Superconductivity (Camerino)

Problem Set 1 | Due Mar. 22, 2013 | Marsiglio

1. Show that the wave function shown in class, that integrates over the phase in the BCS wave function,

$$\psi_{2\nu} = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-i\nu\theta} \Pi_k \big(u_k + e^{i\theta} v_k c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} \big) |0\rangle,$$

describes a system with precisely 2ν electrons.

2. (a) Consider a superconducting plate of thickness δ in the x-direction (it extends in the z and y directions indefintely), with a magnetic field applied in the z-direction. Show that $B(x)$ inside the superconducting plate is given by

$$B(x) = B_a \frac{\cosh{(x/\lambda)}}{\cosh{(\delta/2\lambda)}},$$

where B_a is the field outside the plate; here $x = 0$ is the centre of the plate.
(b) The effective magnetization $M(x)$ in the plate is defined by $B(x) - B_a = \mu_0 M(x)$. Show that $\mu_0 M(x) = -\frac{B_a}{8\lambda^2}(\delta^2 - 4x^2)$, for $\delta << \lambda$.

3. The Ginzburg-Landau equations are:

$$\frac{1}{2m^*}(i\hbar\vec{\nabla} + e^*\vec{A})^2\phi(\vec{r}) + \alpha\phi(\vec{r}) + \beta|\phi(\vec{r})|^2\phi(\vec{r}) = 0,$$

and

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) + i\frac{\hbar e^*}{2m^*}\big(\phi^*(\vec{r})\vec{\nabla}\phi(\vec{r}) - \phi(\vec{r})\vec{\nabla}\phi^*(\vec{r})\big) + \frac{(e^*)^2}{m^*}|\phi(\vec{r})|^2\vec{A} = 0,$$

where all quantities are defined as in class.
(a) Normalize the order parameter and the vector potential, and scale the distances by the Ginzburg-Landau coherence length to derive two dimensionless equations in lieu of those given. Use the gauge $\vec{\nabla} \cdot \vec{A} = 0$. In the resulting equations only the Ginzburg-Landau parameter $\kappa \equiv \lambda_L/\xi_{\rm GL}$ is required, where the London penetration depth is given by

$$\frac{1}{\lambda_L^2} \equiv \frac{(e^*)^2}{m^*}\frac{|\alpha|}{\beta}.$$

(b) Consider an isolated vortex. Take the complex dimensionless order parameter to be

$$\psi(\vec{r}/\xi_{GL}) = f(\rho)e^{i\phi},$$

and the dimensionless vector potential to be in the azimuthal direction ($\hat{\phi}$) only and dependent on ρ only, where ρ is the dimensionless radial coordinate, in cylindrical coordinates. Far away from the vortex the B-field is zero and the order parameter is homogeneous. Derive the behaviour of the order parameter and the vector potential near the origin of the vortex. Sketch qualitatively the dependence of these quantities and the magnetic field as a function of ρ over all values.