Normal Propositional Modal Logics

Propositional modal logics are formed from classical propositional logic by adding two new (interdefinable) sentence operators: $\Box$ ("necessarily", sometimes rendered as L) and $\Diamond$ ("possibly", sometimes rendered as M). These are unary operators: they operate on a single sentence (which of course might be complex, and might even contain other occurrences of the operators). Here's a definition of (most of) the well-known normal modal systems, described by the method of Chellas (1980) of starting with system K, and adding various axioms to it that describe the more complex systems. Thus the modal system KD45 results from adding axioms D, 4, and 5 to system K. As it turns out, some axioms imply others, some combinations of axioms are equivalent to each other, and some combinations are known by other names.

When using axiom systems, normal modal logics are built upon system K, which is:
1. Classical propositional logic (however you wish to present it)
2. $\vdash_k (\Box p \equiv \neg\Diamond\neg p)$ [interdefinability of $\Box$ and $\Diamond$]
3. $\vdash_k \Box(p \supset q) \supset (\Box p \supset \Box q)$ [the **K**-axiom]
4. if $\vdash_k p$ then $\vdash_k \Box p$ [the rule of necessitation, **N**]
5. if $\vdash_k p$ and $\vdash_k (p \supset q)$ then $\vdash_k q$ [Modus Ponens]

Now consider the following six axioms:
- **D.** $\Box p \supset \Diamond p$
- **T.** $\Box p \supset p$
- **G.** $\Diamond\Box p \supset \Box\Diamond p$
- B. $p \supset \Box\Diamond p$
- **4.** $\Box p \supset \Box\Box p$
- **5.** $\Diamond p \supset \Box\Diamond p$

Starting with K (which adds 0 of these axioms), there are 2^6 (=64) different combinations of the six axioms. However, there are certain implications between axioms and equivalences amongst groups of axioms, so we do not get 64 different modal systems. The relevant implications are:

T implies **D** **B** implies **G** **5** implies **G**

and the following equivalences

KB4 is equivalent to **KB5**

KDB4, **KTB4**, **KT45**, **KT5**, **KTB5** are equivalent to one another.

(And any other implications this yields). This leaves us with 21 modal systems. They are listed and diagrammed on the document "ModalLogicDiagram", which is elsewhere on this course page.

Normal modal systems have a semantics described by a binary accessibility relation (Rxy) on a set of "possible worlds" using the definitions of truth for modal statements:

$\Box p$ is true at w_1 iff, for all w_2, if Rw_1w_2, then p is true at w_2
$\Diamond p$ is true at w_1 iff there is a w_2 such that Rw_1w_2 and p is true at w_2

The different modal systems are characterized exclusively by having different requirements on Rxy. Many people use lower-case versions of the letters/symbols that name the axioms to describe the relevant semantic requirement on the accessibility relation. Priest uses Greek letters for some of them.

d. [seriality/connectivity] $\forall w_1 \exists w_2 Rw_1w_2$ [Priest: η]
t. [reflexivity] $\forall w Rww$ [Priest: ρ]
b. [symmetry] $\forall w_1 \forall w_2 (Rw_1w_2 \supset Rw_2w_1$ [Priest: σ]
g. [incestuality] $\forall w_1 \forall w_2 \forall w_3 ((Rw_1w_2 \wedge Rw_1w_3) \supset \exists w_4 (Rw_2w_4 \wedge Rw_3w_4))$
4. [transitivity] $\forall w_1 \forall w_2 \forall w_3 ((Rw_1w_2 \wedge Rw_2w_3) \supset Rw_1w_3)$ [Priest: τ]
5. [euclidean] $\forall w_1 \forall w_2 \forall w_3 ((Rw_1w_2 \wedge Rw_1w_3) \supset Rw_2w_3)$