<div style="text-align: right">GJI Geodesy, potential field and applied geophysics</div>

Decadal variations in gravity caused by a tilt of the inner core

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SUMMARY

A tilt of the geometric figure of the inner core with respect to the mantle results in a global internal mass displacement. This comprises two parts: the redistribution of mass from the rigid equatorial rotation of the elliptical inner core; and that from global elastic deformations that occur to maintain the mechanical equilibrium. This global mass reorganization leads to changes in the moment of inertia tensor and, equivalently, to changes in the degree 2 component of the gravitational field. In this work, we compute the predicted changes in both gravity and in the moment of inertia tensor as a function of inner core tilt. We show that the inclusion of elastic deformations increases the amplitude of the gravity change at the surface by a factor 1.97. The Stokes coefficients that are the most affected are C_{21}, S_{21}: a tilt angle of $0.05°$ leads to a change in these coefficients of $\sim 4 \times 10^{-11}$, while leading to changes in other coefficients of degree 2 that are three orders of magnitude smaller. Observed changes in C_{21}, S_{21} and in polar motion contain decadal variations of undetermined origin; in an effort to determine whether these could be caused by temporal changes in inner core tilt, we compute the changes in C_{21}, S_{21} based on the observed polar motion and compare this prediction against observed variations as determined by satellite laser ranging (SLR) between 1985 and 2005. We show that observed decadal changes in C_{21}, S_{21} and in polar motion suggest that both are predominantly driven by variations in the moment of inertia tensor. The source of these variations cannot be unambiguously determined, nor can we confirm whether they are of internal or surficial origin. Changes in inner core tilt are then not necessarily the cause of these variations, though if they are, our results show that motion in the fluid core must not play a significant role in the global angular momentum balance.

Key words: Time variable gravity; Earth rotation variations; Elasticity and anelasticity.

1 INTRODUCTION

Changes in the Earth's gravitational field occur in response to redistribution of mass in its interior and on its surface. The largest changes are caused by internal processes such as postglacial rebound and mantle convection (e.g. Mitrovica & Forte 2004; Tamisiea *et al.* 2007). These contribute to long-term variations, appearing as a secular trend over a timescale of a few years or decades. Over seasonal timescales, gravity variations are dominantly the result of mass displacements in the atmosphere, oceans and the hydrosphere (Chen & Wilson 2003). A mass exchange between the oceans and continental ice sheets also occurs on seasonal timescales, and in recent years this has also contributed to the secular trend in gravity (e.g. Ramillien *et al.* 2006).

There is also evidence of gravity changes taking place at interannual to decadal timescales. These can be inferred from variations in

the Earth's rotation: changes in the degree 2 spherical harmonic component of gravity are indicative of changes in the Earth's moment of inertia, which, because angular momentum of the whole Earth must be conserved, are accompanied by variations in the Earth's rotation vector. The latter can then be used to monitor changes in Stokes coefficients C_{20}, C_{21} and S_{21}, provided the angular momentum carried by fluid regions of Earth can be effectively modelled (e.g. Chen *et al.* 2000). Changes in C_{21} and S_{21} predicted from models of mass redistribution in the atmosphere, ocean and hydrosphere agree very well with the prediction determined from rotational variations at seasonal and intraseasonal timescales (Chen & Wilson 2003). This agreement persists partially at interannual to decadal timescales, though an important residual signal of amplitude ~ 2–4×10^{-11} remains (Chen *et al.* 2005), suggesting that a part of the gravity signal cannot be explained by surface processes. A consistent result is obtained when one proceeds in reverse: models of motion and redistribution of mass at the Earth's surface are incapable of generating the polar motion observed at interannual to decadal timescales (Gross *et al.* 2005). This may be simply because the decadal mass variations caused by the ensemble of surface precesses are not modelled properly. Alternately, it may indicate

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that a part of the decadal gravity signal is caused by internal mass variations.

Fluid motion with decadal timescale variations are known to occur in the Earth's core (e.g. Jault *et al.* 1988; Jackson *et al.* 1993; Zatman & Bloxham 1997). The convective mass anomalies displaced by these motions must be very small (Stevenson 1987) and this is unlikely to produce the observed gravity signal. However, decadal fluid motions may have the ability to imprint an equatorial torque on the inner core, leading to a tilt of its geometric figure with respect to that of the mantle (e.g. Dumberry & Bloxham 2002; Dumberry 2007). An equatorial rotation of the elliptical inner core displaces the density discontinuity between the fluid and solid inner core, and variations in inner core tilt may be an important source of gravity variations. Based on a recent calculation by Greiner-Mai & Barthelmes (2001), a tilt of $\sim 0.1°$ would be sufficient to explain the required variations of ~ 2–4×10^{-11} in C_{21}, S_{21}.

Variations in inner core tilt may then account for the observed decadal variations in polar motion and in the coefficients C_{21}, S_{21}. If this is the case, the change in C_{21}, S_{21} is directly related to the change in the moment of inertia tensor associated with the tilted inner core. The resulting polar motion is likewise a function of the change in the global moment of inertia, but depends also on the angular momentum carried by fluid motion in the core. This is because an inner core tilt can only result from an equatorial torque acting on it, and the latter must invariably involve motion that carry angular momentum in the fluid core (Dumberry & Bloxham 2002).

This suggests that further clues on the origin of the decadal variations in polar motion and gravity, whether caused by inner core tilt variations or not, can be obtained by a comparison between the observed decadal variations in C_{21}, S_{21} and a prediction of these changes based on the observed polar motion. This should clarify whether the changes in both are predominantly caused by changes in the moment of inertia alone, or whether angular momentum carried by fluid motion in the core, or in any other fluid region, is important in the decadal angular momentum balance.

The first objective of this work is to re-evaluate the change in the degree 2 gravity and in the moment of inertia caused by a tilt of the inner core. The recent calculation of Greiner-Mai & Barthelmes (2001) considered the gravity change due to the mass redistribution associated with the rigid equatorial rotation of the inner core alone. Once the inner core is no longer aligned with the geometric figure of the mantle, global elastic deformations occur in order to maintain the mechanical equilibrium. These elastic displacements result in a global mass reorganization that also contributes to gravity changes. We will show that elastic deformations are important, leading to an increase in the gravity change at the surface by a factor of almost 2.

The second objective is to attempt to determine whether temporal changes in the tilt of the inner core can consistently explain the observed decadal variations in C_{21}, S_{21} and in polar motion. To do so, we compare observed variations in C_{21}, S_{21} as determined by the technique of satellite laser ranging (SLR) with predictions based on the observed polar motion. These predictions are based on whether the equatorial angular momentum of the fluid core participates or not in the decadal timescale balance. As we will see, this comparison does not permit to identify the source of the decadal changes, and thus whether they are caused by inner core tilt variations or by a different mechanism. However, it clarifies the nature of the equatorial angular momentum balance at decadal timescale. Our results suggest that decadal changes in both the polar motion and in C_{21}, S_{21} are predominantly caused by changes in the moment of inertia. The possibility of large inner core tilt variations occurring at decadal timescale is then re-assessed in the light of this result.

2 GRAVITY PERTURBATIONS CAUSED BY A RIGID ROTATION OF THE INNER CORE

2.1 Density displacement

We represent the density structure of the undeformed Earth in terms of surfaces on which the density is constant. These surfaces are defined as

$$r(a, \theta, \phi) = a \left(1 + \epsilon Y_2^0 + \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \varepsilon_l^m Y_l^m \right), \tag{1}$$

where (a, θ, ϕ) are spherical coordinates with a being the mean radius, $Y_l^m = Y_l^m(\theta, \phi)$ are fully normalized spherical harmonic functions (e.g. Edmonds 1960; Dahlen & Tromp 1998), $\epsilon = \epsilon(a)$ is the amplitude of the elliptical flattening, and $\varepsilon_l^m = \varepsilon_l^m(a)$ is the amplitude of the density deformation at degree l and order m.

The largest non-spherical density structure is the ellipticity induced by Earth's rotation, and we expect $\epsilon \gg \varepsilon_l^m$. As a result, the change in the density structure produced by a tilt of the inner core in an equatorial direction should be dominated by the equatorial rotation of the elliptical part of the density. The rotation of the other non-spherical components contributes a small correction to the local change in density, but to simplify, we neglect this contribution. Thus, we consider the reference undeformed density structure of the inner core to be purely ellipsoidal,

$$r_s = a \left(1 + \epsilon Y_2^0 \right). \tag{2}$$

An equivalent definition of the elliptical density surfaces is given by

$$r = a \left[1 - \frac{2}{3} \bar{\epsilon} P_2(\cos \theta) \right], \tag{3}$$

where $P_2(\cos \theta)$ is the Legendre polynomial of degree 2. This later definition is sometimes more convenient to relate our results to familiar quantities in geodesy. $\bar{\epsilon} = \bar{\epsilon}(a)$ represents the flattening, or geometrical ellipticity, defined positive for an oblate spheroid, such that the difference between the equatorial and polar radius is $a\bar{\epsilon}$ (Mathews *et al.* 1991). The ellipticity coefficients defined in (2) and (3) are related through

$$\epsilon = -\frac{2}{3} \sqrt{\frac{4\pi}{5}} \bar{\epsilon}. \tag{4}$$

An equatorial rotation of the inner core by an angle β towards a longitudinal direction α, as pictured in Fig. 1, leads to a rotation of the above defined surfaces within the inner core. In its tilted configuration, the inner core's surfaces of constant density are given by

$$\tilde{r}_s = a \left(1 + \sum_{m=-2}^{2} \tilde{\epsilon}_2^m Y_2^m \right), \tag{5}$$

where the coefficients $\tilde{\epsilon}_l^m = \tilde{\epsilon}_l^m(a)$ are functions of the ellipticity ϵ and the rotation angles β and α. These coefficients can be determined from the rules that govern the rotation of spherical harmonics (e.g. Dahlen & Tromp 1998, section C.8.2). Our specific case is described by 2 successive rotations; first an equatorial rotation of angle β in the direction of longitude $\phi = 0$, followed by an axial rotation of angle α. Any spherical harmonic function $\psi = c_l^m Y_l^m$ subject to two such successive rotations, described symbolically by operators $\mathcal{D}_\beta$ and $\mathcal{D}_\alpha$, is transformed according to

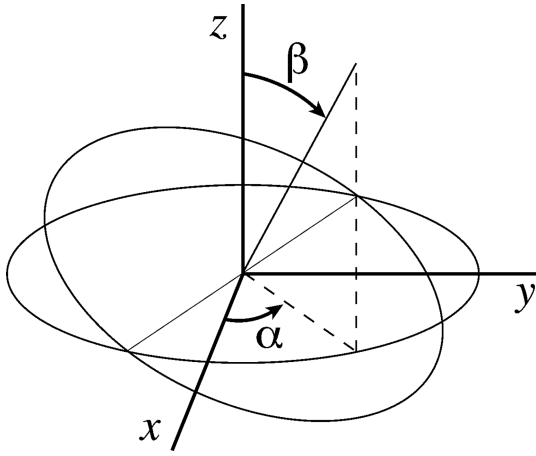$$\tilde{\psi} = \mathcal{D}_\alpha \mathcal{D}_\beta c_l^m Y_l^m \tag{6}$$

Figure 1. Equatorial rotation of the inner core of a tilt angle of $\theta = \beta$ in the longitudinal direction $\phi = \alpha$.

$$= c_l^m \sum_{N=-l}^{l} P_{lm}^N(\cos \beta) e^{-iN\alpha} Y_l^N, \tag{7}$$

where $P_{lm}^N(\cos \beta)$ are generalized associated Legendre functions described by Edmonds (1960) and used in a geophysical context by Phinney & Burridge (1973). For our specific case, where the original spherical harmonic function only comprises a Y_2^0 term, the inner core's rotated surfaces of constant density are described by

$$\tilde{r}_s = a \left[1 + \epsilon \sum_{N=-2}^{2} P_{20}^N(\cos \beta) e^{-iN\alpha} Y_2^N \right], \tag{8}$$

and the relevant generalized associated Legendre functions are

$$P_{20}^0(\cos \beta) = \frac{1}{2}(3\cos^2 \beta - 1),$$

$$P_{20}^2(\cos \beta) = P_{20}^{-2} = \frac{1}{2}\sqrt{\frac{3}{2}}(1 - \cos^2 \beta),$$

$$P_{20}^1(\cos \beta) = -P_{20}^{-1} = -\sqrt{\frac{3}{2}} \sin \beta \, \cos \beta. \tag{9}$$

The displacement of the density structure inside the inner core can thus be conveniently separated into a sum of all orders of the degree 2 harmonic, and can be written as

$$\Delta r_s = \tilde{r}_s - r_s$$

$$= \sum_{m=-2}^{2} \delta_{\beta\alpha}^m a\epsilon Y_2^m, \tag{10}$$

where

$$\delta_{\beta\alpha}^0 = \frac{3}{2}(\cos^2 \beta - 1), \tag{11}$$

$$\delta_{\beta\alpha}^{\pm 1} = \mp\sqrt{\frac{3}{2}} \sin \beta \, \cos \beta e^{\mp i\alpha}, \tag{12}$$

$$\delta_{\beta\alpha}^{\pm 2} = \frac{1}{2}\sqrt{\frac{3}{2}}(1 - \cos^2 \beta)e^{\mp i2\alpha}. \tag{13}$$

2.2 Changes in gravitational potential

The gravitational potential of a slightly non-spherical body whose surfaces of constant density are defined according to (2) is well

known (e.g. Jeffreys 1970). Inside the inner core ($a \leq a_s$), the gravitational potential caused by the density structure of the inner core alone is

$$\Phi_s(a, \theta, \phi) = -4\pi G \left(\frac{1}{a} \int_0^a \rho a'^2 \, \mathrm{d}a' + \int_a^{a_s} \rho a' \, \mathrm{d}a' \right)$$
$$- \frac{4\pi G}{5} \left[\frac{1}{a^3} \int_0^a \rho \frac{\partial}{\partial a'}(a'^5 \epsilon) \, \mathrm{d}a' + a^2 \int_a^{a_s} \rho \frac{\partial \epsilon}{\partial a'} \, \mathrm{d}a' \right] Y_2^0, \tag{14}$$

where G is the gravitational constant and ρ is density. Let us represent the change in the gravitational potential everywhere in the Earth by

$$\Delta \Phi(a, \theta, \phi) = \sum_{l,m} \delta\hat{\varphi}_l^m(a) Y_l^m. \tag{15}$$

An equatorial rotation of the inner core involves only the coefficients of degree 2, $\delta\hat{\varphi}_2^m$. Inside the inner core, these coefficients are obtained by substituting ϵ by $\delta_{\beta\alpha}^m \epsilon$ in (14), and are given by

$$\delta\hat{\varphi}_2^m(a) = \delta_{\beta\alpha}^m U_\epsilon(a), \tag{16}$$

where

$$U_\epsilon(a) = -\frac{4\pi G}{5} \frac{1}{a^3} \int_0^a (\rho - \rho_f') \frac{\partial}{\partial a'}(a'^5 \epsilon) \, \mathrm{d}a'$$
$$- \frac{4\pi G}{5} a^2 \int_a^{a_s} (\rho - \rho_f') \frac{\partial \epsilon}{\partial a'} \, \mathrm{d}a'. \tag{17}$$

In the above expression, ρ_f' is a constant equal to the density of the fluid core at the inner core boundary (ICB). Its presence accounts for the forced displacement of the fluid at the ICB by the equatorial rotation of the inner core.

For the region outside the inner core ($a > a_s$), the gravitational potential due to density displacements within the inner core satisfies Laplace's equation. The coefficients of degree 2 are given by

$$\delta\hat{\varphi}_2^m(a) = \delta_{\beta\alpha}^m U_\epsilon(a_s) \left(\frac{a_s}{a} \right)^3, \tag{18}$$

where

$$U_\epsilon(a_s) = -\frac{4\pi G}{5} \frac{1}{a_s^3} \int_0^{a_s} (\rho - \rho_f') \frac{\partial}{\partial a'}(a'^5 \epsilon) \, \mathrm{d}a'. \tag{19}$$

2.3 Changes in the moment of inertia

The axial and mean equatorial moments of inertia of the undeformed elliptical Earth are given, to the first order in ellipticity, respectively, by

$$C = \frac{8\pi}{3} \int_0^{a_e} \rho \left[a'^4 + \frac{2}{15} \frac{\partial}{\partial a'} \left(a'^5 \bar{\epsilon} \right) \right] \mathrm{d}a', \tag{20}$$

$$A = \frac{8\pi}{3} \int_0^{a_e} \rho \left[a'^4 - \frac{1}{15} \frac{\partial}{\partial a'} \left(a'^5 \bar{\epsilon} \right) \right] \mathrm{d}a'. \tag{21}$$

The moments of inertia of the inner core (C_s, A_s), fluid core (C_f, A_f) and mantle (C_m, A_m) are defined similarly by changing the limits of integration in (20)–(21) to, respectively, $0–a_s$, $a_s–a_f$ and $a_f–a_e$, where a_f and a_e are the mean radii at the core–mantle boundary (CMB) and the Earth's surface, respectively.

An equatorial rotation of the inner core leads to the following changes in the axial and mean equatorial moments of inertia

$$\delta C = \frac{2}{3} \delta_{\beta\alpha}^0 (\mathcal{H}_s - \mathcal{H}_s'), \tag{22}$$

$$\delta A = -\frac{1}{3}\delta_{\beta\alpha}^0(\mathcal{H}_s - \mathcal{H}_s'), \tag{23}$$

where

$$\mathcal{H}_s = \frac{8\pi}{15}\int_0^{a_s} \rho \frac{\partial}{\partial a'}(a'^5\bar{\epsilon})\,\mathrm{d}a', \tag{24}$$

$$\tilde{\mathcal{H}}_s' = \frac{8\pi}{15}\rho_f' a_s^5 \bar{\epsilon}_s. \tag{25}$$

The definitions of $\mathcal{H}_s$, $\mathcal{H}_s'$ and of $U_\epsilon(a_s)$ in (19) imply the following relationship between them,

$$U_\epsilon(a_s) = \sqrt{\frac{4\pi}{5}}\frac{G}{a_s^3}(\mathcal{H}_s - \mathcal{H}_s'). \tag{26}$$

We also note that, with respect to the notation used in the study of forced nutations of Mathews *et al.* (1991),

$$\mathcal{H}_s = A_s e_s, \quad \mathcal{H}_s' = A'e', \quad \mathcal{H}_s - \mathcal{H}_s' = A_s e_s \alpha_3, \tag{27}$$

where

$$\alpha_3 = 1 - \frac{A'e'}{A_s e_s}, \tag{28}$$

and where $e_s = (C_s - A_s)/A_s$ is the dynamic ellipticity of the inner core, A' is the mean equatorial moment of inertia of a body of inner core shape but with uniform density ρ'_f, and e' is the dynamic ellipticity of this body.

3 GRAVITY PERTURBATIONS CAUSED BY ELASTIC DEFORMATIONS

3.1 Elastic density displacement

The global change in gravitational potential described in the previous section leads to a global change in the gravitational force. In addition, the displacement of inner core particles within the elliptical hydrostatic stress field leads to a change in the local pressure force. Thus, the mechanical equilibrium between the hydrostatic stress field and the combined gravitational and centrifugal forces (the equilibrium that describes the reference undeformed Earth) is perturbed. The sum of these induced forces leads to deformations, which then produce further changes in the gravitational force and tractions. In this paper, we make the approximation that the deformations are static; the final deformed state is that which allows the global mechanical equilibrium to be maintained.

We note that solutions to the static deformation problem do not exist in general (e.g Denis *et al.* 1998). However, solutions do exist for the special case of incompressible displacements in the fluid core. This is an approximation that we adopt in this paper, and the elastic deformations that we report are subject to it.

Inside the inner core, we express the total displacement of the density surfaces induced by an equatorial rotation of the inner core as

$$\Delta r_s = \sum_{m=-2}^{2}\delta_{\beta\alpha}^m a\,\epsilon\,(1+\tilde{h}_\epsilon)\,Y_2^m, \tag{29}$$

where the radially variant coefficient $\tilde{h}_\epsilon = \tilde{h}_\epsilon(a)$ is a compliance factor that accounts for elastic deformations. Such a compliance is often referred to as a Love number (Love 1909). In the fluid core and mantle ($a > a_s$), the displacement is solely that due to elastic

deformations,

$$\Delta r_{f,m} = \sum_{m=-2}^{2}\delta_{\beta\alpha}^m a\,\epsilon\,\tilde{h}_\epsilon\,Y_2^m. \tag{30}$$

3.2 Changes in gravitational potential and moments of inertia

The total change in gravitational potential is the sum of that associated with the density displacement of the inner core rotation in (15) and that due to elastic deformations. Inside the inner core, the coefficients of gravitational potential expressing this total change are

$$\delta\varphi_2^m(a) = \delta\hat{\varphi}_2^m(a)(1+\tilde{k}_\epsilon)$$
$$= \delta_{\beta\alpha}^m U_\epsilon(a)(1+\tilde{k}_\epsilon), \tag{31}$$

whereas in the fluid core and mantle,

$$\delta\varphi_2^m(a) = \delta_{\beta\alpha}^m U_\epsilon(a_s)\left(\frac{a_s}{a}\right)^3(1+\tilde{k}_\epsilon). \tag{32}$$

In these expressions, $\tilde{k}_\epsilon = \tilde{k}_\epsilon(a)$, is a gravitational Love number, and it is related to $\tilde{h}_\epsilon$ through

$$\tilde{k}_\epsilon(a) = -\frac{4\pi G}{5}\frac{1}{U_\epsilon(a)\,a^3}\int_0^a \rho\frac{\partial}{\partial a'}(a'^5\epsilon\,\tilde{h}_\epsilon)\,\mathrm{d}a'$$
$$-\frac{4\pi G}{5}\frac{a^2}{U_\epsilon(a)}\int_a^{a_e}\rho\frac{\partial}{\partial a'}\left(\epsilon\,\tilde{h}_\epsilon\right)\,\mathrm{d}a'. \tag{33}$$

This definition of $\tilde{k}_\epsilon$ is valid inside as well as outside the inner core, though when $a > a_s$, $U_\epsilon(a)$ must be replaced by $U_\epsilon(a_s)(a_s/a)^3$.

To express the contribution of elastic deformations to the change in the moments of inertia, it is convenient to define the following quantity,

$$\mathcal{H}^\epsilon = \frac{8\pi}{15}\int_0^{a_e}\rho\frac{\partial}{\partial a'}(a'^5\bar{\epsilon}\tilde{h}_\epsilon)\,\mathrm{d}a'. \tag{34}$$

Using this definition, the total change in the axial and mean equatorial moments of inertia are then

$$\delta C = \frac{2}{3}\delta_{\beta\alpha}^0(\mathcal{H}_s - \mathcal{H}_s' + \mathcal{H}^\epsilon), \tag{35}$$

$$\delta A = -\frac{1}{3}\delta_{\beta\alpha}^0(\mathcal{H}_s - \mathcal{H}_s' + \mathcal{H}^\epsilon). \tag{36}$$

Following the definition of $\mathcal{H}^\epsilon$ in (34), it is related to the value of $\tilde{k}_\epsilon$ at the surface by

$$\tilde{k}_\epsilon(a_e) = \sqrt{\frac{4\pi}{5}}\frac{G}{a_e^3}\left(\frac{a_e}{a_s}\right)^3\frac{\mathcal{H}^\epsilon}{U_\epsilon(a_s)}, \tag{37}$$

and using (26), we also have

$$\left(\mathcal{H}_s - \mathcal{H}_s'\right)\tilde{k}_\epsilon(a_e) = \mathcal{H}^\epsilon. \tag{38}$$

This means that the total change in gravitational potential of degree 2 at the surface can be written as

$$\delta\varphi_2^m(a_e) = \delta_{\beta\alpha}^m U_\epsilon(a_s)\left(\frac{a_s}{a_e}\right)^3[1+\tilde{k}_\epsilon(a_e)], \tag{39}$$

or, alternately, as

$$\delta\varphi_2^m(a_e) = \sqrt{\frac{4\pi}{5}}\frac{G}{a_e^3}\delta_{\beta\alpha}^m(\mathcal{H}_s - \mathcal{H}_s' + \mathcal{H}^\epsilon),$$
$$= \sqrt{\frac{4\pi}{5}}\frac{G}{a_e^3}\delta_{\beta\alpha}^m(\mathcal{H}_s - \mathcal{H}_s')[1+\tilde{k}_\epsilon(a_e)]. \tag{40}$$

Eq. (40), relating changes in the moment of inertia to changes in degree 2 of the gravitational potential, is an expression of the generalized MacCullagh's formula (Chao & Gross 1987).

4 RESULTS

4.1 Changes in Stokes coefficients as a function of inner core tilt

Given a 1-D model of the Earth where the density and elastic parameters are only dependent on radius, the geometrical ellipticity as a function of radius can be determined using the assumption of hydrostatic equilibrium and Clairaut's equation (Jeffreys 1970). The quantities $\mathcal{H}_s, \mathcal{H}'_s$ can thus be readily calculated. To determine the quantities $\mathcal{H}^\epsilon$ and $\tilde{k}_\epsilon(a_e)$, we need to evaluate the elastic displacements $\tilde{h}_\epsilon$ taking place under a tilt of the inner core. The procedure to do so is documented in Appendix A. Here, we simply present their numerical values, which are found in Table 1, along with the numerical values of all the other relevant parameters for subsequent calculations. All parameters are calculated based on the seismic model PREM (Dziewonski & Anderson 1981).

The parameter $\tilde{k}_\epsilon(a_e)$ in (40) expresses the contribution of global elastic deformations to the gravity change at the surface. To simplify the notation, from here onward we refer to this parameter simply as $\tilde{k}_\epsilon$. Its numerical value is 0.9736. This means that elastic deformations contribute almost as much to the gravity change as the mass redistribution from the inner core tilt; they increase the gravity signal by almost a factor 2.

We follow the usual convention in geodesy and express the changes in degree 2 gravitational potential at the surface in terms of changes in the Stokes coefficients, ΔC_{lm} and ΔS_{lm}. These are related to the our coefficients defined in eqs (39)–(40) by

$$\Delta C_{20} = -\frac{1}{\sqrt{4\pi}} \frac{a_e}{GM} \delta\varphi_2^0(a_e), \tag{41}$$

Table 1. Values of the parameters defined in the text and used in the calculations. All the calculated numerical values are based on PREM.

General parameters	
$G = 6.672 \times 10^{-11}$ N m^2 kg^{-2}	
$M = 5.974 \times 10^{24}$ kg	
Radii (m)	
$a_s = 1.222 \times 10^6$	$a_f = 3.480 \times 10^6$
$a_e = 6.371 \times 10^6$	
Moments of Inertia (kg m^2)	
$C = 8.037 \times 10^{37}$	$A = 8.012 \times 10^{37}$
$A_s = 5.853 \times 10^{34}$	$A' = 5.539 \times 10^{34}$
$\mathcal{H}_s = 1.418 \times 10^{32}$	$\mathcal{H}'_s = 1.342 \times 10^{32}$
$\mathcal{H}^\epsilon = 7.397 \times 10^{30}$	$\mathcal{H}^\epsilon_f = 1.794 \times 10^{31}$
$\mathcal{H}^\epsilon_s = -1.583 \times 10^{31}$	
Dynamic ellipticities	
$e = 3.247 \times 10^{-3}$	$e_s = 2.422 \times 10^{-3}$
$e' = 2.422 \times 10^{-3}$	
Coupling constants	
$\alpha_1 = 0.9463$	$\alpha_2 = 0.8294$
$\alpha_3 = 0.0537$	$\alpha_g = 2.1752$
Elastic compliances	
$\tilde{k}_\epsilon(a_e) = \tilde{k}_\epsilon = 0.9736$	$\kappa = 1.039 \times 10^{-3}$
$S_{14} = 9.247 \times 10^{-8}$	$S_{24} = 1.981 \times 10^{-6}$
$S_{34} = -2.705 \times 10^{-4}$	

Table 2. Change in Stokes coefficients as a function of inner core tilt.

Coefficient	$\beta = 0.05°$	$\beta = 0.1°$	$\beta = 1.0°$
ΔJ_2	-7.07×10^{-14}	-2.83×10^{-13}	-2.83×10^{-11}
$\|\Delta C_{20}\|$	3.16×10^{-14}	1.27×10^{-13}	1.27×10^{-11}
$\|\Delta \tilde{C}_{21}\|$	4.19×10^{-11}	8.37×10^{-11}	8.37×10^{-10}
$\|\Delta \tilde{C}_{22}\|$	1.83×10^{-14}	7.31×10^{-14}	7.31×10^{-12}

$$\Delta C_{2m} = -\frac{(-1)^m}{2\sqrt{4\pi}} \frac{a_e}{GM} Re\left[\delta\varphi_2^m(a_e) + (-1)^m \delta\varphi_2^{-m}(a_e)\right], \tag{42}$$

$$\Delta S_{2m} = -\frac{(-1)^m}{2\sqrt{4\pi}} \frac{a_e}{GM} Im\left[-\delta\varphi_2^m(a_e) + (-1)^m \delta\varphi_2^{-m}(a_e)\right], \tag{43}$$

where M is the mass of the Earth. Writing the Stokes coefficients ΔC_{2m} directly in terms of the angles β and α of inner core tilt,

$$\Delta C_{20} = -\frac{3}{2\sqrt{5}}\bar{\mathcal{H}}(\cos^2\beta - 1), \tag{44}$$

$$\Delta C_{21} + i\Delta S_{21} = \Delta\tilde{C}_{21} = -\sqrt{\frac{3}{5}}\bar{\mathcal{H}}\sin\beta\cos\beta\, e^{i\alpha}, \tag{45}$$

$$\Delta C_{22} + i\Delta S_{22} = \Delta\tilde{C}_{22} = \frac{1}{2}\sqrt{\frac{3}{5}}\bar{\mathcal{H}}\left(1 - \cos^2\beta\right) e^{i2\alpha}, \tag{46}$$

where

$$\bar{\mathcal{H}} = \frac{(\mathcal{H}_s - \mathcal{H}'_s + \mathcal{H}^\epsilon)}{M a_e^2} = 6.186 \times 10^{-8}. \tag{47}$$

The elliptical component of gravity is often given in terms of the quantity $J_2 = (C - A)/Ma_e^2$. The changes in J_2 are given by

$$\Delta J_2 = \frac{\delta C - \delta A}{Ma_e^2} = -\sqrt{5}\,\Delta C_{20}. \tag{48}$$

We present in Table 2 the variations in Stokes coefficients that are expected for a change in the tilt of inner core of 0.05°, 0.1° and 1°. A change of 4.2×10^{-11} in C_{21}, S_{21}, of the same order as the observed decadal variations, can be readily produced by an inner core tilt of 0.05°. For the same angle of tilt, the change in C_{20}, C_{22} and S_{22} is three orders of magnitude smaller.

We have also included in Table 2 the expected variations in J_2. With respect to the linear trend caused by tidal dissipation and postglacial rebound, the scale of the typical observed variations in J_2 is of the order 5×10^{-11} (e.g. Cox & Chao 2002), and in order to explain such variations with a tilt of the inner core, a tilt in excess of 1° is required. Such large variations in inner core tilt are clearly not happening, as this would imply variations in C_{21}, S_{21} of the order of 10^{-9}, an order of magnitude larger than the observed variations. It can be safely assumed that, for inner core tilt variations that do not lead to variations in C_{21}, S_{21} in excess of the observed amplitudes, the associated change in J_2 is negligible.

4.2 Variations in Stokes coefficients predicted from polar motion

Predictions of the time-dependent variations in the Stokes coefficients of degree 2 from the mechanism described in the present study require the knowledge of the changes in amplitude and orientation of the inner core tilt as a function of time. Unfortunately, this information is not known directly, but only through indirect means.

One can use the variations in the position of the Earth's rotation axis, or polar motion, as a proxy for the variations in inner core

tilt. Variations in polar motion $\tilde{m} = m_1 + im_2$, where subscripts 1 and 2 refer to the two principal equatorial directions (m_1 is in the direction of Greenwich and m_2 is in the direction of 90°E), must obey (e.g. Lambeck 1980)

$$\frac{i}{\sigma_r}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{m} + \left(1 - \frac{\kappa}{e}\right)\tilde{m} = \tilde{\psi}_{\mathrm{mass}} + \tilde{\psi}_{\mathrm{motion}}, \tag{49}$$

where $e = (C - A)/A$ is the dynamic ellipticity, κ is a compliance factor characterizing elastic deformations that take place under polar motion, and σ_r is the Eulerian nutation frequency, or the Chandler wobble frequency of the rigid Earth ($=e\Omega$, where Ω is the frequency of Earth's rotation). $\tilde{\psi}_{\mathrm{motion}}$ is the excitation function due to motion that carry angular momentum, while $\tilde{\psi}_{\mathrm{mass}}$ is the excitation function due to redistribution of mass. The latter can be expressed in terms of changes in the non-diagonal elements ΔI_{13} and ΔI_{23} of the moment of inertia tensor, which are themselves related to changes in Stokes coefficients of degree 2 through the generalized MacCullagh's formula (Chao & Gross 1987),

$$\tilde{\psi}_{\mathrm{mass}} = \frac{\Delta I_{13} + i\Delta I_{23}}{(C - A)} = -\sqrt{\frac{5}{3}}\frac{Ma_e^2}{(C - A)}(\Delta C_{21} + i\Delta S_{21}). \tag{50}$$

Slow variations in polar motion ($\mathrm{d}/\mathrm{d}t \to 0$) are thus related to changes in C_{21} and S_{21} by

$$\tilde{m} = -\sqrt{\frac{5}{3}}\frac{Ma_e^2}{A(e - \kappa)}(\Delta C_{21} + i\Delta S_{21}). \tag{51}$$

Under the assumption that the decadal changes in C_{21} and S_{21} and in polar motion are both consequent to variations in inner core tilt, using (45) we obtain the following relationship between the polar motion and the tilt of the inner core,

$$\tilde{n}_s = \frac{A(e - \kappa)}{(\mathcal{H}_s - \mathcal{H}_s' + \mathcal{H}^\epsilon)}\tilde{m} = \frac{A(e - \kappa)}{A_s e_s \alpha_3(1 + \tilde{k}_\epsilon)}\tilde{m}, \tag{52}$$

where $\tilde{n}_s = (n_s)_1 + i(n_s)_2 = \beta e^{i\alpha}$, and where we have assumed $\beta \ll 1$. Though we have used a different notation, this relationship is equivalent to that obtained by Greiner-Mai & Barthelmes (2001), except that our expression also includes the effect of elastic deformations accompanying the inner core tilt through the parameter $\tilde{k}_\epsilon$. The inclusion of elastic deformation reduces by a factor 2 the amplitude of the required inner core tilt for a given polar motion amplitude. For example, using the numerical values of the parameters listed in Table 1, a polar motion of 10 mas requires an inner core tilt of 0.065° if elastic deformations are not included, but only 0.033° if they are.

Thus, under the assumption that polar motion can be used as a proxy for inner core tilt, changes in C_{21}, S_{21} are obtained directly from (51). It is important to note, however, that (51) is valid for any internal mass reorganization that leads to a change in the moment of inertia. Eq. (51) simply expresses the consistent changes in gravity and polar motion that occur as a consequence of an internal mass change, regardless of its source.

In Fig. 2, we show the predicted changes in C_{21} and S_{21} based on (51) and the observed variations in polar motion since 1975. We have used the polar motion time-series EOP-C04 of the International Earth Rotation Service (IERS), removed a linear trend, and applied a low-pass third order Butterworth filter to remove changes that occur faster than a threshold periodicity of 5 yr. This is to remove variations in polar motion at interannual timescales and shorter that are known to be caused principally by surface processes (Chen *et al.* 2005; Gross *et al.* 2005). As a reference, the amplitude of the polar motion that produces the changes in C_{21} and S_{21} of the order of 5×10^{-11} shown in Fig. 2 is approximately 10 mas. Assuming these
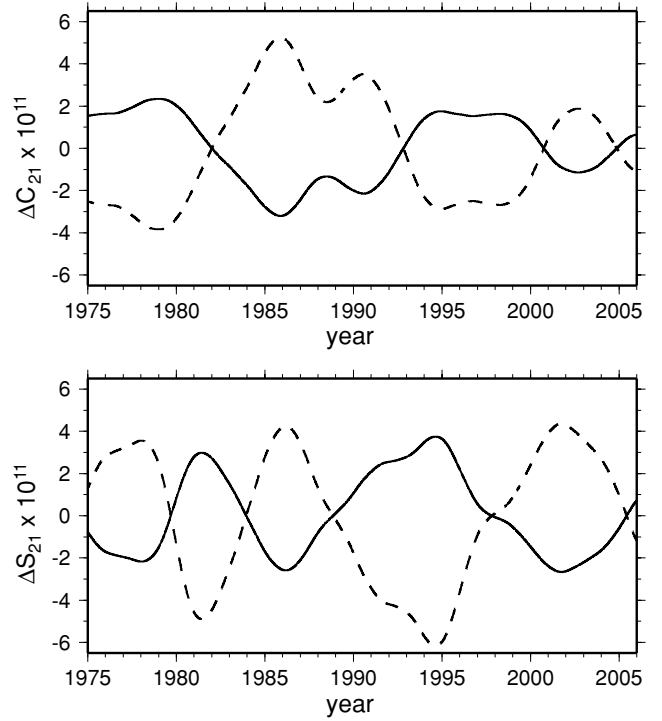

Figure 2. Predicted variations in Stokes coefficients C_{21} and S_{21} based on polar motion according to eq. (51) (solid lines) and eq. (62) (dashed lines). Variations in polar motion are from time-series EOP-C04 of the IERS, on which a third order Butterworth low-pass filter with a period threshold of 5 yr has been applied.

variations reflect changes in the tilt of the inner core, the associated variations in the tilt amplitude are approximately 0.06°.

The study of Dumberry & Bloxham (2002) suggests that the relationship between the inner core tilt and polar motion may be more complex than the simple mass reorganization implied by (52). The reason, they argue, is because one must take into account the torque that gives rise to the inner core tilt in the first place. If this torque is from surface tractions at the ICB, the angular momentum dynamics of the fluid core must be consistent with the generation of this torque. To obtain the relationship between $\tilde{m}$ and $\tilde{n}_s$, one must solve a system of four coupled equations, which in matrix notation is given as

$$\mathbf{M} \cdot \mathbf{x} = \mathbf{b}. \tag{53}$$

The vector $\mathbf{x}$, defined as $[\tilde{m}, \tilde{m}_f, \tilde{m}_s, \tilde{n}_s]^T$ where $\tilde{m}_f$ and $\tilde{m}_s$ represent the individual rotation of the fluid and inner core, respectively, is the solution of the equations and $\mathbf{b}$ includes the applied torque on the inner core and its reaction on the fluid core. The elements of matrix $\mathbf{M}$ are given in Dumberry & Bloxham (2002) and also in Mathews *et al.* (1991), though these do not include the elastic deformations caused by a tilted inner core that we have calculated in the present study. The inclusion of these in (53) can be accomplished through the changes in the moment of inertia of the whole Earth, of the fluid core, and of the inner core that enter the elements of matrix $\mathbf{M}$,

$$\Delta I_{13} + i\Delta I_{23} = A\left(\kappa\tilde{m} + \xi\tilde{m}_f + \zeta\tilde{m}_s + S_{14}\tilde{n}_s\right), \tag{54}$$

$$\Delta I_{13}^f + i\Delta I_{23}^f = A_f\left(\gamma\tilde{m} + \beta\tilde{m}_f + \delta\tilde{m}_s + S_{24}\tilde{n}_s\right), \tag{55}$$

$$\Delta I_{13}^s + i \Delta I_{23}^s = A_s \left(\theta \tilde{m} + \chi \tilde{m}_f + \nu \tilde{m}_s + S_{34} \tilde{n}_s \right). \tag{56}$$

The three set of compliances $(\kappa, \xi, \zeta), (\gamma, \beta, \delta), (\theta, \chi, \nu)$ characterize elastic deformations of the whole Earth, the fluid core, and the inner core, respectively, which arise through independent rotation of these three regions. These are already included in **M**. The parameters S_{j4} are the additional compliances describing the elastic deformations accompanying a tilted inner core, and follow the notation introduced by Buffett *et al.* (1993). These can be expressed as

$$S_{14} = \frac{\mathcal{H}_s^\epsilon}{A}, \quad S_{24} = \frac{\mathcal{H}_f^\epsilon}{A_f}, \quad S_{34} = \frac{\mathcal{H}_s^\epsilon}{A_s}, \tag{57}$$

where the quantities $\mathcal{H}_s^\epsilon$ and $\mathcal{H}_f^\epsilon$ are defined as in (34) but with integration limits changed to, respectively, 0 to a_s, and a_s to a_f. Numerical values for S_{14}, S_{24}, S_{34}, $\mathcal{H}_s^\epsilon$ and $\mathcal{H}_f^\epsilon$ are given in Table 1. With the inclusion of these new compliances, the following 3 elements of the matrix **M** must be changed to

$$\mathbf{M}(1,4) \to (1+\sigma)(A_s e_s \alpha_3/A + S_{14}), \tag{58}$$

$$\mathbf{M}(2,4) \to -\sigma(A_s e_s \alpha_1/A_f + S_{24}), \tag{59}$$

$$\mathbf{M}(3,4) \to (1 + \sigma - \alpha_2 + \sigma S_{34}), \tag{60}$$

where σ is the frequency of oscillation and parameters α_1 and α_2 are defined in Mathews *et al.* (1991).

Solutions of (53) are obtained for an applied torque on the inner core of a given amplitude and periodicity. If we exclude the free oscillations from the solution, the relationship between $\tilde{m}$ and $\tilde{n}_s$ for slow variations ($\sigma \to 0$) is well approximated by

$$\tilde{n}_s \approx -\frac{A(e-\kappa)}{(\mathcal{H}_s - \mathcal{H}_s')(\alpha_g - \tilde{k}_\epsilon)} \tilde{m} = -\frac{A(e-\kappa)}{A_s e_s \alpha_3(\alpha_g - \tilde{k}_\epsilon)} \tilde{m}, \tag{61}$$

where the factor α_g represents the ability of the rest of the Earth to exert a gravitational torque on the inner core (Mathews *et al.* 1991). In this case, the inclusion of elastic deformations lead to an increase of the amplitude of the inner core tilt for a given polar motion: a polar motion of 10 mas requires an inner core tilt of $0.054°$ (in the reverse direction) if elastic deformations are included, but $0.030°$ if they are not.

The inclusion of the dynamics associated with the fluid core leads to a very different relationship between $\tilde{m}$ and $\tilde{n}_s$ than the one predicted by (52). The reason is because the fluid core contributes to the global Earth angular momentum balance and enters through $\tilde{\psi}_{\text{motion}}$ in (49). If this contribution is neglected in the system (53), we retrieve (52) exactly.

Assuming that the polar motion resulting from a tilted inner core must satisfy (61), the change in gravity caused by an inner core tilt can be computed directly from polar motion with

$$\tilde{m} = \sqrt{\frac{5}{3}} \frac{M a_e^2 (\alpha_g - \tilde{k}_\epsilon)}{A(e-\kappa)(1+\tilde{k}_\epsilon)} (\Delta C_{21} + i \Delta S_{21}). \tag{62}$$

We show in Fig. 2 the predicted changes in C_{21} and S_{21} based on (62). This prediction is different from that based on eq. (51) by factor $-(1+\tilde{k}_\epsilon)/(\alpha_g - \tilde{k}_\epsilon) \approx -1.63$.

4.3 Observed variations in gravity

Under the assumption that decadal changes in polar motion and gravity are caused by temporal variations in the tilt of the inner core, variations in C_{21} and S_{21} should follow (51) or (62), depending on whether the fluid core participates or not in the global angular momentum balance. To determine whether the actual variations in C_{21} and S_{21} that have taken place in the last few decades are compatible with either (51) or (62), we need measurements of the variations in C_{21} and S_{21} that are independent from the polar motion. We present in Fig. 3 the temporal variations in C_{21} and S_{21} as determined by SLR tracking (C. M. Cox, personal communication, 2006). This model is an updated version of the one presented in Cox *et al.* (2004), and after a correction for the IB-NCEP atmospheric gravity series. Data for satellite Starlette prior to 1984 are sparse, and we show only the variations between 1985 and 2005. A linear trend between 1985 and 2005 has been removed.

A clear annual signal is visible in both ΔC_{21} and ΔS_{21}. This annual signal is driven by a combination of atmospheric, oceanic, and hydrological seasonal mass variations (Chen & Wilson 2003). Interannual and decadal variations in both ΔC_{21} and ΔS_{21} are also visible in Fig. 3. Interannual variations of up to a few years are also caused predominantly by mass variations at the surface of the Earth, though variations at longer timescale, such as the 10-yr timescale variations visible in Fig. 3 do not appear to be related to surface processes (Chen *et al.* 2005). To extract the variations in C_{21} and S_{21} at timescales longer than a few years, we applied a third order low-pass Butterworth filter with a period threshold of 5 yr to the SLR time-series. These 'decadal' variations in C_{21} and S_{21} are shown by the red lines on Fig. 3 and are of the order of 5×10^{-11}.

In principle, decadal gravity variations in Fig. 3 do not rely on any *a priori* assumptions about a relationship between ΔC_{21}, ΔS_{21} and $\tilde{m}$. These can then be compared with the predictions of the gravity variations determined by (51) or (62) (Fig. 4). The SLR-derived changes in C_{21} agree generally well with the prediction from (51). Similarly, changes in S_{21} from 1995 onward are also correlated with the variations predicted by (51). Though the changes in S_{21} prior to 1995 do not follow (51) well, there is no clear indication that (62) provides a better prediction. Thus, gravity variations predicted on the basis of (51) are in general agreement with the observed variations determined by SLR tracking.

Part of the differences between the observed and predicted variations in C_{21}, S_{21} are undoubtedly caused by the limited quality of the SLR gravity model and polar motion observations. However, that a reasonable correlation exists between SLR-derived variations in C_{21} and S_{21} and the prediction from polar motion is reassuring. It indicates that the recovered decadal variations in these two independent signals are likely robust, at the very least after 1995.

The connection between the polar motion and changes in gravity field through the relation (51) applies for any internal mass redistribution that causes a change in the moment of inertia tensor. Therefore, this does not constitute a demonstration that the decadal polar motion and changes in C_{21}, S_{21} are both caused by variations in the tilt of the inner core. In fact, based on Fig. 4, it is even unclear whether the mass change is internal rather than occurring at the surface of the Earth. A surface mass reorganization loads the Earth, and involves a load Love number in the relationship (51). This would increase the amplitude of the polar-motion-predicted ΔC_{21} and ΔS_{21} by a factor 1.43. This clearly would provide a better fit to the SLR-derived ΔC_{21} after 1990, though it is less clear that this is also the case for ΔS_{21} (see grey lines on Fig. 4). To address
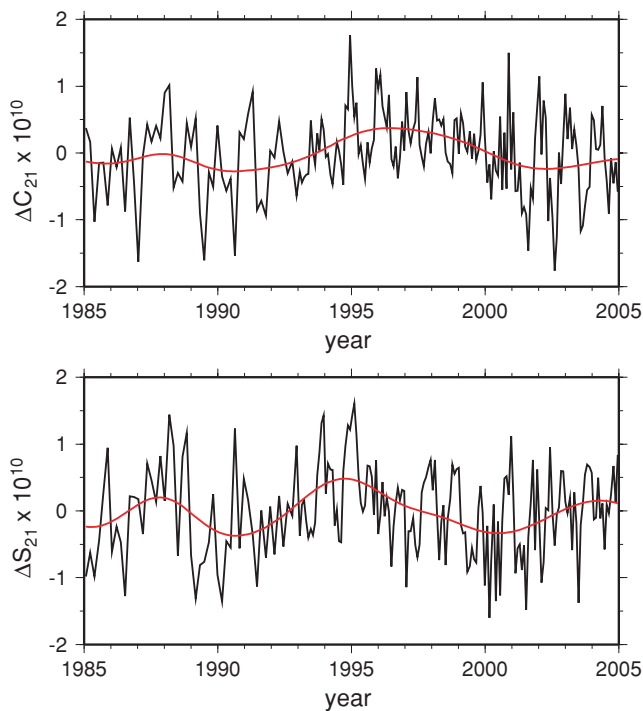
Figure 3. Changes in C_{21} and S_{21} determined by SLR tracking (black lines) after removal of the IB-NCEP atmospheric gravity series. A linear trend has also been subtracted. The red lines correspond to the decadal variations in C_{21} and S_{21} obtained by applying a third order Butterworth low-pass filter with a period threshold of 5 yr to the SLR signal.

Figure 4. Decadal (>5 yr) changes in C_{21} and S_{21} derived from SLR tracking measurements (red lines), and the predictions based on (51) (solid black lines) and (62) (dashed black lines). The solid grey lines represent the prediction based on (51) but multiplied by a factor 1.43, as appropriate for a surface mass change.

this point in more detail, the known interannual contribution from surface processes should be removed from the polar motion and to the observed ΔC_{21}, ΔS_{21}. The comparison between the residual of each should allow to determine more accurately whether the remaining decadal mass reorganization is internal or surficial.

Though Fig. 4 does not allow to determine the source of the mass change, it nevertheless clarifies the role of the fluid core on decadal timescales. In view of the reasonably good agreement based on (51), changes in equatorial angular momentum of the fluid core must remain small on decadal timescales. The primary global angular momentum balance simply involves a polar motion in reaction to a change in the moment of inertia tensor. Thus, if decadal changes in both polar motion and ΔC_{21} and ΔS_{21} are caused by temporal changes in inner core tilt, this can only be the case if the temporal change in angular momentum of the fluid core remains small.

5 DISCUSSION AND CONCLUSION

The largest variations in gravity caused by a tilt of the elliptical inner core are in the coefficients C_{21} and S_{21}. In order to explain observed decadal variations in C_{21}, S_{21} of $\sim 4 \times 10^{-11}$, a tilt of the inner core of $0.05°$ is required. Slightly more than half of the mass redistribution is from the rigid equatorial rotation of the inner core, while the remaining part is from the accompanying global elastic deformations.

A prediction of the decadal changes in C_{21}, S_{21} for the past 20 yr based on the observed polar motion agrees generally well with observed changes in C_{21}, S_{21} as determined by SLR tracking measurements. This is especially true after 1995. This prediction assumes no contribution from the angular momentum of the fluid core or
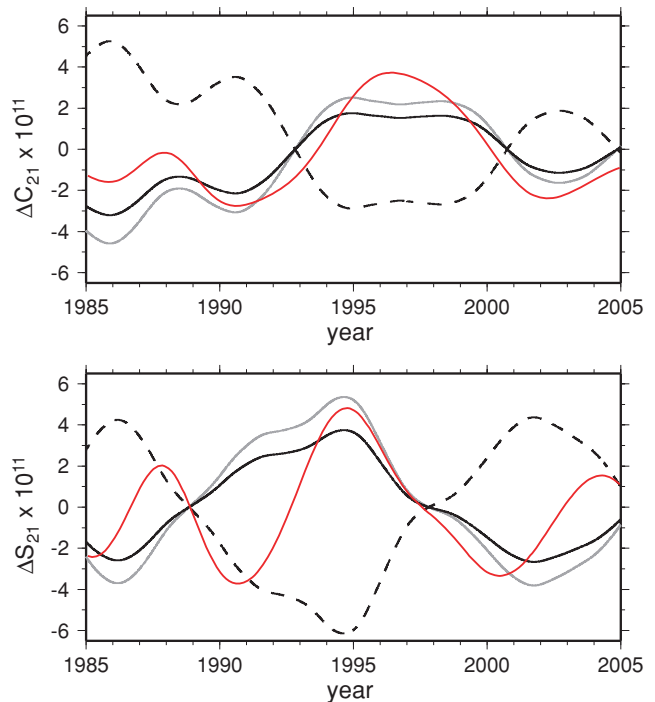
any other fluid regions. This suggests that the decadal variations in the moment of inertia tensor, which are induced by the mass redistribution that causes the change in C_{21}, S_{21}, represent the leading contribution to polar motion at decadal timescales.

By itself, this result does not prove or disprove that temporal variations in inner core tilt are the cause of the observed decadal changes in gravity and polar motion; any mass redistribution can accomplish both. Nor are we able to determine whether the source of the decadal mass change is predominantly internal or occurring at the surface of the Earth. To do this would require a more thorough treatment of the known contributions to both polar motion and gravity changes from surface processes.

However, in the light of this result, the likelihood that inner core tilt variations are indeed responsible for these decadal variations can be assessed through dynamical arguments. On the one hand, if mass variations are shown to be internal, it is difficult to conceive of a different mechanism that can generate the required decadal mass fluctuations. Mass changes in the mantle through convection and postglacial rebound can be significant, but occur on much longer timescales [$\mathcal{O}(10^6)$ and $\mathcal{O}(10^3)$ yr, respectively]. Motions in the fluid core are known to occur at decadal timescales, however the amplitude of the moving density anomalies are very small. The existence of these decadal fluid motions is however important, as it provides mechanisms to generate the required decadal equatorial torques on the inner core (Dumberry & Bloxham 2002; Dumberry 2007). As the observed changes in gravity and polar motion can be accomplished by relatively small variations in the tilt of the inner core (of the order of $0.05°$), one may argue that this remains the most plausible explanation.

On the other hand, the very requirement of a decadal torque on the inner core by fluid motion can be used as the counter argument

for the occurrence of inner core tilt variations of 0.05°. The study of Dumberry & Bloxham (2002) argues that if the torque is due to surface stresses at the ICB, the reverse torque must be applied on the fluid core, and the angular momentum carried by the latter contributes significantly to the global angular momentum budget. The results presented here suggest that this is not the case. The implication is that the fluid core does not have the ability to generate a large enough torque on the inner core such that the required mass reorganization from its tilt can be achieved.

A possible way out of this last argument is if the fluid core also exerts a torque on the mantle which approximately balances the torque that the fluid exerts on the inner core. In this way, the total torque on the fluid core remains small, and so does its angular momentum variation. However, attempts at evaluating the equatorial torque at the CMB suggest that the latter is too small to provide the balance (Greff-Lefftz & Legros 1995; Hide *et al.* 1996; Hulot *et al.* 1996). Another possibility is if the torque on the inner core is not from surface stresses but from a gravitational volume force. In this case, the torque applied on the fluid core may remain small. Such a scenario was presented by Dumberry (2007), though the amplitude of the torque was found to be insufficient to generate the required tilt variations.

Considering the difficulty of generating the required inner core tilt without involving large variations in the angular momentum of the fluid core, it is perhaps more likely that decadal variations in polar motion and in C_{21}, S_{21} may not result from inner core tilt variations. However, if they do, our results imply that the process governing the temporal variations of inner core tilt must be associated with a small changes in the angular momentum carried by the fluid core.

ACKNOWLEDGMENTS

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APPENDIX A: ELASTIC DEFORMATIONS

The density displacement associated with an inner core tilt results in a global change in the gravitational force. Inside the inner core, the displacement of particles within the elliptical hydrostatic stress field results in a further forcing from the change in pressure. The combination of these forces perturb the mechanical equilibrium and leads to additional deformations. If we assume that a static equilibrium is maintained in the deformed state, these additional deformations must be such that the changes in the gravitational potential and stress tensor that they induce are exactly those required to maintain the mechanical equilibrium.

The goal of this appendix is to evaluate the amplitude of these additional deformations, which we assume to be elastic. To do so, we follow the standard procedure of integrating throughout the whole Earth the elastic-gravitational equations associated with small displacements produced by a known forcing. This procedure is well documented in the literature (e.g. Alterman *et al.* 1959; Dahlen 1972, 1974), and here we only provide the details that are relevant to our specific problem. We closely follow the procedure outlined in Dumberry & Bloxham (2004), where more details on the approximations can be found as well as further references to the literature.

Results of similar calculations have been previously published by Buffett *et al.* (1993) in the context of the forced nutations of the Earth, and were derived assuming a diurnal periodicity for the forcing and deformations. Here, we are interested in the deformations resulting from a forcing that varies on a much longer timescale, so much that we approximate this forcing to be static.

A1 Undeformed reference Earth

The displacement associated with the tilt of the inner core and its accompanying elastic deformations are taking place with respect to an undeformed reference Earth. In Section 2, we expressed this undeformed Earth in terms of elliptical surfaces of constant density. These surfaces were defined according to the hydrostatic equilibrium that prevails for a self-gravitating Earth under steady rotation.

Here, however, it is more convenient to define the equilibrium undeformed reference Earth to be purely spherically symmetric. This is because the geometrical ellipticity is $\mathcal{O}(10^{-3})$ and is a small perturbation of an otherwise spherical Earth. Thus, to the first order in ellipticity, the mechanical equilibrium in the deformed state is achieved solely by deforming the spherical Earth. We retain the effect of the ellipticity only in the evaluation of the forces that perturbs the equilibrium. Proceeding as such simplifies the formulation of the problem while only introducing an error of the order of the geometrical ellipticity on the elastic deformations.

Since the ellipticity is entirely a consequence of the effect of the Earth's rotation, neglecting it amounts to eliminating the centrifugal potential in the hydrostatic equilibrium. The hydrostatic equilibrium of the spherically symmetric undeformed Earth is given by

$$0 = -\frac{\partial}{\partial r}p_o(r) - \rho_o(r)g_o(r), \tag{A1}$$

where p_o is the hydrostatic pressure, ρ_o is density, g_o is the gravitational acceleration defined as $g_o = \frac{\partial \phi_o}{\partial r}$, and where the gravitational potential $\phi_o(r)$ satisfies Poisson's equation,

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)\phi_o(r) = 4\pi G \rho_o(r). \tag{A2}$$

Throughout this appendix, the variable r describes the spherical radius. It may still be viewed as describing surfaces of constant density, as in (2), but which are here defined to be purely spherical. The values of ρ_o, p_o, ϕ_o, g_o as a function of radius are obtained from models of the Earth's interior inferred by seismic observations. In this study, we use the model PREM (Dziewonski & Anderson 1981), although we neglect the presence of oceans at the surface.

A2 Rigid rotation of the inner core

Elastic deformations with respect to the above spherical reference equilibrium arise as a result of the forcing associated with the tilt of the inner core, and are function of the density displacement inside the inner core and at the ICB. As we have defined our reference undeformed Earth to be spherically symmetric, we need to determine the radial displacement of the spherical density surfaces that produce the equivalent change in density. To first order, it is given by (10),

$$\Delta r_s = \sum_{m=-2}^{2} \delta_{\beta\alpha}^m r \epsilon(r) Y_2^m. \tag{A3}$$

This displacement causes a local density change of

$$\rho_s = -\Delta r_s \frac{\partial \rho_o}{\partial r}. \tag{A4}$$

The gravitational potential inside the inner core ($r \leq a_s$) associated with this perturbation in density is

$$\phi_s = \sum_{m=-2}^{2} \delta_{\beta\alpha}^m \phi_\epsilon Y_2^m, \tag{A5}$$

where

$$\phi_\epsilon = \frac{4\pi G}{5}\left(\frac{1}{r^3}\int_0^r \epsilon \frac{\partial \rho_o}{\partial r'} r'^5 \, dr' + r^2 \int_r^{a_s} \epsilon \frac{\partial \rho_o}{\partial r'} \, dr'\right). \tag{A6}$$

Outside the inner core, it is

$$\phi_s = \sum_{m=-2}^{2} \delta_{\beta\alpha}^m \phi_\epsilon(a_s)\left(\frac{a_s}{r}\right)^3 Y_2^m. \tag{A7}$$

Similarly, the displacement of the density discontinuity at the ICB leads to a change in gravitational potential inside and outside the inner core given, respectively, by

$$\phi'_s = \sum_{m=-2}^{2} \delta_\beta^m \phi'_\epsilon Y_2^m, \tag{A8}$$

and

$$\phi'_s = \sum_{m=-2}^{2} \delta_\beta^m \phi'_\epsilon(a_s)\left(\frac{a_s}{r}\right)^3 Y_2^m, \tag{A9}$$

where

$$\phi'_\epsilon = \frac{4\pi G}{5}r^2\epsilon(a_s)\left[\rho_o^f(a_s) - \rho_o^s(a_s)\right], \tag{A10}$$

and where $\rho_o^f(a_s)$ and $\rho_o^s(a_s)$ are, respectively, the density of the fluid core and the inner core at the ICB.

The rotation of the elliptical inner core within the elliptical hydrostatic pressure field leads to a change in the local pressure. With respect to our reference hydrostatic pressure defined to be spherically symmetric, the equivalent pressure change is, to first order,

$$p_s = \rho_o g_o \Delta r_s. \tag{A11}$$

A3 Elastic deformations in the inner core

We seek to determine the elastic deformations **u** inside the inner core in response to a specified forcing. We assume that a static equilibrium is maintained in the deformed state. In other words, we neglect all inertial accelerations in the momentum balance. We also assume that the rotation does not influence the deformations in the sense that we neglect their associated Coriolis acceleration. The static equilibrium is governed by an ensemble of conditions that comprise the momentum equation which determines the mechanical equilibrium,

$$\mathbf{0} = \nabla \cdot \mathbf{T} - \nabla(\rho_o \mathbf{u} \cdot \nabla \phi_o) - \rho_o \nabla \phi_1 - \rho_1 g_o \mathbf{e_r} + \mathbf{f}_t, \tag{A12}$$

an elastic constitutive relation,

$$\mathbf{T} = \lambda_o \mathbf{I}(\nabla \cdot \mathbf{u}) + \mu_o(\nabla \mathbf{u} + (\nabla \mathbf{u})^T), \tag{A13}$$

an equation for continuity,

$$\rho_1 = -\rho_o \nabla \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{e_r} \frac{\partial \rho_o}{\partial r}, \tag{A14}$$

and Poisson's equation, which determines the changes in the gravity field

$$\nabla^2 \phi_1 = 4\pi G \rho_1. \tag{A15}$$

The above set of four coupled equations are known as the linearized elastic-gravitational equations for small displacements **u**. The incremental Cauchy stress tensor **T** involves the second rank identity tensor **I**, the Lamé parameter λ_o and the modulus of rigidity μ_o. The latter two define the elastic state of the reference undeformed Earth. The density perturbation ρ_1 is solely a consequence of the elastic displacements **u**. The change in gravitational potential ϕ_1 includes two parts: the change due to elastic deformations inside the inner core ϕ_u, which satisfies Poisson's equation $\nabla^2 \phi_u = 4\pi G \rho_1$; and the change due to mass displacement outside the inner core. This latter part satisfies Laplace's equation which can be added to Poisson's equation for ϕ_u to form (A15).

Neither ϕ_1 nor ρ_1 include the effect of mass redistribution associated with the equatorial rotation of the inner core. This is included in the vector $\mathbf{f}_t$, the forcing induced by the inner core tilt. It is given by

$$\mathbf{f}_t = -\nabla p_s - \rho_o \nabla \phi_s - \rho_s \nabla \phi_o,$$
$$= -\rho_o \nabla(\phi_s + \Delta r \, \mathbf{e_r} \cdot \nabla \phi_o), \tag{A16}$$

where ρ_s, ϕ_s and p_s are given, respectively, in (A4), (A5) and (A11). This forcing does not include the change in potential from the displacement of the density jump at the ICB: this contribution enters our formulation through the boundary conditions.

Solutions of the above system are found by an expansion of the perturbation variables and the forcing vectors in terms of fully nor-

malized surface spherical harmonics of degree 2,

$$\mathbf{u} = \sum_{m=-2}^{2} \left[U_2^m(r) Y_2^m \mathbf{e_r} + V_2^m(r) \nabla_1 Y_2^m \right],$$

$$\phi_1 = \sum_{m=-2}^{2} \Phi_2^m(r) Y_2^m,$$

$$\mathbf{e}_r \cdot \mathbf{T} = \sum_{m=-2}^{2} \left[R_2^m(r) Y_2^m \mathbf{e_r} + S_2^m(r) \nabla_1 Y_2^m \right],$$

$$\mathbf{f}_t = \sum_{m=-2}^{2} (f_2^t)_2^m Y_2^m \mathbf{e_r} + (f_4^t)_2^m \nabla_1 Y_2^m, \tag{A17}$$

where ∇_1 represents the gradient operator on the unit sphere. Using the above decomposition, eqs (A12)–(A15) are written as a system of 6 coupled first order ODE's in radius for each harmonic order m,

$$\frac{\partial}{\partial r} \mathbf{y^s} = \mathbf{A} \cdot \mathbf{y^s} - \mathbf{f}_t, \tag{A18}$$

where the vector $\mathbf{y^s} = [y_1^s, y_2^s, y_3^s, y_4^s, y_5^s, y_6^s]^T$ contains the following elements

$$y_1^s = U_2^m,$$

$$y_2^s = R_2^m = \lambda_o \left(\frac{\partial}{\partial r} U_2^m + 2\frac{U_2^m}{r} - 6\frac{V_2^m}{r} \right) + 2\mu_o \frac{\partial}{\partial r} U_2^m,$$

$$y_3^s = V_2^m,$$

$$y_4^s = S_2^m = \mu_o \left(\frac{\partial}{\partial r} V_2^m - \frac{V_2^m}{r} + \frac{U_2^m}{r} \right),$$

$$y_5^s = \Phi_2^m,$$

$$y_6^s = \frac{\partial}{\partial r} \Phi_2^m + \frac{3}{r} \Phi_2^m + 4\pi G \rho_o U_2^m, \tag{A19}$$

The elements of matrix **A** are given in Dumberry & Bloxham (2004). The elements of the force vector $\mathbf{f}_t = [0, (f_2^t)_2^m, 0, (f_4^t)_2^m, 0, 0]^T$ are

$$(f_2^t)_2^m = -\delta_{\beta\alpha}^m \rho_o \frac{\partial}{\partial r} \phi_\epsilon - \delta_{\beta\alpha}^m \rho_o \frac{\partial}{\partial r}(r\epsilon g_o),$$

$$(f_4^t)_2^m = -\delta_{\beta\alpha}^m \rho_o \frac{\phi_\epsilon}{r} - \delta_{\beta\alpha}^m \rho_o \epsilon g_o. \tag{A20}$$

A4 Elastic deformations in the mantle

The perturbations in the mantle are treated in an similar manner as those in the inner core. The forcing $\mathbf{f}_t$ associated with the inner core tilt is equal to $-\rho_o \nabla(\phi_s + \phi_s')$, where ϕ_s and ϕ_s' are specified in (A7) and (A9), respectively. As both these potentials satisfy Laplace's equation in the mantle, they can be added to Poisson's equation. Thus, we can recast our formulation into one where $\mathbf{f}_t$ equals zero but where the potential is redefined to also include the contribution from ϕ_s and ϕ_s'. The latter two enter the system through our specification of the boundary conditions. The system of equations can be written as

$$\frac{\partial}{\partial r} \mathbf{y^m} = \mathbf{A} \cdot \mathbf{y^m}, \tag{A21}$$

where $\mathbf{y^m}$ is the solution vector in the mantle.

A5 Deformations in the fluid core

We assume that the fluid core is inviscid and remains in hydrostatic equilibrium in the deformed state. Thus, tangential stresses vanish everywhere. As a result, surfaces of constant density, constant fluid

pressure and constant gravitational potential always coincide. The displacement of individual fluid particles is undetermined, and we only seek to determine the radial displacement u_r of the equipotential surfaces. The static equilibrium in the core is governed by the linearized first order perturbations in the hydrostatic balance,

$$\mathbf{0} = -\nabla p_1 - \rho_o \nabla \phi_1 - \rho_1 g_o \mathbf{e_r}, \tag{A22}$$

from which one can deduce that $p_1 = -\rho_o \phi_1$ and $u_r = -\frac{\phi_1}{g_o}$. Thus, the perturbation in pressure p_1 and the radial displacement of equipotential surfaces can be determined uniquely from the perturbation in gravitational potential ϕ_1. The latter satisfies the linearized first order Poisson's equation,

$$\nabla^2 \phi_1 = 4\pi G \rho_1. \tag{A23}$$

and an equation for continuity

$$\rho_1 = -u_r \frac{\partial \rho_o}{\partial r} = \frac{\phi_1}{g_o} \frac{\partial \rho_o}{\partial r}, \tag{A24}$$

in which we have used the divergence-free condition, $\nabla \cdot \mathbf{u} = 0$. As for the mantle, $\mathbf{f}_t$ does not appear explicitly in (A22) because we have defined the potential ϕ_1 such that it includes ϕ_s and ϕ'_s.

By inserting (A14) into (A23), we write Poisson's equation as a coupled set of ODE's, written in matrix form as

$$\frac{\partial}{\partial r} \mathbf{y^c} = \mathbf{B} \cdot \mathbf{y^c}, \tag{A25}$$

where $\mathbf{y^c} = [y_5^c, y_6^c]^T$ and

$$y_5^c = \Phi_2^m$$
$$y_6^c = \frac{\partial y_5^c}{\partial r} - \frac{4\pi G \rho_o}{g_o} y_5^c + \frac{3}{r} y_5^c, \tag{A26}$$

The elements of matrix $\mathbf{B}$ are given in Dumberry & Bloxham (2004).

A6 Boundary conditions

The solution of (A18), (A21) and (A25) require a total of 14 constants of integration, which represent as many degrees of freedom. These are determined by a set of boundary conditions at the origin, the ICB, the CMB and the Earth's surface.

The conditions near the origin are determined by the requirement that the solution remains finite as $r \to 0$. These conditions are found by expanding the variables in power-series of r near the origin $r = \xi$ (Crossley 1975), and can be written as

$$\mathbf{y^s}(\xi) = C_1 \mathbf{y^{\xi 1}} + C_2 \mathbf{y^{\xi 2}} + C_3 \mathbf{y^{\xi 3}}, \tag{A27}$$

where the coefficients C_1, C_2 and C_3 are three constants of integration. The elements of the vectors $\mathbf{y^{\xi 1}}$, $\mathbf{y^{\xi 2}}$ and $\mathbf{y^{\xi 3}}$ are given in appendix C of Dumberry & Bloxham (2004).

At the top of the mantle ($r = a_e$), in the absence of any external load, the radial stress, the tangential stress and the gravitational flux all vanish, which specifies 3 additional conditions:

$$y_2(a_e) = y_4(a_e) = y_6(a_e) = 0. \tag{A28}$$

At the CMB ($r = a_f$), the gravitational potential and the gravitational flux $\mathbf{e_r} \cdot (\nabla \phi + 4\pi G \rho_o \mathbf{u})$ must be continuous. The radial displacement is also continuous, but a discontinuity in tangential displacement is allowed. The normal traction force must be continuous, and since an inviscid fluid cannot support shear the tangential

traction on the solid side of the boundary must vanish. The complete set of boundary conditions at the CMB is

$$y_1^m(a_f) = -\frac{y_5^c(a_f)}{g_o(a_f)} + C_5,$$
$$y_2^m(a_f) = C_5 g_o(a_f) \rho_o^c(a_f),$$
$$y_3^m(a_f) = C_4,$$
$$y_4^m(a_f) = 0,$$
$$y_5^m(a_f) = y_5^c(a_f)$$
$$y_6^m(a_f) = y_6^c(a_f) + 4\pi G \rho_o^c(a_f) C_5, \tag{A29}$$

The constant C_4 is the arbitrary tangential displacement at the base of the mantle. The constant C_5 represents the 'apparent' jump between the radial displacement of material particles on the mantle side of the CMB and the radial displacement of equipotential surfaces in the fluid core. At static equilibrium, once all oscillations have decayed, these two different quantities do not have to coincide at the fluid–solid boundaries. This is because adopting divergence-free displacements in the fluid core necessarily implies a neutrally stratified core. The radial displacement of the solid boundary also displaces fluid particles, but the density of the latter can adjust adiabatically. Thus the radial displacement of the CMB does not require an equivalent displacement of density surfaces in the core (e.g. Dahlen 1974; Dahlen & Fels 1978, and references therein).

A similar set of conditions applies at the ICB. However, here we must also introduce the gravitational potential that results from the displacement of the tilted elliptical density discontinuity, ϕ'_s. In our formulation, it does not introduce an additional displacement y_1: we have specified the effect of the displacement Δr inside the inner core in terms of a forcing applied on an undeformed Earth and we follow the same philosophy at the boundaries. Instead, it enters the condition on y_6. Additionally, the gravitational potential ϕ_1 in the fluid core and mantle have been defined to contain the part due to the displacement of the density surfaces within the inner core, ϕ_s. Outside the inner core, this potential satisfies Laplace's equation and its effect is included by adding it to the boundary conditions at the ICB. The set of conditions at the ICB are then

$$y_1^s(a_s) = -\frac{y_5^c(a_s)}{g_o(a_s)} + \frac{\delta_{\beta\alpha}^m \phi_\epsilon(a_s)}{g(a_s)} + A_1,$$
$$y_2^s(a_s) = A_1 g_o(a_s) \rho_o^c(a_s),$$
$$y_3^s(a_s) = A_2,$$
$$y_4^s(a_s) = 0,$$
$$y_5^s(a_s) = y_5^c(a_s) - \delta_{\beta\alpha}^m \phi_\epsilon(a_s),$$
$$y_6^s(a_s) = y_6^c(a_s) + 4\pi G \rho_o^c(a_s) A_1 + \delta_{\beta\alpha}^m \frac{5\phi'_\epsilon(a_s)}{a_s}. \tag{A30}$$

The 21 boundary conditions are specified in terms of seven constants. Once these are known, it removes 7 degrees of freedom in the system and allows a unique solution for the whole problem.

A7 Solutions

All the variables in the above equations are non-dimensionalized. Solution are found by numerically integrating the system of equations in each region, from a small r to the surface, and applying the appropriate boundary conditions between each region. The integration is iterated by varying the constants until all boundary conditions are simultaneously satisfied.
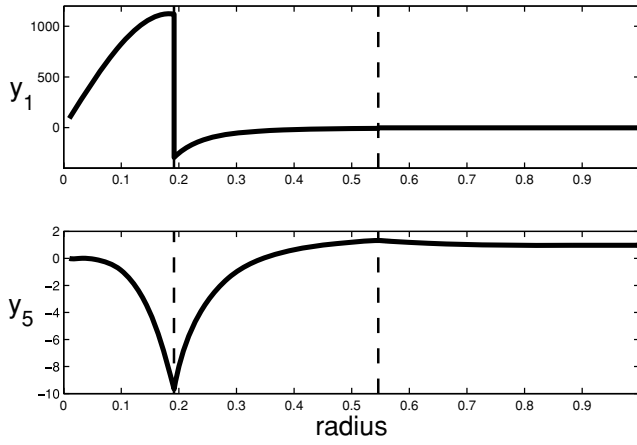
Figure A1. Solution of the radial elastic displacement y_1 and change in gravitational potential y_5 as a function of radius resulting from a tilt of the inner core. All values are dimensionless with respect to the scales defined in our numerical integration. The solution of y_1 does not include the rigid displacement associated with the inner core tilt. Likewise, the gravitational potential from the rigid displacement has been removed from the solution of y_5, so that the value plotted is only that due to elastic deformations.

We compute the solution for a forcing normalized such that the change in gravitational potential from the rigid rotation of the inner core equals 1 at the surface. In other words, such that $\delta_{\beta\alpha}^m \bar{\phi}(a_e) = 1$, where $\bar{\phi}(a_e) = (a_s/a_e)^3(\phi_\epsilon(a_s) + \phi'_\epsilon(a_s))$. The solution for the radial elastic displacements and the change in gravitational potential is presented in Fig. A1. The values shown do not include the rigid part of the displacement of the inner core nor the gravitational potential associated with it. As a reference, at the ICB, the rigid displacement and its associated gravitational potential are equal to -9847 and 140.7 in the non-dimensional units of Fig. A1. The elastic displacement near the ICB is, therefore, slightly larger than 10 per cent value of the rigid displacement. At the ICB, the additional gravitational potential is about 7 per cent the value of the potential from the rigid deformation. However, at the surface, they are almost equal. To get the dimensional value, one must multiply the solutions of y_5

by $\delta_{\beta\alpha}^m \bar{\phi}(a_e)$ and the solution of y_1 by $\delta_{\beta\alpha}^m \bar{\phi}(a_e)/g_o(a_e)$. For example, a tilt of the inner core by an angle of $0.1°$, leads to a degree 2 order 1 rigid deformation at the ICB of approximately 7 m and an elastic deformation of approximately 0.7 m. The rigid and elastic deformations of degree 2 order 0 at the ICB for the same inner core tilt are, respectively, 14 and 1.4 mm.

The quantities $\mathcal{H}^\epsilon$, $\mathcal{H}_f^\epsilon$ and $\mathcal{H}_s^\epsilon$ are computed from the solution displayed in Fig. A1. In Section 3, these were defined in terms of displacement of density surfaces $a\epsilon\tilde{h}_\epsilon$. It is only within the fluid core that our solutions for y_1 correspond to these displacements. In the inner core and mantle, the surfaces of constant density are displaced by y_1 as in the fluid core, but a local change in pressure also contributes to the change in density (see eq. A14). To calculate the parameters $\mathcal{H}_j^\epsilon$, we instead use the following transformation

$$\int_{a_1}^{a_2} \rho \frac{\partial}{\partial a'} \left(a'^5 \epsilon \tilde{h}_\epsilon\right) \mathrm{d}a' = \int_{a_1}^{a_2} \rho_1 a'^4 \mathrm{d}a' - \sum_i [\rho_o]_i a_i^4 y_{1i}, \quad (\text{A31})$$

where a_1 and a_2 delineate the region of integration, ρ_1 is obtained from (A14), and the second term on the right-hand side is the contribution from the displacement y_1 at all density discontinuity of amplitude $[\rho_o]$ within the region. The final values for $\mathcal{H}_j^\epsilon$ are obtained by converting ϵ to $\bar{\epsilon}$ and multiplying by the appropriate dimensional constants. Their numerical values are given in Table 1 of Section 4.

Our choice of normalization allows to write the change in the coefficients of the gravitational potential as

$$\delta\varphi_2^m(a) = \left[1 + \tilde{k}_\epsilon(a)\right] \delta_{\beta\alpha}^m \bar{\phi}(a), \quad (\text{A32})$$

where the coefficient $\tilde{k}_\epsilon(a)$ is the elastic compliance defined in Section 3. At the surface, its value is given directly by the solution of y_5 at $r = 1$. We obtain $\tilde{k}_\epsilon(a_e) = 0.9736$. This value can be used to test our solution of $\mathcal{H}^\epsilon$, as according to (38) we should have

$$\mathcal{H}^\epsilon = \left(\mathcal{H}_s - \mathcal{H}'_s\right) \tilde{k}_\epsilon(a_e). \quad (\text{A33})$$

We have verified that our results agree to within four parts in ten thousand.