

Figure 7. (a) Hypothetical configuration of the magnetic field (black arrows) and azimuthal flow (grey arrows) near the inner core. (b) The surface force **F** produced by the shear of the flow on the radial part of the magnetic field is directed into (out of) the page in the northern (southern) hemisphere. The resulting torque on the inner core is $\mathbf{\Gamma} = \mathbf{r} \times \mathbf{F}$.

the location of the field patches. The direction of the resulting torque on the inner core is perpendicular to the plane defined by the position vector (from the center of the inner core to the field patch) and the surface force vector. This torque has an equatorial component, as shown in Fig. 7(b). A reversed flow would produce a torque in the opposite direction. Thus, an azimuthal flow oscillating at decade timescales, a torsional oscillation flow, produces variations in the direction of the torque at the same timescale.

An order of magnitude estimate of the amplitude of the equatorial torque can be made from an oscillating azimuthal flow acting on a tilted dipole field at the ICB. Although the field at ICB is more complicated than a tilted dipole, for the simple calculation that follows, the contribution of the multipole components of the field to the electromagnetic torque can be incorporated in terms of an 'effective' dipole field. The details of this calculation are presented in Appendix A. The torque that is obtained is

$$\Gamma \approx -\frac{\pi a_s^3 g_1^0 g_1^1 \Delta u_\phi}{\mu(\eta\sigma_d)^{1/2}}, \tag{20}$$

where a_s is the radius of the inner core, μ is the permeability of free space, η is the magnetic diffusivity (which is related to the electrical conductivity σ_e by $\eta = 1/\mu\sigma_e$), σ_d is the frequency of oscillation of the azimuthal flow, Δu_ϕ is the difference in axial velocity between the inner core and the fluid cylindrical shells. g_1^0 and g_1^1 are the Gauss coefficients of the axial and equatorial 'effective' dipole field, defined for the present context at the surface of the inner core. We emphasize that this 'effective' dipole field is largely unrelated to the dipole field at the CMB that is observed at the surface.

The rigid azimuthal velocity of the fluid shells inside the tangent cylinder can be evaluated with the flow inversions at the CMB (Jault *et al.* 1988; Jackson *et al.* 1993; Zatman & Bloxham 1997; Pais & Hulot 2000; Hide *et al.* 2000). The amplitude of the velocity in this region of the core is not well constrained because it represents a small fraction of the total area of the CMB. Recent studies have estimated a steady westward component of the flow of 0.5–1° per year (Olson & Aurnou 1999; Pais & Hulot 2000), but the oscillating part of this flow is unknown. Nevertheless, a reasonable estimate of the amplitude of this oscillating flow is 10^{-4} m s^{-1}, in agreement with typical torsional oscillations amplitude in the bulk of the core. With this estimate for the velocity, together with typical values of electrical conductivity in the core of $\sigma_e = 5 \times 10^5$ S m^{-1} (Gubbins & Roberts 1987), which gives $\eta = 1.6$ m^2 s^{-1}, and with $a_s = 1200$ km and $\sigma_d = 2\pi/30$ yr^{-1}, eq. (20) implies that in order to get a torque of 10^{20} N m, we need $g_1^0 g_1^1 \approx 20$ mT^2. For instance, if we partition this value into $g_1^0 = 6.3$ mT and $g_1^1 = 3$ mT, this gives an 'effective' dipole tilt of about 25 degrees and a RMS radial magnetic field of roughly 4 mT. With the same dipole tilt and parameters as above but with a velocity of 2×10^{-4} m s^{-1}, the RMS field is reduced to ∼3 mT. These estimates for the radial magnetic field are large but not unreasonable. Results of the Kuang–Bloxham numerical simulation of the geodynamo (Kuang & Bloxham 1997b, 1999) suggest that the RMS amplitude of B_r may be as large as 3 mT at the ICB. Inferences from the forced nutations suggest a minimum RMS field of 4.6 mT (Buffett *et al.* 2002). Therefore, on that order of magnitude basis, we conclude that electromagnetic torques produced by torsional oscillations could be capable of imposing an equatorial torque of 10^{20} N m on the inner core.

A detailed investigation of the equatorial torque according to this scenario is definitely not possible due to our limited knowledge of the radial field structure and the differential velocity at the ICB. Another avenue would be to investigate the equatorial torque that emerges in the numerical simulations of the geodynamo. Typical values of the electromagnetic equatorial torque in the Kuang–Bloxham model (Kuang &

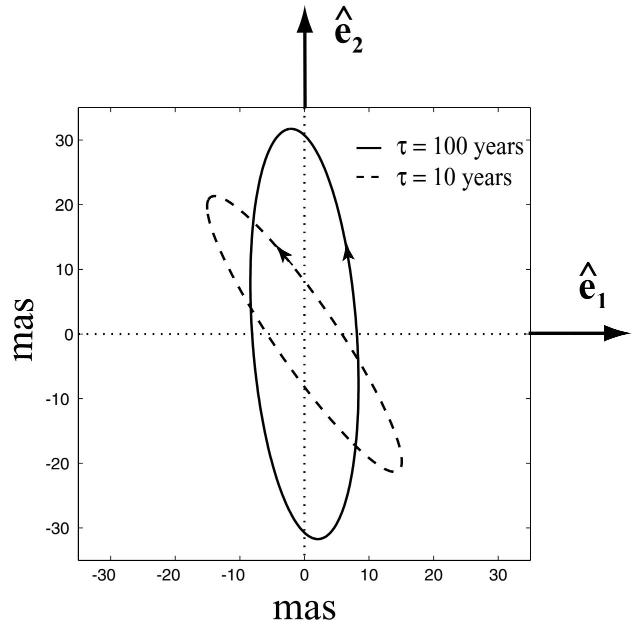

Figure 8. Polar motion at the surface which results from a periodical applied torque of 10^{20} N m oscillating along $\hat{\mathbf{e}}_1$. The solid trajectory corresponds to $\tau = 100$ yr and the dashed trajectory to $\tau = 10$ yr.

Bloxham 1997b, 1999) are of the order of the required 10^{20} N m. It is thus tempting to confirm that the mechanism described above is indeed supported by the dynamics of the fluid core. However, such a claim would be misguided: care has to be taken when inferring results from the geodynamo models. The parameter regime in which the Earth's dynamo operates is not yet attainable in the numerical simulations (see Dormy *et al.* 2000, for a recent review) and it is unclear whether the former result can be appropriately scaled to the Earth's core. The ability of the Kuang–Bloxham model to reproduce the required torque on the inner core may be promising, but remains inconclusive.

As we can see in Fig. 7, the electromagnetic torque produced in the fashion described above has a specific orientation determined by the morphology of the radial magnetic field at the ICB. If the radial magnetic field is steady over several periods of oscillation of the flow, the resulting torque will then oscillate along a specific longitudinal plane. This represents an appealing aspect of our model because the observed Markowitz wobble is indeed highly eccentric and polarized along a specific longitude. In fact, a good test for any mechanism that tries to explain the Markowitz wobble is its ability to produce a polarized polar motion at the surface.

In Fig. 8, we present the polar motion which is generated from an applied torque at the ICB that oscillates with a period of 30 yr along a specific orientation (in this case, direction $\hat{\mathbf{e}}_1$). The resulting polar motion is polarized, with an orientation that is at 90° to the torque for large inner core viscosity. The ellipticity of the motion is roughly 0.92 and the sense of rotation is prograde. The fact that the resulting polar motion is not perfectly polarized is a consequence of an enhancement of the motion in the prograde direction which is due to the resonant effect of the inner core free wobble mode, a purely prograde mode. (The ratio of amplitudes of a polar motion produced by a prograde versus a retrograde torque is roughly 1.69 at a 30 yr period.)

Dickman (1981) has suggested that the Markowitz wobble is predominantly a retrograde motion. If further analyses of the decade polar motion signal confirm this fact, then the scenario presented above cannot be reconciled with the observation. However, at the current level of uncertainties in the data, a prograde motion cannot be ruled out Dickman (1983).

Hence, electromagnetic coupling on the inner core tilt provides not only a torque which might be of sufficient magnitude, but also one that produces an eccentric polar motion with a preferred orientation. We therefore conclude that electromagnetic coupling on the inner core due to the action of torsional oscillations cannot be discounted as the driving mechanism for the observed Markowitz wobble.

6.3 Excitation of the Chandler wobble

The Chandler wobble has been observed in the Earth rotation data for more than a century, but the mechanism responsible for its excitation is still not fully resolved. Lately, seafloor pressure at the bottom of the oceans (Gross 2000) and barometric pressure and winds (Celaya *et al.* 1999) have been shown to be the most promising candidates.

The results of Fig. 6 might suggest the possibility that the inner core participates in the excitation process. A required torque of amplitude 10^{19} N m on the inner core is not unreasonable from considerations of electromagnetic coupling, as we have demonstrated in the previous section. The problem, of course, is to generate a torque of such amplitude near the periodicity of the Chandler wobble (435 days). Typical timescales for the dynamics in the fluid core are much longer than a year. However, a recent study indicates that higher wavenumber torsional oscillations with short period can explain the sudden variations in the geomagnetic field data (Bloxham *et al.* 2002). The short period torsional oscillations could then generate an electromagnetic torque on the inner core by the mechanism described in the previous section and provide a forcing near the Chandler frequency. Other possible mechanism might involve non-linear interactions in the dynamics that could produce a rapid torque but this is of course highly speculative and the details of how this mechanism might operate are certainly not known.

An additional word of caution comes from the fact that dissipation of rotational energy in the mantle and the oceans is not considered in our model. Therefore, one expects that the value of the required torque that we have calculated is underestimated. Nevertheless, this finding is interesting and further consideration should be given to both the potential mechanism that can produce a rapid torque and the proper incorporation of mantle and ocean dissipation in our model.

7 CONCLUSIONS

In this work we have developed a model to investigate the possibility that a tilted inner core could participate in the irregularities in the direction of the Earth's rotation axis with respect to the mantle. We have shown that the required torque produced at the ICB in order to explain the Markowitz wobble of decade periods is of the order of 10^{20} N m if the homogeneous viscosity of the inner core exceeds 5×10^{17} Pa s. This torque produces inner core tilt amplitudes of 0.07 degrees. Torques of larger magnitude are required if the viscosity of the inner core is smaller.

A torque of such amplitude with decade periods could be produced by electromagnetic coupling between the inner core and the torsional oscillations in the fluid core. Such a torque has the appealing feature of producing an elliptical and polarized polar motion, two defining characteristics of the observed Markowitz wobble.

A good test to determine if such a mechanism is indeed responsible for the Markowitz wobble is the detection of a 6-yr period in the polar motion data and gravimetric measurements. This period corresponds to the free inner core wobble mode and we have shown that this mode is easily excited by torques at the ICB. The absence of such a signal in the data would most likely indicate that decade torques at the ICB are probably too small to explain the amplitude of the Markowitz wobble.

If the Markowitz wobble results from a different mechanism, then the model presented in this study can be used to constrain the equatorial torque at the ICB and the inner core viscosity. For instance, if the viscosity can be shown by independent means to be larger than 5×10^{17} Pa s, it would imply that the equatorial torque at the ICB is less than $\sim 10^{20}$ N m. Alternatively, if one shows that the torque at the ICB is of the order of 10^{20} N m, then our model would constrain the viscosity of the inner core to be smaller than 10^{17} Pa s.

ACKNOWLEDGMENTS

The authors would like to thank Steve R. Dickman for a thorough review which greatly improved this paper. Mathieu Dumberry is supported by scholarships from NSERC/CRSNG and FCAR. This work was also supported by NSF Awards EAR-0073988 and EAR-0112469 and NASA grant NAG5-7616.

REFERENCES

Aurnou, J. & Olson, P., 2000. Control of inner core rotation by electromagnetic, gravitational and mechanical torques, *Phys. Earth planet. Inter.,* **117,** 111–121.

Bergman, M., 1998. Estimates of the Earth's inner core grain size, *Geophys. Res. Lett.,* **25,** 1593–1596.

Bloxham, J., Zatman, S. & Dumberry, M., 2002. The origin of geomagnetic jerks, *Nature,* in press.

Braginsky, S.I., 1970. Torsional magnetohydrodynamic vibrations in the Earth's core and variations in day length, *Geomag. Aeron.,* **10,** 1–10.

Buffett, B.A., 1992. Constraints on magnetic energy and mantle conductivity from the forced nutations of the Earth, *J. geophys. Res.,* **97,** 19 581–19 597.

Buffett, B.A., 1997. Geodynamic estimates of the viscosity of the Earth's inner core, *Nature,* **388,** 571–573.

Buffett, B.A., 1998. Free oscillations in the length of day: inferences on physical properties near the coremantle boundary, in *The Core-Mantle Boundary Region, Geodynamics series,* Vol. 28, pp. 153–165, eds, Gurnis, M., Wysession, M.E., Knittle, E. & Buffett, B.A., *Am. geophys. Un.,* Geophysical Monograph.

Buffett, B.A., Mathews, P.M. & Herring, T.A., 2002. Modeling of nutation-precession: effects of electromagnetic coupling, *J. geophys. Res.,* **107(84)**.

Busse, F.H., 1970. The dynamical coupling between inner core and mantle of the Earth and the 24-year libration of the pole, in *Earthquake Displacement Fields and the Rotation of the Earth, Astrophysics and Space Science Library,* Vol. 20, pp. 88–98, eds, Mansinha, D., Smylie, D.E. & Beck, A.E., Springer-Verlag, New York.

Celaya, M.A., Wahr, J.M. & Bryan, F.O., 1999. Climate-driven polar motion, *J. geophys. Res.,* **104,** 12 813–12 829.

Courtier, N. *et al.*, 2000. Global superconducting gravimeter observations and the search for the translational modes of the inner core, *Phys. Earth planet. Inter.,* **117,** 3–20.

Dehant, V., Hinderer, J., Legros, H. & Lefftz, M., 1993. Analytical approach to the computation of the Earth, the outer core and inner core rotational motions, *Phys. Earth planet. Inter.,* **76,** 259–282.

Dickman, S.R., 1981. Investigation of controversial polar motion features using homogeneous international latitude service data, *J. geophys. Res.,* **86,** 4904–4912.

Dickman, S.R., 1983. The rotation of the ocean-solid Earth system, *J. geophys. Res.,* **88,** 6373–6394.

Dickman, S.R., 1985. Comments on 'Normal modes of the coupled Earth system' by John M. Wahr, *J. geophys. Res.,* **90,** 11 553–11 556.

Dormy, E., Valet, J.-P. & Courtillot, V., 2000. Numerical models of the geodynamo and observational constraints, *Geochem. Geophys. Geosyst.,* **1,** paper number 2000GC000062.

Dziewonski, A.M. & Anderson, D.L., 1981. Preliminary reference Earth model, *Phys. Earth planet. Inter.,* **25,** 297–356.

Frost, H.J. & Ashby, M.E., 1982. *Deformation Mechanism Maps,* Pergamon Press, Oxford.

Greff-Lefftz, M. & Legros, H., 1995. Core mantle coupling and polar motion, *Phys. Earth planet. Inter.,* **91,** 273–283.

Greff-Lefftz, M., Legros, H. & Dehant, V., 2000. Influence of the inner core viscosity on the rotational eigenmodes of the Earth, *Phys. Earth planet. Inter.,* **122,** 187–204.

Greiner-Mai, H. & Barthelmes, F., 2001. Relative wobble of the Earth's inner core derived from polar motion and associated gravity variations, *Geophys. J. Int.,* **144,** 27–36.

Gross, R.S., 1990. The secular drift of the rotation pole, in *Earth Rotation and Coordinate Reference Frames,* pp. 146–153, eds, Boucher, C. & Wilkins, G.A., Springer-Verlag, New York, NY.

Gross, R.S., 2000. The excitation of the Chandler wobble, *Geophys. Res. Lett.,* **27,** 2329–2332.

Gubbins, D. & Roberts, P.H., 1987. Magnetohydrodynamics of the Earth's core, in *Geomagnetism,* vol. 2, ed. Jacobs, J.A., Academic Press, London.

Hide, R., Boggs, D.H. & Dickey, J.O., 2000. Angular momentum fluctuations within the Earth's liquid core and torsional oscillations of the core-mantle system, *Geophys. J. Int.,* **143,** 777–786.

Hide, R., Boggs, D.H., Dickey, J.O., Dong, D., Gross, R.S. & Jackson, A., 1996. Topographic core-mantle coupling and polar motion on decadal time-scales, *Geophys. J. Int.,* **125,** 599–607.

Hinderer, J. & Crossley, D., 2000. Time variations in gravity and inferences on the Earth's structure and dynamics, *Surv. Geophys.,* **21,** 1–45.

Hinderer, J., Gire, C., Legros, H. & Le Mouël, J.-L., 1987. Geomagnetic secular variation, core motions and implications for the Earth's wobble, *Phys. Earth planet. Inter.,* **49,** 121–132.

Hinderer, J., Legros, H., Jault, D. & Le Mouël, J.-L., 1990. Core-mantle topographic torque: a spherical harmonic approach and implications for the excitation of the Earth's rotation by core motions, *Phys. Earth planet. Inter.,* **59,** 329–341.

Hulot, G., Le Huy, M. & Le Mouël, J.-L., 1996. Influence of core flows on the decade variations of the polar motion, *Geophys. astrophys. Fluid. Dyn.,* **82,** 35–67.

Jackson, A., Bloxham, J. & Gubbins, D., 1993. Time-dependent flow at the core surface and conservation of angular momentum in the coupled core-mantle system, in *Dynamics of the Earth's Deep Interior and Earth Rotation,Geophysical Monograph*Vol. 2, 72, pp. 97–107, eds, Le Mouël, J.-L., Smylie, D.E. & Herring, T., American Geophysical Union, Washington DC.

Jault, D., Gire, C. & Le Mouël, J.-L., 1988. Westward drift, core motions and exchanges of angular momentum between core and mantle, *Nature,* **333,** 353–356.

Kakuta, C., Okamoto, I. & Sasao, T., 1975. Is the nutation of the solid inner core responsible for the 24-year libration of the pole?, *Publ. astr. Soc. Jpn.,* **27,** 357–365.

Kuang, W. & Bloxham, J., 1997a. On the dynamics of topographical core-mantle coupling, *Phys. Earth planet. Inter.,* **99,** 289–294.

Kuang, W. & Bloxham, J., 1997b. An Earth-like numerical dynamo model, *Nature,* **389,** 371–374.

Kuang, W. & Bloxham, J., 1999. Numerical modeling of magnetohydrodynamic monvection in a mapidly rotating spherical shell: weak and strong field dynamo action, *J. Comp. Phys.,* **153,** 51–81.

Lambeck, K., 1980. *The Earth's Variable Rotation: Geophysical Causes and Consequences,* Cambridge University Press, Cambridge.

Laske, G. & Masters, G., 1999. Limits on differential rotation of the inner core from an analysis of the Earth's free oscillations, *Nature,* **402,** 66–69.

Markowitz, W., 1960. Latitude and longitude and the secular motion of the pole, in *Methods and Techniques in Geophysics,* pp. 325–361, ed. Runcorn, S.K., Interscience Publishers, London.

Markowitz, W., 1968. Concurrent astronomical observations for studying continental drift, polar motion, and the rotation of the Earth, in *Continental Drift, Secular Motion of the Pole and Rotation of the Earth,* pp. 25–32, eds, Markowitz, W. & Guinot, B., Springer-Verlag, Dordrecht.

Mathews, P.M. & Shapiro, I.I., 1992. Nutations of the Earth, *Ann. Rev. Earth planet. Sci.,* **20,** 469–500.

Mathews, P.M., Buffett, B.A., Herring, T.A. & Shapiro, I.I., 1991a. Forced nutations of the Earth: Influence of inner core dynamics. 1. theory, *J. geophys. Res.,* **96,** 8219–8242.

Mathews, P.M., Buffett, B.A., Herring, T.A. & Shapiro, I.I., 1991b. Forced nutations of the Earth: Influence of inner core dynamics. 2. numerical results, *J. geophys. Res.,* **96,** 8243–8257.

McCarthy, D.D. & Luzum, B.J., 1996. Path of the mean rotational pole from 1899 to 1994, *Geophys. J. Int.,* **125,** 623–629.

Olson, P. & Aurnou, J., 1999. A polar vortex in the Earth's core, *Nature,* **402,** 170–173.

Pais, A. & Hulot, G., 2000. Length of day decade variations, torsional oscillations and inner core superrotation: evidence from recovered core surface zonal flows, *Phys. Earth planet. Inter.,* **118,** 291–316.

Poma, A., Proverbio, E. & Uras, S., 1987. Long term variations in the Earth's motion and crustal movements, *J. Geodyn.,* **8,** 245–261.

Poupinet, G., Souriau, A. & Coutant, O., 2000. The existence of an inner core super-rotation questioned by teleseismic doublets, *Phys. Earth planet. Inter.,* **118,** 77–88.

Rochester, M.G., 1960. Geomagnetic westward drift and irregularities in the Earth's rotation, *Phil. Trans. R. Soc. Lond., A.* **252,** 531–555.

Sasao, T., Okubo, T. & Saito, M., 1980. A simple theory on the dynamical effects of a stratified fluid core upon nutational motion of the Earth, in *Proceedings of IAU Symposium,* Vol. 78, pp. 165–183, eds, Federov, E.P., Smith, M.L. & Bender, P.L., D. Reidel, Hingham, Mass.

Smith, M.L. & Dahlen, F.A., 1981. The period and Q of the Chandler wobble, *Geophys. J. R. astr. Soc.,* **64,** 223–281.

Smylie, D.E. & McMillan, D.G., 2000. The inner core as a dynamic viscometer, *Phys. Earth planet. Inter.,* **117,** 71–79.

Smylie, D.E., Szeto, A.M.K. & Rochester, M.G., 1984. The dynamics of the Earth's inner and outer cores, *Rep. Prog. Phys.,* **47,** 855–906.

Song, X.D. & Richards, P.G., 1996. Seismological evidence for differential rotation of the Earth's inner core, *Nature,* **382,** 221–224.

Souriau, A. & Poupinet, G., 2000. Inner core rotation: a test at the worldwide scale, *Phys. Earth planet. Inter.,* **118,** 13–27.

Sovers, O.J., Fanselow, J.L., & Jacobs, C. S., 1998. Astrometry and geodesy with radio interferometry: experiments, models, results, *Rev. Mod. Phys.,* **70,** 1393–1454.

Su, W.J., Dziewonski, A.M. & Jeanloz, R., 1996. Planet within a planet: Rotation of the inner core of the Earth, *Science,* **274,** 1883–1887.

Sumita, I., Yoshida, S., Kumazawa, M. & Hamano, Y., 1996. A model for sedimentary compaction of a viscous medium and its application to inner-core growth, *Geophys. J. Int.,* **124,** 502–524.

Taylor, J.B., 1963. The magneto-hydrodynamics of a rotating fluid and the Earth's dynamo problem, *Proc. R. Soc. Lond., Ser.,* A. **274,** 274–283.

Wahr, J.M., 1981. The forced nutations of an ellitical, rotating, elastic and oceanless Earth, *Geophys. J. R. astr. Soc.,* **87,** 633–668.

Wahr, J.M., 1984. Normal modes of the coupled Earth and ocean system, *J. geophys. Res.,* **89,** 7621–7630.

Wilson, C.R. & Vicente, R.O., 1980. An analysis of homogeneous ILS polar motion series, *Geophys. J. R. astr. Soc.,* **62,** 605–616.

Xu, S. & Szeto, A.M.K., 1998. The coupled rotation of the inner core, *Geophys. J. Int.,* **133,** 279–297.

Yoshida, S., Sumita, I. & Kumazawa, M., 1996. Growth model of the inner core coupled with the outer core dynamics and the resulting elastic anisotropy, *J. geophys. Res.,* **101,** 28 085–28 103.

Zatman, S. & Bloxham, J., 1997. Torsional oscillations and the magnetic field whithin the Earth's core, *Nature,* **388,** 760–763.

APPENDIX A: ELECTROMAGNETIC TORQUE FROM TORSIONAL OSCILLATIONS ACTING ON A TILTED DIPOLE FIELD

The electromagnetic torque on the inner core that results from an oscillating azimuthal flow is given by

$$\boldsymbol{\Gamma} = \frac{1}{\mu}\int_{\mathrm{ICB}} (\mathbf{r}\times\mathbf{b})\mathbf{B}\cdot\mathbf{dS}, \tag{A1}$$

where $\mathbf{B}$ is the magnetic field, $\mathbf{b}$ is the magnetic field perturbation created by the shear in the flow at the ICB, μ is the permeability of free space, $\mathbf{r}$ is the position vector directed away from the center and $\mathbf{dS} = a_s^2 \sin\theta \, d\phi \, d\theta \hat{\mathbf{e}}_\mathbf{r}$ is a surface element at the approximately spherical ICB. The torque can be expanded in spherical coordinates as

$$\mathbf{\Gamma} = \frac{a_s^3}{\mu} \int_0^\pi \int_0^{2\pi} (B_r b_\theta \hat{\mathbf{e}}_\phi - B_r b_\phi \hat{\mathbf{e}}_\theta) \sin\theta \, d\phi \, d\theta. \tag{A2}$$

In terms of the Cartesian unit vectors $\hat{\mathbf{e}}_\mathbf{1}$, $\hat{\mathbf{e}}_\mathbf{2}$ and $\hat{\mathbf{e}}_\mathbf{3}$, where $\hat{\mathbf{e}}_\mathbf{1}$ and $\hat{\mathbf{e}}_\mathbf{2}$ are the equatorial directions, the unit vectors $\hat{\mathbf{e}}_\phi$ and $\hat{\mathbf{e}}_\theta$ are given by

$$\hat{\mathbf{e}}_\phi = -\sin\phi \hat{\mathbf{e}}_\mathbf{1} + \cos\phi \hat{\mathbf{e}}_\mathbf{2}, \tag{A3}$$

$$\hat{\mathbf{e}}_\theta = \cos\theta \cos\phi \hat{\mathbf{e}}_\mathbf{1} + \cos\theta \sin\phi \hat{\mathbf{e}}_\mathbf{2} - \sin\theta \hat{\mathbf{e}}_\mathbf{3}, \tag{A4}$$

and using these definitions in (A2), the equatorial component in, say, direction $\hat{\mathbf{e}}_\mathbf{1}$ is

$$\Gamma_1 = \frac{a_s^3}{\mu} \int_o^\pi \left(\int_0^{2\pi} -B_r b_\theta \sin\phi \, d\phi \right) \sin\theta \, d\theta + \left(\int_0^{2\pi} -B_r b_\phi \cos\phi \, d\phi \right) \cos\theta \sin\theta \, d\theta. \tag{A5}$$

In the scenario that a torque is produced by an oscillating azimuthal flow shearing the radial magnetic field at the ICB, the induced magnetic field will be in the azimuthal direction. We therefore neglect the b_θ term. The solution for b_ϕ at the ICB that results from periodic tangential motion across the boundary was calculated by Buffett (1998) and its amplitude is given by

$$b_\phi = \frac{B_r \Delta u_\phi \sin\theta}{2\sqrt{\eta \sigma_d}}, \tag{A6}$$

where η is the magnetic diffusivity, σ_d is the frequency of oscillation of the azimuthal flow, and Δu_ϕ is the azimuthal velocity difference between the inner core and the flow. This solution takes into account the effects of magnetic diffusion. The equatorial torque in direction $\hat{\mathbf{e}}_\mathbf{1}$ becomes

$$\Gamma_1 = -\frac{a_s^3}{\mu} \frac{\Delta u_\phi}{2\sqrt{\eta \sigma_d}} \int_o^\pi \left(\int_0^{2\pi} B_r^2 \cos\phi \, d\phi \right) \cos\theta \sin^2\theta \, d\theta. \tag{A7}$$

The precise amplitude of the torque depends on the morphology of the radial magnetic field. As an example, we can calculate the torque that results when the radial magnetic field is a superposition of an axial dipole and an equatorial dipole field oriented in $\hat{\mathbf{e}}_\mathbf{1}$. At the ICB, in terms of Gauss coefficients defined at the surface of the inner core, this field is described by

$$B_r = -2g_1^0 \cos\theta - 2g_1^1 \cos\phi \sin\theta. \tag{A8}$$

Taking the square of (A8) and inserting it in (A7), the integrals of the $(g_1^0)^2$ term and $(g_1^1)^2$ term are both zero and only the cross term, $8g_1^0 g_1^1 \cos\phi \cos\theta \sin\theta$, contributes. The integration of

$$\Gamma_1 = -\frac{a_s^3}{\mu} \frac{\Delta u_\phi}{2\sqrt{\eta \sigma_d}} \int_o^\pi \left(\int_0^{2\pi} 8g_1^0 g_1^1 \cos^2\phi \, d\phi \right) \cos^2\theta \sin^3\theta \, d\theta, \tag{A9}$$

gives

$$\Gamma_1 = -\frac{16\pi a_s^3 g_1^0 g_1^1 \Delta u_\phi}{15\mu(\eta\sigma_d)^{1/2}}. \tag{A10}$$