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Influence of elastic deformations on the inner core wobble

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SUMMARY

The Earth's longest free mode of nutation, the inner core wobble (ICW), consists of a prograde precession of the tilted figure of the inner core with respect to a fixed mantle. The dynamics of the ICW is controlled by coupling at the inner-core boundary (ICB) and by the torque exerted by the rest of the Earth on a tilted inner core. This mode has not yet been observed either directly or indirectly, though theoretical estimates suggest its period should be 2410 solar days or 6.6 yr. However, these estimates were based on models that did not properly take into account elastic deformations associated with a tilted inner core. In this work, we incorporate these elastic deformations in a model of free nutations that rests on an angular momentum formalism. We find that based on an elastic, oceanless and dissipationless earth model corresponding to PREM, elastic deformations within the inner core contribute to a lengthening of the period of the ICW from 2410 to 2715 solar days. The internal forcing caused by the misalignment between surfaces of constant density and centrifugal potential is the most important contribution to elastic deformations.

Key words: Earth rotation variations; Core, outer core and inner core; Elasticity and anelasticity.

1 INTRODUCTION

The Chandler wobble (CW) is the most well known of the Earth's free modes of nutation. It involves an offset between the rotation axis and the axis of symmetry of the oblate geometric figure of the solid Earth (Fig. 1a). As a result of this misalignment, a torque is applied on the elliptical Earth, leading to a gyroscopic effect and to a prograde precession of the rotation axis around the figure axis (e.g. Lambeck 1980). In a mantle reference frame, the observed period of the CW is approximately 433 solar days (Gross 2007). Were the Earth a single rigid body with an identical mass distribution, its CW period would be closer to 300 d (e.g. Smith 1977; Smith & Dahlen 1981). The Earth departs from a rigid body because it has a fluid core, which to a good approximation does not participate in the CW. In addition, elastic deformations of the solid Earth take place during the CW, effectively reducing the misalignment between the rotation axis and mantle figure axis, thus leading to a reduced gyroscopic effect and to a lengthening of the period of the wobble.

The presence of the fluid core also allows for a second free mode, the free core nutation (FCN), also referred to as the nearly diurnal free wobble (NDFW; e.g. Lambeck 1980). This mode is characterized by a misalignment between the rotation axes of the mantle and core (Fig. 1b), resulting in a latitudinal pressure gradient on the elliptical core–mantle boundary (CMB) and to an internal torque between them. The gyroscopic effect from this torque leads to a precession between the rotation vectors of the core and mantle, the latter remaining nearly in alignment with the geometric figure axis of the mantle. As observed from a mantle-fixed perspective, the precession of the core rotation axis is retrograde at a nearly diurnal period. From a reference frame fixed in space, the observed period of this mode is close to 430 solar days, in a retrograde direction (Herring *et al.* 1986; Mathews *et al.* 2002). For this reason, this mode is also referred to as the retrograde free core nutation (RFCN).

The presence of the solid inner core gives rise to two additional free modes. In the first, the rotation axis and the geometric figure axis of the oblate inner core are aligned together and both are offset with respect to the figure axis of the rest of the Earth (Fig. 1c). As is the case for the RFCN, the fluid pressure acting on the elliptical inner core boundary (ICB) results in an internal torque and to a precession. Gravitational forces on the tilted inner core also contribute to the internal torque and the sum of these effects gives rise to a second nearly diurnal retrograde mode, when observed from a mantle-fixed frame. In a space-fixed frame, contrary to the RFCN, the motion of this new mode is prograde and is thus referred to as the prograde free core nutation (PFCN) or alternately as the free inner core nutation (FICN) (de Vries & Wahr 1991; Mathews *et al.* 1991a; Dehant *et al.* 1993). Inferences from the resonance of this mode with the forced nutations of the Earth suggest that its period should be close to ∼1000 d (Mathews *et al.* 2002; Koot *et al.* 2008).

The second free mode associated with the inner core is also characterized by an offset of the geometric figure axis of the inner core from that of the mantle, though for this mode, the rotation vector of the inner core remains nearly aligned with the mantle figure

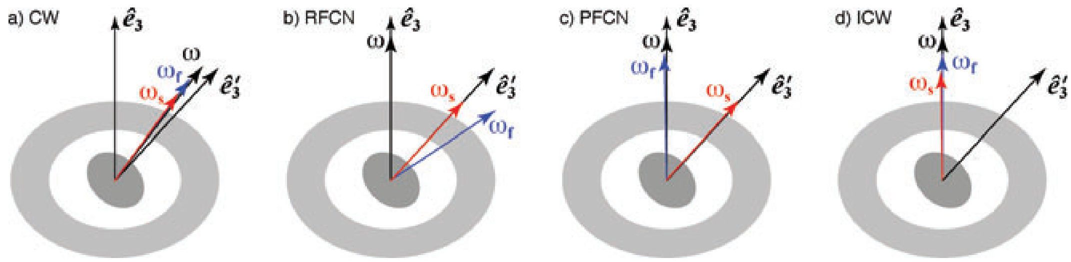

Figure 1. Schematic description of the free nutations: (a) CW; (b) RFCN; (c) PFCN and (d) ICW. $\boldsymbol{\omega}$, $\boldsymbol{\omega_f}$ and $\boldsymbol{\omega}_\mathrm{s}$ are the rotation vectors of the whole Earth, fluid core and inner core, respectively, and $\hat{\boldsymbol{e}}_3'$ represents the direction of the tilted figure axis of the inner core. All modes are shown with respect to a frame fixed to the mantle (with figure axis $\hat{\boldsymbol{e}}_3$). See text for further details. Not drawn to scale.

(Fig. 1d). As is the case for the CW, the offset between the rotation and figure axes of the inner core leads to a gyroscopic effect and to a prograde precession between the two. For this reason, this mode has been referred to as the Chandler wobble of the inner core (Smith 1977), or more simply as the inner core wobble (ICW) (Mathews *et al.* 1991a). From a mantle fixed frame, this mode consists essentially of a slow prograde precession of the tilted figure of the inner core.

The period of this last mode is the main topic of this paper. It depends crucially on the internal coupling dynamics taking place at the ICB and on the torque exerted by the rest of the Earth on a tilted inner core. For a rigid body with a dynamic ellipticity equal to that of the inner core, the gyroscopic effect alone should result in a ICW period of approximately 400 d. However, because the inner core is immersed in the fluid core, the period of the ICW is reduced by a factor involving the density contrast at the ICB (Busse 1970; Kakuta *et al.* 1975). An additional dynamic effect that contributes to the ICW is the gravitational torque from the rest of the Earth, acting on the tilted figure of the inner core (Mathews *et al.* 1991a). When all these internal couplings are taken into account, the period of the ICW, based on the reference earth model PREM (Dziewonski & Anderson 1981), is predicted to be approximately 2410 solar days (6.6 yr; Mathews *et al.* 1991a,b; Dehant *et al.* 1993; Xu & Szeto 1998).

However, in modelling the period of the ICW, the above studies did not properly take into account elastic deformations that accompany a tilt of the inner core. As is the case for the CW, elastic deformations within the inner core during the wobble should lead to a reduced effective gyroscopic effect and to a lengthening of the period of the ICW. Indeed, in a recent study of the long period wobbles by Rochester & Crossley (2009), elastic deformations were found to increase the period of the ICW from 6.6 to 7.5 yr.

The recent results of Rochester & Crossley (2009) provided the motivation for this paper. Their theoretical model is based on a Lagrangian description, where particle displacements from both rotation and elastic deformation are allowed to vary as a function of radius. In contrast, the model of Mathews *et al.* (1991a) is based on an angular momentum approach, where the free modes of nutation are computed from the coupled angular momentum equations of the whole Earth, fluid core and inner core. The rotation vector within each of these regions is by definition uniform, and elastic deformations are modelled in terms of perturbations in the moments of inertia. Here, we show that a lengthening of the ICW period similar to that found by Rochester & Crossley (2009) is retrieved when elastic deformations associated with a tilted inner core are incorporated into the model of Mathews *et al.* (1991a).

2 THEORY

2.1 Free nutations of the Earth

To determine the free nutations of the Earth, we follow the procedure outlined in Mathews *et al.* (1991a) and updated in Mathews *et al.* (2002). We give here only a brief description of their model and refer the interested reader to these papers. The model describes the free and forced nutations taking place in response to external torques from the Sun and the Moon (and to a smaller extent from other planets) acting on the equatorial bulge of the Earth. It assumes an axisymmetric, oceanless, dissipationless, rotating Earth, comprised of a mantle, fluid core and inner core. The reference equilibrium state is one of uniform rotation $\boldsymbol{\Omega_o} = \Omega_o \hat{\boldsymbol{e}}_3$ with respect to a reference frame fixed to the mantle. The nutations are the departures from this reference state and are found by solving a system of four equations, the first three describing respectively the evolution of the angular momentum of the whole Earth ($\boldsymbol{H}$), the fluid core ($\boldsymbol{H}_\mathrm{f}$) and the inner core ($\boldsymbol{H}_\mathrm{s}$), and a fourth for the kinematic relation governing the tilt of the inner core relative to the mantle,

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H} + \boldsymbol{\Omega} \times \boldsymbol{H} = \boldsymbol{0}\,, \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}_\mathbf{f} - \boldsymbol{\omega}_\mathbf{f} \times \boldsymbol{H}_{\boldsymbol{f}} = -\boldsymbol{\Gamma}^{\mathbf{CMB}} - \boldsymbol{\Gamma}^{\mathbf{ICB}}\,, \tag{2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}_\mathbf{s} + \boldsymbol{\Omega} \times \boldsymbol{H}_\mathbf{s} = \boldsymbol{\Gamma}_\mathbf{s} + \boldsymbol{\Gamma}^{\mathbf{ICB}}\,, \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{n}_\mathbf{s} = \boldsymbol{\omega}_\mathbf{s} \times \hat{\boldsymbol{e}}_3\,. \tag{4}$$

In these equations, $\boldsymbol{\Omega} = \boldsymbol{\Omega_o} + \boldsymbol{\omega} = \Omega_{\boldsymbol{o}} + \Omega_o \boldsymbol{m}$ is the instantaneous rotation vector of the Earth with respect to the mantle, $\boldsymbol{\omega}_\mathrm{f} = \Omega_o \boldsymbol{m}_\mathrm{f}$ and $\boldsymbol{\omega}_\mathrm{s} = \Omega_o \boldsymbol{m}_\mathrm{s}$ are vectors describing, respectively, the departure in the rotation of the fluid core and of the inner core with respect to $\boldsymbol{\Omega}$. The tilt of the inner core $\boldsymbol{n}_\mathrm{s}$ is defined as the difference between the unit vectors $\hat{\boldsymbol{e}}_3'$ and $\hat{\boldsymbol{e}}_3$ pointing in the directions of the geometrical figure axis of the inner core and the mantle, respectively. The four rotation variables $\boldsymbol{m}$, $\boldsymbol{m}_\mathrm{f}$, $\boldsymbol{m}_\mathrm{s}$ and $\boldsymbol{n}_\mathrm{s}$ are the unknowns to be solved for and are expressed in complex notation ($\tilde{m} = m_1 + \mathrm{i}m_2$, etc.) where the directions 1 and 2 refer to the two equatorial directions in the mantle reference frame.

The angular momentum vectors $\boldsymbol{H}$, $\boldsymbol{H}_\mathrm{f}$, $\boldsymbol{H}_\mathrm{s}$ are expanded in terms of the four rotation variables. They involve the equatorial moments of inertia and dynamic ellipticities of the whole Earth (A and e, respectively), the fluid core (A_f, e_f) and the inner core

(A_s, e_s). The vector $\boldsymbol{\Gamma}_s$ represents the torque exerted by the rest of the Earth on a tilted inner core. An expansion of this torque in terms of the four rotation variables is presented in Mathews *et al.* (1991a); in the next section we show how the inclusion of elastic deformations associated with a tilted inner core affects this expression. The torque Γ^{CMB} contains the influence from surface tractions acting on the mantle at the CMB and, equivalently, Γ^{ICB} contains surface tractions on the inner core at the ICB. These include the effects of Maxwell stresses (e.g. Buffett 1992; Buffett *et al.* 2002) and/or viscous friction (e.g. Mathews & Guo 2005; Deleplace & Cardin 2006). They can be expressed in terms of dimensionless coupling constants K^{CMB} and K^{ICB} (Buffett *et al.* 2002)

$$K^{\mathrm{CMB}} = \frac{\tilde{\Gamma}^{\mathrm{CMB}}}{\mathrm{i}\Omega_o^2 A_\mathrm{f}\tilde{m}_\mathrm{f}}\,, \qquad K^{\mathrm{ICB}} = \frac{\tilde{\Gamma}_{\mathrm{ICB}}}{\mathrm{i}\Omega_o^2 A_\mathrm{s}(\tilde{m}_\mathrm{s} - \tilde{m}_\mathrm{f})}\,. \tag{5}$$

In the forced nutation problem, an external torque is prescribed on the right-hand side of (1). Our focus is on the free modes of nutation, which involve no external torque; the angular momentum of the whole Earth is conserved. To solve for the free modes, a time-dependence of the form $\mathrm{e}^{\mathrm{i}\sigma\Omega_o t}$ is assumed, where σ is the frequency of nutation in cycles per sidereal day. The eqs (1)–(4) can then be written in a compact form as

$$\mathbf{M}\cdot \boldsymbol{x} = \mathbf{0}\,, \tag{6}$$

where $\boldsymbol{x} = [\tilde{m}, \tilde{m}_\mathrm{f}, \tilde{m}_\mathrm{s}, \tilde{n}_\mathrm{s}]^T$. The elements of matrix $\mathbf{M}$ involve σ, A, A_f, A_s, e, e_f and e_s, as well as elastic compliances and other coupling parameters. The free modes, with eigenvalues σ, are the eigensolutions of (6).

2.2 Influence of elastic deformations on $\boldsymbol{\Gamma}_s$

The dynamics that govern the ICW and the PFCN is crucially dependent on $\boldsymbol{\Gamma}_s$. We follow Mathews *et al.* (1991a), and separate $\boldsymbol{\Gamma}_s$ into the following contributions

$$\boldsymbol{\Gamma}_\mathrm{s} = \mathbf{N} + \mathbf{P} + \mathbf{Q}\,, \tag{7}$$

where

$$\mathbf{N} = \int_V \mathbf{r}\times\rho\nabla\phi_\mathrm{c}^\mathrm{f}\,\mathrm{d}V\,, \tag{8}$$

$$\mathbf{P} = \int_V \mathbf{r}\times(\phi_\mathrm{g}+\phi_\mathrm{c}^\mathrm{f})\nabla\rho\,\mathrm{d}V\,, \tag{9}$$

$$\mathbf{Q} = \int_S (\rho_\mathrm{f}-\rho_\mathrm{s})(\phi_\mathrm{g}+\phi_\mathrm{c}^\mathrm{f})\mathbf{r}\times\mathbf{dS}\,, \tag{10}$$

and where V is the volume occupied by the solid inner core with surface S, $\mathbf{r}$ is the coordinate vector from the origin, ϕ_g is the gravitational potential, $\phi_\mathrm{c}^\mathrm{f}$ is the centrifugal potential associated with the angular velocity of the fluid core, ρ is the density within the inner core and ρ_s and ρ_f are the densities on the solid and fluid side of the ICB, respectively. Expressions for these torques were developed in appendix B of Mathews *et al.* (1991a),

$$\tilde{N} = -\mathrm{i}\Omega_o^2 A_\mathrm{s}e_\mathrm{s}(\tilde{m}+\tilde{m}_\mathrm{f}-\tilde{n}_\mathrm{s}) + \mathrm{i}\Omega_o^2\tilde{c}_3^\mathrm{s}\,, \tag{11}$$

$$\tilde{P} = \mathrm{i}\Omega_o^2 A_\mathrm{s}e_\mathrm{s}\left(\frac{\rho_\mathrm{s}}{\rho_\mathrm{f}}\alpha_1 - 1\right)\left[-\tilde{m}-\tilde{m}_\mathrm{f}+(1+\alpha_\mathrm{g})\tilde{n}_\mathrm{s}\right], \tag{12}$$

$$\tilde{Q} = -\alpha_1\left(\frac{\rho_\mathrm{s}}{\rho_\mathrm{f}} - 1\right)\left[\tilde{N} + \mathrm{i}\Omega_o^2 A_\mathrm{s}e_\mathrm{s}\alpha_\mathrm{g}\tilde{n}_\mathrm{s}\right]. \tag{13}$$

Here, we have written the torques in terms of the conventional complex notation and have not included the contribution from external lunisolar forces. In these expressions, α_1 is defined as

$$\alpha_1 = \frac{A'e'}{A_\mathrm{s}e_\mathrm{s}}\,, \tag{14}$$

where A' is the mean equatorial moment of inertia of a body of inner core shape but with uniform density ρ_f, and e' is the dynamic ellipticity of this body. The coefficient α_g captures the effect of gravitational coupling from the rest of the Earth on a tilted inner core and is defined as

$$\alpha_g = \frac{8\pi G}{5\Omega_o^2}\left[\int_{a_\mathrm{s}}^{a_\mathrm{e}} \rho(a')\frac{\mathrm{d}\epsilon(a')}{\mathrm{d}a'}\mathrm{d}a' + \rho_\mathrm{f}\epsilon_\mathrm{s}\right], \tag{15}$$

where a_s and a_e are the mean radii of the inner core and the whole Earth, respectively, G is the gravitational constant and ϵ is the geometrical ellipticity of the surfaces on which density is constant (ϵ_s is the geometrical ellipticity at the ICB). The variable $\tilde{c}_3^\mathrm{s} = c_{31}^\mathrm{s} + \mathrm{i}c_{32}^\mathrm{s}$ represents the off-diagonal contribution to the moment of inertia of the inner core that are caused by elastic deformations. The sum of $\tilde{N}$, $\tilde{P}$ and $\tilde{Q}$ yields a total torque on the inner core of (eq. 17 of Mathews *et al.* 1991a)

$$\tilde{\Gamma}_s = \mathrm{i}\Omega_o^2 A_\mathrm{s}e_\mathrm{s}\left[-\alpha_1(\tilde{m}+\tilde{m}_\mathrm{f}) + \alpha_2\tilde{n}_\mathrm{s}\right] + \mathrm{i}\Omega_o^2\tilde{c}_3^\mathrm{s}\,, \tag{16}$$

where a factor $[1-\alpha_1(\rho_\mathrm{s}/\rho_\mathrm{f}-1)]$ multiplying $\tilde{c}_3^\mathrm{s}$ has been approximated to 1.

In the work of Mathews *et al.* (1991a), $\tilde{c}_3^\mathrm{s}$ included the effects from changes in the centrifugal force associated with changes in $\tilde{m}$, $\tilde{m}_\mathrm{f}$ and $\tilde{m}_\mathrm{s}$. The contribution to $\tilde{c}_3^\mathrm{s}$ from elastic deformations consequent to a tilted inner core were not included. In addition, as an inspection of (11)–(13) reveals that elastic deformations were only considered in modelling the $\tilde{N}$-part of the torque, the part that involves the centrifugal potential $\phi_\mathrm{c}^\mathrm{f}$; they were not included in modelling the gravitational torque on a tilted inner core by the rest of the Earth, nor in the part of $\tilde{P}$ involving the centrifugal potential.

The second of these two omissions is easily addressed. The torque $\tilde{P}$ involves the rigid part of the torque $\tilde{N}$ multiplied by a factor. Therefore, elastic deformations from the centrifugal potential part of $\tilde{P}$ can be taken into account simply by writing

$$\tilde{P} = \left(\frac{\rho_\mathrm{s}}{\rho_\mathrm{f}}\alpha_1 - 1\right)\left[\tilde{N} + \mathrm{i}\Omega_o^2 A_\mathrm{s}e_\mathrm{s}\alpha_\mathrm{g}\tilde{n}_\mathrm{s}\right], \tag{17}$$

with the understanding that $\tilde{N}$ is given by (11). The total torque associated with the centrifugal potential, which we write $\tilde{\Gamma}_\mathrm{s}^\mathrm{c}$, is obtained by adding $\tilde{N}$ to the centrifugal contributions of $\tilde{P}$ and $\tilde{Q}$ and is

$$\tilde{\Gamma}_\mathrm{s}^\mathrm{c} = \alpha_1\tilde{N} = -\mathrm{i}\Omega_o^2 A_\mathrm{s}e_\mathrm{s}\alpha_1(\tilde{m}+\tilde{m}_f-\tilde{n}_\mathrm{s}) + \mathrm{i}\Omega_o^2\alpha_1\tilde{c}_3^\mathrm{s}\,. \tag{18}$$

We now need to add elastic deformations in the part of the torque associated with the gravitational potential, $\tilde{\Gamma}_\mathrm{s}^\mathrm{g}$. The rigid component of this torque is given by the sum of the gravitational parts of $\tilde{P}$ and $\tilde{Q}$ and is

$$\tilde{\Gamma}_\mathrm{s}^\mathrm{g} = -\mathrm{i}\Omega_o^2 A_\mathrm{s}e_\mathrm{s}\alpha_3\alpha_\mathrm{g}\tilde{n}_\mathrm{s}\,, \tag{19}$$

where $\alpha_3 = 1-\alpha_1$. The contribution to this torque from elastic deformations can be separated in two parts. The first is from elastic deformations taking place within the inner core. As above, these are modelled in terms of their contribution to the off-diagonal elements of the moment of inertia tensor of the inner core. To include this part, one must replace $A_\mathrm{s}e_\mathrm{s}\tilde{n}_\mathrm{s}$ in (19) by $A_\mathrm{s}e_\mathrm{s}\tilde{n}_\mathrm{s}+\tilde{c}_3^\mathrm{s}$. The second part results from elastic deformations taking place outside the inner core, thereby altering the geometry of the elliptical surfaces of constant density and modifying the parameter α_g. This parameter is akin to a moment of inertia and the effect of elastic deformations can be modelled in terms of an off-diagonal contribution. We write this off-diagonal contribution as $\tilde{\alpha}_g = \alpha_g \tilde{n}_\epsilon$; so, $\tilde{n}_\epsilon$ is equivalent to a rigid rotation the whole of the elliptical fluid core and mantle with respect to a stationary inner core. Since the gravitational torque on the inner core produced by $\tilde{n}_\epsilon$ should be equal and opposite to the gravitational torque exerted by the rest of the Earth on an inner core tilted by the same amount, it is given by (19), but replacing $\tilde{n}_s$ by $-\tilde{n}_\epsilon$. The total torque associated with the gravitational potential, including elastic deformations, can be written as

$$\tilde{\Gamma}_s^g = -i\Omega_o^2 A_s e_s \alpha_3 \alpha_g \tilde{n}_s - i\Omega_o^2 \alpha_3 \alpha_g \tilde{c}_3^s + i\Omega_o^2 A_s e_s \alpha_3 \alpha_g \tilde{n}_\epsilon \,. \tag{20}$$

The total torque on the inner core is then the sum of the centrifugal and gravitational parts and is

$$\tilde{\Gamma}_s = i\Omega_o^2 A_s e_s \left[-\alpha_1(\tilde{m} + \tilde{m}_f) + \alpha_2 \tilde{n}_s\right] + i\Omega_o^2 \alpha_2 \tilde{c}_3^s + i\Omega_o^2 A_s e_s \alpha_3 \alpha_g \tilde{n}_\epsilon \,, \tag{21}$$

where

$$\alpha_2 = \alpha_1 - \alpha_g + \alpha_1 \alpha_g \,. \tag{22}$$

A comparison of the expression for $\tilde{\Gamma}_s$ in (21) versus that in (16) reveals that, as a result of including elastic deformations in $\tilde{\Gamma}_s$, the term involving $\tilde{c}_3^s$ is now multiplied by a factor α_2. Additional differences are from the inclusion of elastic deformations consequent to a tilted inner core. This includes $\tilde{n}_\epsilon$ and also additional contributions to $\tilde{c}_3^s$, which are presented in the next section.

2.3 Elastic compliances

The remaining task is to express $\tilde{c}_3^s$ and $\tilde{n}_\epsilon$ in terms of the variables $\tilde{m}$, $\tilde{m}_f$, $\tilde{m}_s$ and $\tilde{n}_s$ to close the system (6). Eqs (1) and (2) involve $\tilde{c}_3$ and $\tilde{c}_3^f$, which represent the contributions from elastic deformations to the moment of inertia of the whole Earth and the fluid core, respectively. These too must also be expressed in terms of $\tilde{m}$, $\tilde{m}_f$, $\tilde{m}_s$ and $\tilde{n}_s$. Following Mathews *et al.* (1991a), this is done by writing

$$\tilde{c}_3 = A\left[\kappa\tilde{m} + \xi\tilde{m}_f + \zeta\tilde{m}_s + S_{14}^g \tilde{n}_s + S_{14}^p(\tilde{n}_s - \tilde{m} - \tilde{m}_s)\right] \,, \tag{23}$$

$$\tilde{c}_3^f = A_f\left[\gamma\tilde{m} + \beta\tilde{m}_f + \delta\tilde{m}_s + S_{24}^g \tilde{n}_s + S_{24}^p(\tilde{n}_s - \tilde{m} - \tilde{m}_s)\right] \,, \tag{24}$$

$$\tilde{c}_3^s = A_s\left[\theta\tilde{m} + \chi\tilde{m}_f + \nu\tilde{m}_s + S_{34}^g \tilde{n}_s + S_{34}^p(\tilde{n}_s - \tilde{m} - \tilde{m}_s)\right] \,, \tag{25}$$

$$\tilde{n}_\epsilon = S_{\epsilon 1}\,\tilde{m} + S_{\epsilon 2}\,\tilde{m}_f + S_{\epsilon 3}\,\tilde{m}_s + S_{\epsilon 4}^g\,\tilde{n}_s + S_{\epsilon 4}^p\,(\tilde{n}_s - \tilde{m} - \tilde{m}_s)\,. \tag{26}$$

The three sets of centrifugal compliances (κ, ξ, ζ), (γ, β, δ), (θ, χ, ν) characterize elastic deformations of the whole Earth, the fluid core and the inner core, respectively, which arise through independent rotation of these three regions. They were calculated in Mathews *et al.* (1991b) and Buffett *et al.* (1993), and their numerical values are presented in Table 1 when using PREM. The calculations reported in these studies pertain to diurnal timescale deformations, as appropriate for nutation studies. We have recalculated the compliances for when the timescale of deformation is long enough that the inertial contribution is negligible and deformations can be considered static, as is more appropriate for the long period wobbles (CW and ICW). These static compliances are also presented in Table 1. Their numerical values differ very little from the diurnal case because inertial contributions to the mechanical force balance are already small at diurnal frequencies (e.g. Saito 1974) and both calculations are based on the same earth model and, thus, the same elastic moduli. In PREM, the elastic moduli are representative of deformations occurring with a period of 1 s. Ideally, the diurnal and static compliances should each be calculated on the basis of earth models where the elastic moduli have values that are more appropriate for these deformation timescale. However, such earth models are not available. We will return to this point in the discussion.

Table 1. Elastic compliances calculated using PREM.

Compliance	Diurnal value	Static value
κ	1.039×10^{-3}	1.038×10^{-3}
ξ	2.222×10^{-4}	2.219×10^{-4}
ζ	4.964×10^{-9}	5.134×10^{-9}
γ	1.965×10^{-3}	1.962×10^{-3}
β	6.160×10^{-4}	6.151×10^{-4}
δ	-4.869×10^{-7}	-4.865×10^{-7}
θ	6.794×10^{-6}	7.024×10^{-6}
χ	-7.536×10^{-5}	-7.529×10^{-5}
ν	7.984×10^{-5}	7.984×10^{-5}
S_{14}^g	9.732×10^{-8}	1.092×10^{-7}
S_{24}^g	3.406×10^{-7}	3.412×10^{-7}
S_{34}^g	-1.722×10^{-6}	-1.813×10^{-6}
S_{14}^p	-4.140×10^{-8}	-1.683×10^{-8}
S_{24}^p	1.641×10^{-6}	1.639×10^{-6}
S_{34}^p	-2.687×10^{-4}	-2.686×10^{-4}
$S_{\epsilon 1}$	5.254×10^{-1}	5.247×10^{-1}
$S_{\epsilon 2}$	1.779×10^{-1}	1.777×10^{-1}
$S_{\epsilon 3}$	1.733×10^{-5}	1.339×10^{-5}
$S_{\epsilon 4}^g$	5.281×10^{-3}	5.328×10^{-4}
$S_{\epsilon 4}^p$	-5.722×10^{-5}	-4.396×10^{-5}

Note: The diurnal values for (κ, ξ, ζ), (γ, β, δ) and (θ, χ, ν) are those given in table 1 of Mathews *et al.* (1991b). The sum of static values of S_{i4}^g and S_{i4}^p are equal to the values presented in Dumberry (2008).

Elastic deformations arising as a result of a tilted inner core were first considered by Buffett *et al.* (1993), where three additional compliances were introduced, (S_{14}, S_{24}, S_{34}), to characterize the deformations of the whole Earth, the fluid core and the inner core, respectively. Their study focused on whether these additional compliances, as well as further corrections to the centrifugal compliances, could explain some of the discrepancies between the observed and predicted forced nutations. Elastic deformations caused by a tilted inner core were also considered in a different context in the study of Dumberry (2008), where the aim was to obtain a prediction of the gravity variations associated with decadal timescale changes in the tilt of the inner core. In both of these studies, however, the consequence of these additional compliances on the free nutations was not considered.

To investigate their effect on the free nutations, we found necessary to split each compliance S_{i4} into two separate contributions, S_{i4}^g and S_{i4}^p, to reflect the different contributions to the forcing on the inner core, when the latter is in a tilted state. To model this forcing, we follow the study of Dumberry (2008), in which the displacement field from a tilted oblate inner core is modelled as an equivalent radial deformation Δr_s of a spherically symmetric reference earth model. This deformation is responsible for local density variations ρ_s and the latter leads to global variations in the gravitational potential ϕ_s. The forcing on the inner core is given by

$$\mathbf{f}_t = -\nabla(\rho_o g_o \Delta r_s) - \rho_o \nabla\phi_s - \rho_s \nabla\phi_o \,, \tag{27}$$

in which ρ_o, ϕ_o and $g_o = \partial\phi_o/\partial r$ are, respectively, the density, gravitational potential and gravitational acceleration of the spherically symmetric reference earth model. The second and third terms on the right-hand side of (27) represent the 'gravitational' forcing, arising from the misalignment of elliptical surfaces of constant density and gravitational potential. There is also an additional contribution to the gravitational forcing caused by the displacement of the density jump at the ICB; this latter contribution enters the formulation through the boundary conditions (see Dumberry 2008 for details). Elastic deformations caused by this combined gravitational forcing account for the compliances ($S^{\rm g}_{14}$, $S^{\rm g}_{24}$, $S^{\rm g}_{34}$) in (23)–(25).

The second contribution, characterized by the compliances ($S^{\rm p}_{14}$, $S^{\rm p}_{24}$, $S^{\rm p}_{34}$) in (23)–(25), is that resulting solely from the first term on the right-hand side of (27). This term represents the forcing caused by a misalignment between the oblate inner core and the elliptical surfaces of constant centrifugal potential. This results in an equivalent 'pressure' force acting everywhere inside the inner core to reestablish the alignment. The torque on the inner core produced by this force is analogous to the torque on the mantle that sustains the CW. We note that this forcing should not be confused with the centrifugal force associated with the centrifugal compliances. The latter involves perturbations in the rotation vectors alone and would occur even if the reference undeformed Earth were spherical. The forcing described here is entirely a consequence of the rotated elliptical shape of the inner core described through $\Delta r_{\rm s}$ and would vanish for a spherical Earth. This forcing would also vanish if the tilt ($\tilde{n}_{\rm s}$) and rotation vector of the inner core ($\tilde{m} + \tilde{m}_{\rm s}$) are in alignment; the compliances ($S^{\rm p}_{14}$, $S^{\rm p}_{24}$, $S^{\rm p}_{34}$) in (23)–(25) are then factors of this difference. Numerical values for the compliances $S^{\rm g}_{i4}$ and $S^{\rm p}_{i4}$ are presented in Table 1 for static and diurnal timescale deformations, following the method described in the appendix of Dumberry (2008). Differences between the static and diurnal compliances are small.

In the study of Dumberry (2008), where forced decadal variations of inner core tilt were considered, it was assumed that the rotation vector of the inner core remained aligned with the geometric axis of the mantle ($\tilde{m} + \tilde{m}_{\rm s} = 0$). In this case, $S^{\rm g}_{i4}\tilde{n}_{\rm s} + S^{\rm p}_{i4}\tilde{n}_{\rm s} = S_{i4}\tilde{n}_{\rm s}$; the numerical values for S_{i4} presented by Dumberry (2008) are equivalent to $S^{\rm g}_{i4} + S^{\rm p}_{i4}$. Although the assumption $\tilde{m} + \tilde{m}_{\rm s} = 0$ may be appropriate for that scenario and although this may also appropriately describe the ICW, it is clearly not appropriate when considering the PFCN for which $\tilde{m} + \tilde{m}_{\rm s} \approx \tilde{n}_{\rm s}$. This is the reason why splitting S_{i4} into $S^{\rm g}_{i4}$ and $S^{\rm p}_{i4}$ is necessary when considering the free modes of nutations.

Finally, the five new compliances $S_{\epsilon 1}$, $S_{\epsilon 2}$, $S_{\epsilon 3}$, $S^{\rm g}_{\epsilon 4}$ and $S^{\rm p}_{\epsilon 4}$ introduced in (26) capture the relative changes in gravitational coupling parameter $\alpha_{\rm g}$ as a result of elastic deformations in the fluid core and mantle caused by changes in the rotation variables. Their numerical values for both static and diurnal timescale deformations are presented in Table 1.

2.4 Changes in matrix $\mathbf{M}$

Substituting (25) and (26) in (21) and in the left-hand side of (3), the terms on the third row of matrix $\mathbf{M}$ in (6) become

$$\begin{aligned}\mathbf{M}(3,1) \to\ & \sigma(1+\theta - S^{\rm p}_{34}) + (\theta - S^{\rm p}_{34})(1-\alpha_2) - e_{\rm s}\alpha_3 \\ & - e_{\rm s}\alpha_3\alpha_{\rm g}(S_{\epsilon 1} - S^{\rm p}_{\epsilon 4}),\end{aligned} \tag{28}$$

$$\mathbf{M}(3,2) \to \sigma\chi + \chi(1-\alpha_2) + e_{\rm s}\alpha_1 - e_{\rm s}\alpha_3\alpha_{\rm g}S_{\epsilon 2} - K^{\rm ICB}, \tag{29}$$

$$\begin{aligned}\mathbf{M}(3,3) \to\ & 1 + \sigma(1+\nu - S^{\rm p}_{34}) + (\nu - S^{\rm p}_{34})(1-\alpha_2) \\ & - e_{\rm s}\alpha_3\alpha_{\rm g}(S_{\epsilon 3} - S^{\rm p}_{\epsilon 4}) + K^{\rm ICB},\end{aligned} \tag{30}$$

$$\mathbf{M}(3,4) \to (1+\sigma-\alpha_2)(e_{\rm s} + S^{\rm g}_{34} + S^{\rm p}_{34}) - e_{\rm s}\alpha_3\alpha_{\rm g}(S^{\rm g}_{\epsilon 4} + S^{\rm p}_{\epsilon 4}). \tag{31}$$

The introduction of the compliances $S^{\rm g}_{14}$, $S^{\rm g}_{24}$, $S^{\rm p}_{14}$ and $S^{\rm p}_{24}$ also leads to changes in the following matrix elements

$$\mathbf{M}(1,1) \to \sigma + (1+\sigma)(\kappa - S^{\rm p}_{14}) - {\rm e}, \tag{32}$$

$$\mathbf{M}(1,3) \to (1+\sigma)\left(\frac{A_{\rm s}}{A} + \zeta - S^{\rm p}_{14}\right), \tag{33}$$

$$\mathbf{M}(1,4) \to (1+\sigma)\left(\frac{A_{\rm s}e_{\rm s}\alpha_3}{A} + S^{\rm g}_{14} + S^{\rm p}_{14}\right), \tag{34}$$

$$\mathbf{M}(2,1) \to \sigma(1+\gamma - S^{\rm p}_{24}), \tag{35}$$

$$\mathbf{M}(2,3) \to \sigma(\delta - S^{\rm p}_{24}) - \frac{A_{\rm s}}{A_f}K^{\rm ICB}, \tag{36}$$

$$\mathbf{M}(2,4) \to -\sigma\left(\frac{A_{\rm s}e_{\rm s}\alpha_1}{A_{\rm f}} + S^{\rm g}_{24} + S^{\rm p}_{24}\right). \tag{37}$$

3 RESULTS

The free modes of nutations are the eigensolutions of the system (6). Their frequencies, measured in cycles per sidereal day (cpsd) in the rotating mantle reference frame, are given in Table 2 for four different models of elastic deformations. We also give the associated period in solar days, following the usual convention of reporting the periods of the CW and ICW with respect to a mantle-fixed frame and the periods of the RFCN and PFCN with respect to a space-fixed frame. In the first model (RIGID), all compliances are set to zero; this corresponds to a rigid Earth. The second model (ELASTIC1) is equivalent to the one presented in Mathews *et al.* (1991b), where the diurnal values of the centrifugal compliances are used, all compliances related to the inner core tilt are set to zero (i.e. $S^{g}_{i4} = S^{p}_{i4} = S_{\epsilon i} = 0$); and the elements of matrix $\mathbf{M}$ are as given in eq. (26b) of Mathews *et al.* (1991a). In the third model (ELASTIC2), the values for all compliances are equivalent to those of the ELASTIC1 model, but the matrix $\mathbf{M}$ now includes the corrections in (28)–(37) that arise from considering the elastic deformations in the torque $\boldsymbol{\Gamma}_s$. Finally, in the last model (FULL), we have used the numerical values for all compliances as given in Table 1, as well as the correction to matrix $\mathbf{M}$ in (28–37). For this last model,

Table 2. Frequencies and periods of free modes based on PREM.

Model	CW	RFCN	PFCN	ICW
		Frequency (cpsd)		
RIGID	0.003663	−1.002882	−0.997984	0.0004138
ELASTIC1	0.002489	−1.002189	−0.997903	0.0004138
ELASTIC2	0.002489	−1.002189	−0.997917	0.0004138
FULL	0.002490	−1.002191	−0.997916	0.0003674
APPROX	0.002493	−1.002178	−0.997924	0.0003669
		Period (solar days)		
RIGID	272.3	−346.1	494.7	2410
ELASTIC1	400.6	−455.7	475.5	2410
ELASTIC2	400.6	−455.7	478.7	2410
FULL	400.5	−455.2	478.6	2715
APPROX	400.1	−457.8	480.5	2718

Note: The frequencies are in cycles per sidereal days (cpsd). The periods are given in solar days, with respect to a mantle-fixed frame for the CW and ICW and with respect to a space-fixed frame for the RFCN and PFCN.

Table 3. Eigenvectors of the free modes.

	CW	RFCN	PFCN	ICW
ELASTIC1				
$\tilde{m}$	8.823×10^{-1}	-2.476×10^{-4}	6.680×10^{-7}	-1.123×10^{-4}
$\tilde{m}_{\mathrm{f}}$	-2.189×10^{-3}	1.000	-3.621×10^{-3}	5.252×10^{-8}
$\tilde{m}_{\mathrm{s}}$	-2.489×10^{-3}	6.113×10^{-1}	9.979×10^{-1}	-4.138×10^{-4}
$\tilde{n}_{\mathrm{s}}$	1.000	6.100×10^{-1}	1.000	1.000
FULL				
$\tilde{m}$	9.426×10^{-1}	-2.478×10^{-4}	4.353×10^{-7}	-4.266×10^{-5}
$\tilde{m}_{\mathrm{f}}$	-2.340×10^{-3}	1.000	-4.591×10^{-3}	2.181×10^{-8}
$\tilde{m}_{\mathrm{s}}$	-2.490×10^{-3}	5.983×10^{-1}	9.979×10^{-1}	-3.674×10^{-4}
$\tilde{n}_{\mathrm{s}}$	1.000	5.970×10^{-1}	1.000	1.000

Note: The amplitudes of $\tilde{m}$, $\tilde{m}_{\mathrm{f}}$, $\tilde{m}_{\mathrm{s}}$ and $\tilde{n}_{\mathrm{s}}$ are normalized such that the largest of them equals unity.

the static compliances have been used to calculate the frequencies and periods of the CW and ICW reported in Table 2, whereas the diurnal compliances have been used to calculate those of the RFCN and PFCN. The numerical values for all other parameters entering matrix $\mathbf{M}$ are taken as those in Table 1 of Mathews *et al.* (1991b) for PREM. The eigenvectors of the free modes of the ELASTIC1 and FULL models are presented in Table 3. To focus on the effects introduced by elastic deformations involved with a tilted inner core, we have set K^{ICB} and K^{CMB} to zero in all four models.

A comparison between the periods of the modes for the ELASTIC1 and ELASTIC2 models reveals that the corrections to matrix $\mathbf{M}$ introduced by the revised expression of $\tilde{\Gamma}_{\mathrm{s}}$ makes very little difference when elastic deformations associated with a tilted inner core are not included. When they are (FULL model), the periods of the CW, RFCN and PFCN remain similar, but the period of the ICW is significantly different. The basic nature of the ICW remains unchanged: to first order, it is still a simple precession of $\tilde{n}_{\mathrm{s}}$ (see the eigenvector in Table 3), though there are changes in the relative amplitude of the rotation variables. We note that the eigenvectors of the other modes are also slightly modified. The compliances $S_{\epsilon 1}$, $S_{\epsilon 2}$ and S^{p}_{24} are the most important contributions to the changes in the CW, RFCN and PFCN, respectively.

The role of elastic deformations in the free modes can be further understood with the help of simple algebraic expressions that closely approximate their frequencies. These were first developed by Mathews *et al.* (1991a); with the inclusion of elastic effects associated with the inner core tilt, the approximate frequencies are

$$\sigma_{\mathrm{cw}} = \frac{A}{A_m}\,(e - \kappa)\ , \tag{38}$$

$$\sigma_{\mathrm{rfcn}} = -1 - \left(1 + \frac{A_f}{A_m}\right)\left(e_f - \beta\right)\ , \tag{39}$$

$$\sigma_{\mathrm{pfcn}} = -1 + \left(1 + \frac{A_s}{A_m}\right)\left[\alpha_2\left(e_s + \nu + S^g_{34}\right)\right]\ , \tag{40}$$

$$\sigma_{\mathrm{icw}} = (1 - \alpha_2)(e_s + S^g_{34} + S^p_{34})\,. \tag{41}$$

The frequencies and periods of each mode based on these approximations is also given in Table 2 (APPROX). From these expressions, it is clear that it is the rotation-induced elastic deformations through the compliances κ, β and ν that are responsible for modifying the frequencies of the CW, PFCN and RFCN between the RIGID and ELASTIC models. The ICW is not influenced by rotation-induced elastic deformations; its period is unchanged between the RIGID and ELASTIC models, as was found by Rochester & Crossley (2009). The difference in the frequency of the ICW between the FULL and ELASTIC models is almost entirely a consequence of elastic deformations taking place within the inner core as a result of its tilt through the compliances S^{g}_{34} and S^{p}_{34}. Setting $S^{\mathrm{g}}_{14} = S^{\mathrm{p}}_{i4} = S^{\mathrm{g}}_{24} = S^{\mathrm{p}}_{24} = 0$ and also all of the parameters $S_{\epsilon i}$ to 0 results in differences smaller than 0.1 per cent in the frequency of the modes.

The parameters S^{g}_{34} and S^{p}_{34} are both negative; so, their combined effect in (41) is to reduce the frequency of the ICW. The approximate expression for the ICW is now similar to that of the CW: both involve a factor that contains a dynamic ellipticity reduced by a compliance. Since $S^{\mathrm{p}}_{34} \gg S^{\mathrm{g}}_{34}$, elastic deformations in the ICW are predominantly caused by the internal pressure gradient that develops as a result of the misalignment between the oblate inner core and its rotation vector. Based on (41), the period of the ICW is increased by a factor $(1 - \tilde{k}^{\mathrm{s}}_{\epsilon})^{-1} = 1.126$, where we have defined $\tilde{k}^{\mathrm{s}}_{\epsilon} = -(S^{\mathrm{g}}_{34} + S^{\mathrm{p}}_{34})/e_{\mathrm{s}} = 0.1117$. This corresponds to an increase from 2410 to 2715 solar days. This is similar to the results of Rochester & Crossley (2009): they find ICW periods of 2764, 2749 or 2732 days, depending on whether the fluid core is treated as an incompressible, compressible or neutrally stratified fluid, respectively. Based on their ICW periods for an incompressible and compressible fluid core when assuming a rigid inner core, respectively, 2428 and 2414 days, the inclusion of elastic deformations in the work of Rochester & Crossley (2009) results in an increase by a factor 1.139.

The slight difference with the factor 1.126 we have found, may be explained, in parts, by different parameter values. More likely, they are caused by a difference in modelling approach. In the angular momentum approach adopted in the present study, the rotation vector is assumed uniform within each region. Similarly, although variations with radius are taken into account in the calculation of the compliances, their values represent an integrated average of elastic deformations distributed equally within a given region. Adopting these approximations allows to investigate the free nutations with a simple set of equations, though at the price of introducing errors, which can be significant if departures from uniform rotation within a region are important. In contrast, in the Lagrangian formulation used by Rochester & Crossley (2009), rotation and elastic deformations are allowed to vary radially. The fact that we have recovered a similar ICW period signifies that the departures from uniform rotation within each region are small, though not completely negligible.

Elastic deformations as a result of a tilted inner core also result in a lengthening of the period of the PFCN (see eq. 40). However, contrary to the ICW, only those caused by gravitational forces through the compliance S^{g}_{34} play a role. This is because the rotation vector and geometric figure axis of the inner core remain essentially in alignment, as can be verified by the eigenvector of the PFCN in Table 3. With no misalignment between surfaces of constant density and centrifugal potential, there is no internal 'pressure' torque and no contribution from S_{34}^{p}. Since $S_{34}^{\mathrm{g}} \ll \nu$, elastic deformations induced by a tilted inner core only contribute to a very small change in the period of the PFCN, an increase from 475.5 solar days (ELASTIC1 model) to 478.6 solar days (FULL model).

The PFCN period that we have found is slightly longer than 473.9 sidereal days (472.6 solar days), the value found in the study of Rogister (2001) based on a Lagrangian approach. In this latter study, it is clearly shown that the total motion in the PFCN involves important departures from uniform rotation in the fluid core, an aspect that cannot be captured by our angular momentum approach. Since the amplitude of the motion within the fluid core in the PFCN is only a small fraction of the motion of the inner core (see Table 3), the assumption of uniform rotation within each region that we adopt in our study does not lead to large differences in the computed PFCN period. Undoubtedly though, the differences that remain between our value of the PFCN period and that computed by Rogister (2001), are caused by departures from uniform rotation.

Finally, we note that the changes in the matrix $\mathbf{M}$ also bring small changes in the results of the study of Dumberry (2008). In this study, predictions of the decadal changes in gravity resulting from time-dependent variations in inner core tilt were presented. The historical variations in inner core tilt were inferred directly from the observed decadal polar motion, according to two different scenarios. In the second of these, the inner core tilt results from a torque applied by the fluid core at the ICB. The relationship between the polar motion $\tilde{m}$ and the tilt angle of the inner core $\tilde{n}_{\mathrm{s}}$ is obtained by solving (6) with this applied torque on the right-hand side; this relationship is thus dependent on the elements of $\mathbf{M}$. Elastic deformations accompanying a tilted inner core were included in the definitions of $\tilde{c}_3$, $\tilde{c}_3^{\mathrm{f}}$ and $\tilde{c}_3^{\mathrm{s}}$, but the appropriate changes in $\tilde{\Gamma}_{\mathrm{s}}$, as done here in (21), were not. When the elements of the matrix $\mathbf{M}$ are changed according to (28)–(37), the approximate relationship between $\tilde{m}$ and $\tilde{n}_{\mathrm{s}}$ for slow variations (eq. 61 of Dumberry 2008) must be changed to

$$\tilde{n}_{\mathrm{s}} \approx -\frac{A(e-\kappa)}{A_{\mathrm{s}} e_{\mathrm{s}} \alpha_3 \left[\alpha_{\mathrm{g}} - \tilde{k}_\epsilon - \left(1+\alpha_{\mathrm{g}}\right)\tilde{k}_\epsilon^{\mathrm{s}}\right]}\,\tilde{m}\,, \tag{42}$$

where $\tilde{k}_\epsilon = 0.9736$. With this correction, the relationship between the polar motion and gravity (eq. 62 of Dumberry (2008)) becomes

$$\tilde{m} = \sqrt{\frac{5}{3}}\,\frac{M a_e^2\left[\alpha_{\mathrm{g}} - \tilde{k}_\epsilon - \left(1+\alpha_{\mathrm{g}}\right)\tilde{k}_\epsilon^{\mathrm{s}}\right]}{A(e-\kappa)(1+\tilde{k}_\epsilon)}(\Delta C_{21} + \mathrm{i}\Delta S_{21})\,, \tag{43}$$

where ΔC_{21} and ΔS_{21} are Stokes coefficients of gravitational potential and M is the mass of the Earth. The factor difference between this new expression and the equivalent expression for the first scenario (eq. 52 of Dumberry 2008) is then changed from -1.63 to -2.32, and the predicted gravity signal shown by the dashed-lines in Fig. 4 of Dumberry (2008) should have a slightly larger amplitude. Besides these relatively small changes, none of the conclusions in the study of Dumberry (2008) are altered.

4 DISCUSSION AND CONCLUSION

Elastic deformations in the whole Earth in response to a tilted inner core do not affect the periods of the CW, the RFCN and the PFCN. However, the period of the ICW is changed significantly, principally by elastic deformations taking place within the inner core. The misalignment of the geometric figure of the inner core with surfaces of constant gravitational potential and with surfaces of constant centrifugal potential are both sources of elastic deformations, though the latter of these two contributions is the most important. We find that based on an elastic, oceanless and dissipationless earth model corresponding to PREM, the predicted period for the ICW is now 2715 solar days or 7.4 yr, a lengthening by 12.6 per cent from its period of 2410 solar days computed by Mathews *et al.* (1991b), and in agreement with the recent prediction by Rochester & Crossley (2009).

Several factors not considered in our study can influence the ICW. First, the dynamic ellipticities e, e_{f} and e_{s} that we have used are based on the assumption that the Earth is in hydrostatic equilibrium. Non-equilibrium processes such as mantle convection contribute to small deviations in their values. Indeed, the observed period of the RFCN, inferred through its resonance with the forced nutations, is approximately -430 solar days (Herring *et al.* 1986; Mathews *et al.* 2002), shorter than the prediction of -456 days found in Table 2. This is usually attributed to a non-equilibrium modification of e_{f} (e.g. from 0.002548 to 0.0002665, Mathews *et al.* 1991b), corresponding to a change of approximately 500 m in the elliptical shape of the CMB at the equator (Gwinn *et al.* 1986; Mathews *et al.* 2002). Since the period of the ICW depends on e_{s}, we can expect that non-equilibrium processes similarly affect the period of the ICW. For a change in e_{s} in similar proportion as the change in e_{f} suggested by Mathews *et al.* (1991b), the ICW period would be shortenned to 2581 solar days.

The second factor is the effect of surface tractions at the ICB. The observed period of the PFCN, also inferred through its resonance with the forced nutations, is approximately a factor 2 longer than the theoretical estimate of $\sim$475 d in Table 2. Since the PFCN involves a differential rotation at the ICB, non-zero surface tractions can affect this mode significantly and possibly explain the difference between the observed and predicted period. The effect of surface tractions are modelled through the coupling constant K^{ICB}, modifying the frequency of the PFCN to (Mathews *et al.* 2002)

$$\sigma_{\mathrm{pfcn}} = -1 + \left(1 + \frac{A_{\mathrm{s}}}{A_m}\right)\left(\alpha_2\left(e_{\mathrm{s}} + \nu + S_{34}^{\mathrm{g}}\right) - K^{\mathrm{ICB}}\right)\,. \tag{44}$$

A value of Re $[\mathrm{K}^{\mathrm{ICB}}] = (1.11 \pm 0.10) \times 10^{-3}$ has been reported by Mathews *et al.* (2002) in order to match the period of the PFCN of $\sim$1025 solar days. A more recent estimate by Koot *et al.* (2008) gives a slightly lower value of $\mathrm{Re}[\mathrm{K}^{\mathrm{ICB}}] = (1.00 \pm 0.16) \times 10^{-3}$ (and thus a period of the PFCN closer to $\sim$915 days). Surface tractions at the CMB, through the coupling parameter K^{CMB} play a similar (though less dramatic) role in the period of the RFCN (Mathews *et al.* 2002).

However, surface tractions are not likely to affect the period of the ICW greatly. This is because, unlike the PFCN, the torque sustaining the wobble is not from a differential rotation at the ICB but is instead distributed over the volume of the inner core. If we include surface tractions in the matrix $\mathbf{M}$ as prescribed in Buffett (1992) and Mathews *et al.* (2002), the approximate frequency of the ICW is modified to

$$\sigma_{\mathrm{icw}} = \frac{(1-\alpha_2)(e_{\mathrm{s}} + S_{34}^{\mathrm{g}} + S_{34}^{\mathrm{p}})}{(1 + K^{\mathrm{ICB}})}\,. \tag{45}$$

For $\mathrm{Re}[\mathrm{K}^{\mathrm{ICB}}]$ of the order of 10^{-3}, this would involve a lengthening of the ICW period of at most a couple of days. The influence of surface tractions on a precessing tilted inner core alone (i.e. no differential rotation at the ICB) were not considered in Mathews *et al.* (2002); these may not affect the period of the ICW greatly but may be effective at attenuating the wobble motion.

A third factor that can alter the theoretical estimate of the period of the ICW, and potentially the most important, is the effect of anelastic deformations. The elastic moduli from which our elastic compliances are calculated (those of PREM) are appropriate for a timescale of deformation of 1 s. If anelastic deformations are important, their values should depend on the frequency of the imposed forcing. For instance, it is well known that the discrepancy between the predicted period of the CW based on an elastic, oceanless Earth ($\sim$400 solar days) and its observed period of $\sim$433 solar days is caused by ocean tides (increasing the CW period from $\sim$400 to $\sim$425 d) and mantle anelasticity (responsible for the remaining discrepancy of $\sim$8 to 10 days) (e.g. Smith & Dahlen 1981). Anelastic deformations within the inner core should similarly lengthen the period of the ICW. A proper prediction is difficult without *a priori* knowledge of the elastic properties of the inner core at a timescale of several years. As an order of magnitude estimate, for an increase of the compliances $S_{34}^{\rm g}$ and $S_{34}^{\rm p}$ by 5 per cent, identical to the relative change of in the compliance κ required to explain a 10 day change in the CW, the ICW period would be lengthened to 2732 solar days, a relatively small increase. However, anelastic deformations inside the inner core over a timescale of several years are likely more important than those in the mantle over a 14 months timescale. Indeed, geodynamic models of the seismically observed inner core superrotation suggest that viscous deformations within the inner core may take place with a typical timescale of the order of a few years or less (Buffett 1997; Dumberry 2007). Therefore, we expect that anelastic deformations should lengthen the period of the ICW to a value larger than the above crude estimate of 2732 solar days.

An observation of the ICW would provide an important constraint on the combined parameters that govern its frequency, including the ellipticity of the ICB and elastic properties of the inner core. An attempt by Guo *et al.* (2005) to extract the ICW from polar motion data did not prove successful, though their search was focused on a signal with a period of 6.6 yr. As we saw here, confirming the results Rochester & Crossley (2009), a more appropriate period for the ICW may be closer to 7.5 yr. If the dynamic ellipticity of the inner core departs from equilibrium in a similar proportion to that of the fluid core we expect a slight decrease in its period. Conversely, if anelastic effects are important, we expect the period of the wobble to be slightly longer, though the associated dissipation in the amplitude of the wobble would also decrease the chance of its detection.

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