Day 5: Inscribing and Circumscribing

Getting Closer to π: Inscribing and Circumscribing Polygons - Archimedes Method

Goals:

- Construct an inscribed hexagon and dodecagon.
- Construct a circumscribed hexagon and dodecagon.
- Calculate the perimeters of the inscribed hexagon and dodecagon accurately and compare it to the circumference.
- Calculate the perimeters of the circumscribed hexagon and dodecagon accurately and compare it to the circumference.
- Explain Archimedes Method and its validity
- Learn about the history of Archimedes Method – Palimpsests

Getting Closer to π: Inscribing and Circumscribing Polygons - Archimedes Method

Archimedes Historical Note:

Archimedes used the idea of inscribing and circumscribing polygons to come up with an estimate for π.

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An **inscribed polygon** is a polygon whose vertices are on the edge of a circle.

A **circumscribed polygon** is a polygon whose vertices are outside the edge of a circle and have every edge of the polygon touch the circle exactly once.

Archimedes used two ideas to draw regular inscribed and circumscribed regular polygons on a circle:

1) Archimedes used a compass and a straight edge to create a hexagon.
2) Archimedes used a compass and a straight-edge to make perpendicular bisectors. He used perpendicular bisectors to draw tangent lines as well.

Using only these two ideas, Archimedes was able to draw inscribed and circumscribed polygons with 6 sides (hexagon), 12 sides (dodecagon), 24 sides, 48 sides and 96 sides.

Inscribing and Circumscribing.

How to inscribe a hexagon

1) Mark a starting point anywhere along the outside of your circle.
2) Set your compass so that it is equal to the length from the center to your starting point (i.e. the length is the radius).
3) With your compass point (the point without the pencil) on the starting point, rotate your compass to mark off two more points on the outside of your circle. You should now have three marks on the edge of your circle (your starting point and two others).
4) Using your straight edge, line up each point on the edge of the circle with the centre of the circle. Draw a diameter to make another point on the other side of the circle. Once you have finished with all of your points you should have drawn 3 diameters and have six vertices on the outside edge of the circle. Join the points on the edge of the circle to create a perfect regular hexagon!

How to create a perpendicular bisector

A **perpendicular bisector** of a line segment is the line that bisects (cuts perfectly in half) a line segment at an angle of 90° (perpendicular).

1) Set your compass to be a little longer than half of the line segment you wish to bisect with a perpendicular line.
2) With your compass point on one of the endpoints of the line segment, draw a circle.

3) Without changing your compass settings, draw another circle by placing your compass point on the other endpoint.
4) The two circles that were drawn will intersect at two points. Draw a line connecting these two intersection points. This line will be a perpendicular bisector.

How to draw a tangent line

A **tangent line** is a line that touches the edge of a circle only once.

1) Draw a diameter of a circle, but instead of stopping at the end of the circle continue your line beyond the circle.
2) Set your compass to any measure less than the radius of your circle. Place the compass point on the point where your line hits the edge of the circle and draw a circle.
3) Your circle will intersect your line twice.
4) Use these points as the endpoints of a line segment and create a perpendicular bisector. Your perpendicular bisector will be a tangent to the circle.

Creating inscribed hexagons and dodecagons

1) Follow the instructions above to inscribe a hexagon on both the inscribed hexagon and dodecagon circle.
2) On the dodecagon circle, create a perpendicular bisector for each of the edges of the hexagon you created in 1).
3) The perpendicular bisectors will intersect the edge of the circle. Altogether, you will have 12 points on the edge of the circle now (6 from the hexagon and 6 from the perpendicular bisectors).
4) Join all 12 of these points to create a perfect dodecagon!
5) Highlight the edges of your inscribed hexagon and dodecagon.

Creating a circumscribed hexagon

1) Follow the instructions above to inscribe a hexagon on both the circumscribed hexagon and dodecagon circle, but don't join the lines to actually make the inscribed hexagon.
2) Extend the lines of the 3 diameters you made beyond the edge of the circle. Follow the instructions above to make tangent lines at each of the six points on the edge of the circle.
3) The six tangent lines should intersect each other creating a circumscribed hexagon!

Creating a circumscribed dodecagon

1) Using the circumscribed hexagon you already created, draw three lines connecting the vertices and going through the center of the circle.
2) Each of these lines will intersect the edge of the circle. Mark each of these intersection points on the edge of the circle. There should be six points. Create tangent lines at each of these points. You should now have 12 tangent lines drawn (6 from the hexagon and 6 more).
3) Highlight the tangent lines to create a dodecagon (a 12-sided polygon)!

Inscribed and Circumscribed Polygon Data Sheet

Archimedes used inscribed and circumscribed polygons to estimate how many diameters were needed to go around the edge of a circle. He was able to use mathematics that he knew about triangles to find out the length of one side of the polygon. Then he could just multiply by the number of sides to find the perimeter of the polygon. By looking at how many diameters were in the perimeter, Archimedes could get an estimate for how many diameters were needed in a circle.

We will use a ruler to accurately estimate the side lengths and diameter. Using grade 9 trigonometry of right triangles, students could figure out the length of one side of both an inscribed hexagon and dodecagon.

Inscribed Polygons					
Type of Polygon	Number of Sides	Length of One Side	Perimeter of Polygon	Diameter of the Circle	Number of diameters in the perimeter of the polygon
Hexagon					
Dodecagon					

Mathematical experiments:

Circumscribed Polygons					
Type of Polygon	Number of Sides	Length of One Side	Perimeter of Polygon	Diameter of the Circle	Number of diameters in the perimeter of the polygon
Hexagon					
Dodecagon					

Mathematical experiments:

1. A couple of classes ago we taped diameters around the edge of a circle to conclude that we needed 3 and a bit diameters to go around the circumference. Unfortunately, our results were not that specific. Look at the results from your charts. Archimedes used these results to state:

   The ratio of the circumference of any circle to its diameter is less than $3^{1}/_{7}$ but greater than $3^{10}/_{71}$.[1]

   Explain whether or not you think $3\frac{1}{7}$ is a good estimation the number of diameters in a circle.

   Math Experiments: Math Explanation:

2. We now use the number $\pi \approx 3.141592654\ldots$ to represent the number of diameters around the circumference of a circle. Below is a chart of Archimedes results from his inscribed and circumscribed polygons. How accurate do you think that Archimedes could get to estimating π? Explain your reasoning.

	Inscribed Polygons		**Circumscribed Polygons**
Number of Sides	Number of diameters in the perimeter of the polygon	π	Number of diameters in the perimeter of the polygon
6	3.0	π	3.4641
12	3.108	π	3.2154
24	3.1326	π	3.1597
48	3.1394	π	3.1461
96	3.1410	π	3.1427

[1] http://itech.fgcu.edu/faculty/clindsey/mhf4404/archimedes/archimedes.html on April 6, 2007.

Inscribing and Circumscribing Polygons - Archimedes Method

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6-Sided Polygon□
inscribed perimeter = 3.0□
circumscribed perimeter = 3.4641

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12-Sided Polygon□
inscribed perimeter = 3.1058□
circumscribed perimeter = 3.2154

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24-Sided Polygon□
inscribed perimeter = 3.1326□
circumscribed perimeter = 3.1597

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48-Sided Polygon
inscribed perimeter = 3.1394
circumscribed perimeter = 3.1461

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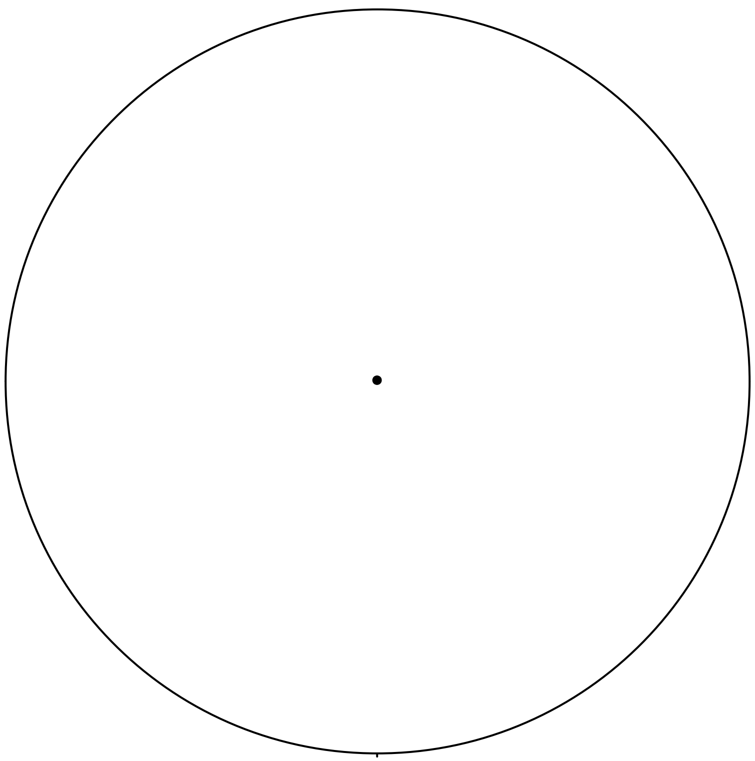
96-Sided Polygon
inscribed perimeter = 3.1410
circumscribed perimeter = 3.1427

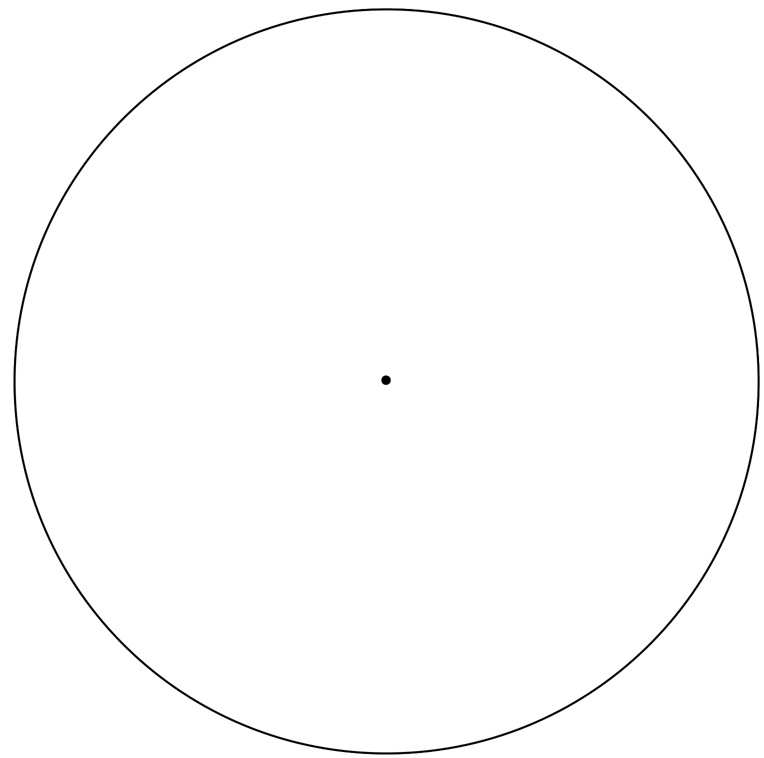
Source: http://www.pbs.org/wgbh/nova/archimedes/pi.html

Inscribing Polygons

Inscribed Hexagon

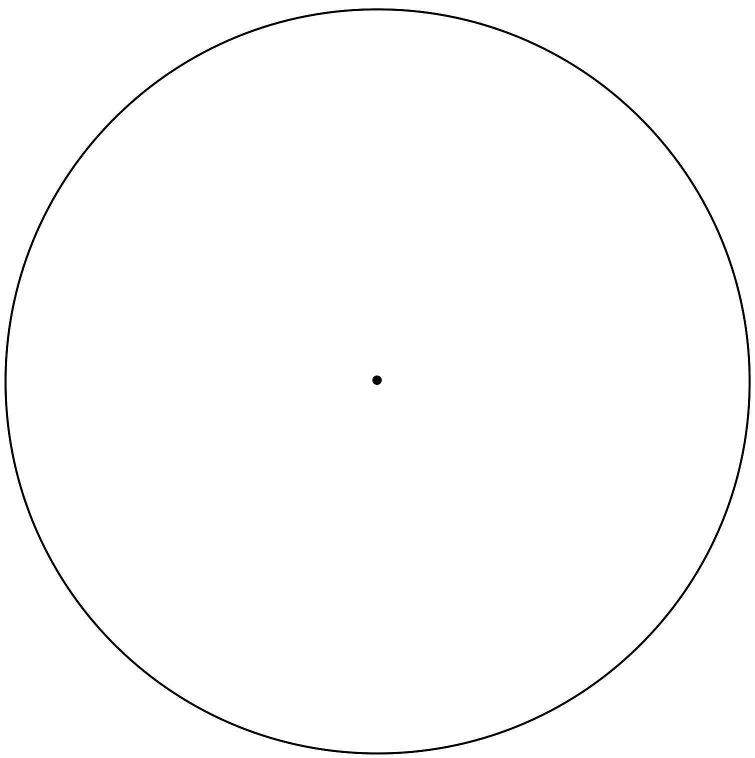


Inscribed Dodecagon



Circumscribed Polygons

Circumscribed Hexagon



Circumscribed Dodecagon