Unary negation:

$$
\begin{array}{c|c}
\varphi_{1} & \neg \varphi_{1} \\
\hline T & F \\
F & T
\end{array}
$$



$$
\begin{array}{cc|c}
\varphi_{1} & \varphi_{2} & \left(\varphi_{1} \supset \varphi_{2}\right) \\
\hline T & T & T \\
T & F & F \\
F & T & T \\
F & F & T
\end{array}
$$

$$
\begin{array}{cc|c}
\varphi_{1} & \varphi_{2} & \left(\varphi_{1} \equiv \varphi_{2}\right) \\
\hline T & T & T \\
T & F & F \\
F & T & F \\
F & F & T
\end{array}
$$

How many (two-valued) binary truth functions are there?

| $\varphi_{1}$ | $\varphi_{2}$ | 1 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ |
| $T$ | $F$ | $T$ | $T$ | $T$ | $T$ | $F$ | $F$ | $F$ | $F$ | $T$ | $T$ | $T$ | $T$ | $F$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $T$ | $F$ | $F$ | $T$ | $T$ | $F$ | $F$ | $T$ | $T$ | $F$ | $F$ | $T$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | $F$ | $T$ | $F$ | $T$ | $F$ | $T$ | $F$ | $T$ | $F$ | $T$ | $F$ | $T$ | $F$ | $T$ | $F$ |

How many (two-valued) binary truth functions are there?

| $\varphi_{1}$ | $\varphi_{2}$ |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ |
| $T$ | $F$ | $T$ | $T$ | $T$ | $T$ | $F$ | $F$ | $F$ | $F$ | $T$ | $T$ | $T$ | $T$ | $F$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $T$ | $F$ | $F$ | $T$ | $T$ | $F$ | $F$ | $T$ | $T$ | $F$ | $F$ | $T$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | $F$ | $T$ | $F$ | $T$ | $F$ | $T$ | $F$ | $T$ | $F$ | $T$ | $F$ | $T$ | $F$ | $T$ | $F$ |

$16 \quad\left(=2^{\left(2^{2}\right)}\right)$

Which ones are in common use?

How many (two-valued) unary truth functions are there?

| $\varphi_{1}$ |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
| $T$ | $T$ | $T$ | $F$ | $F$ |
| $F$ | $T$ | $F$ | $T$ | $F$ |

$4 \quad\left(=2^{\left(2^{1}\right)}\right)$

How many 0-ary connectives are there?
|T F
$2 \quad\left(=2^{\left(2^{0}\right)}\right)$

What's the difference between $\perp, \quad \perp\left(\varphi_{1}\right)$, and $\perp\left(\varphi_{1}, \varphi_{2}\right)$ ?

Logic textbooks don't usually discuss connectives of higher adicity, although the following one is sometimes mentioned (e.g., by Alonzo Church, 1956). He called it "conditional disjunction", although most of the rest of us call it "if-then-else":

| $\varphi_{1}$ | $\varphi_{2}$ | $\varphi_{3}$ | $*\left(\varphi_{1}, \varphi_{2}, \varphi_{3}\right)$ |
| :---: | :---: | :---: | :---: |
| $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $T$ |
| $T$ | $F$ | $T$ | $F$ |
| $T$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $F$ | $F$ |

How many ternary connectives are there?

How many ternary connectives are there?
$256 \quad\left(=2^{\left(2^{3}\right)}\right)$

How many quadrinary? $65,536 \quad\left(=2^{\left(2^{4}\right)}\right)$

How many truth functions are there all told?

So-why do logic textbooks stick with just the few common ones???

A combination of: These connectives correspond (sort of) to natural language connectives, and they are functionally complete [also known as "expressively complete"]

This means that all of the infinite number of truth functions can be expressed as some combination of the ones they chose.

How do we prove that??

By showing how to construct a formula that corresponds to any arbitrary truth table:

Here's an example truth function (represented by this arbitrary truth table. (I'm just using a ternary truth function, so as to be able to fit it on the page... the method works with any adicity).

| $\varphi_{1}$ | $\varphi_{2}$ | $\varphi_{3}$ | $\Psi$ |
| :---: | :---: | :---: | :---: |
| $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $F$ |
| $T$ | $F$ | $T$ | $F$ |
| $T$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $F$ | $F$ |

Step 1: For each line of the truth table where $\Psi$ is T , write a formula that is true in that row and in no other row.

| $\varphi_{1}$ | $\varphi_{2}$ | $\varphi_{3}$ | $\Psi$ |  |
| :---: | :---: | :---: | :---: | :---: |
| $T$ | $T$ | $T$ | $T$ | $\longmapsto\left(\varphi_{1} \wedge \varphi_{2} \wedge \varphi_{3}\right)$ |
| $T$ | $T$ | $F$ | $F$ |  |
| $T$ | $F$ | $T$ | $F$ |  |
| $T$ | $F$ | $F$ | $F$ |  |
| $F$ | $T$ | $T$ | $T$ | $\longmapsto\left(\neg \varphi_{1} \wedge \varphi_{2} \wedge \varphi_{3}\right)$ |
| $F$ | $T$ | $F$ | $F$ |  |
| $F$ | $F$ | $T$ | $T$ | $\longmapsto\left(\neg \varphi_{1} \wedge \neg \varphi_{2} \wedge \varphi_{3}\right)$ |
| $F$ | $F$ | $F$ | $F$ |  |

Note that each of those formulas is true in just exactly their one corresponding row of the truth table.

Now, put a $\vee$ between each of those formulas:

$$
\left(\left(\varphi_{1} \wedge \varphi_{2} \wedge \varphi_{3}\right) \vee\left(\neg \varphi_{1} \wedge \varphi_{2} \wedge \varphi_{3}\right) \vee\left(\neg \varphi_{1} \wedge \neg \varphi_{2} \wedge \varphi_{3}\right)\right)
$$

This is true in one of the three rows, but not in any other row. Thus it characterizes $\psi$.

Did it really work?? Did the demonstration cover all the cases??

What do you think?

The formula thus generated is in what is called Disjunctive Normal Form (DNF):
(a) The main connectives are disjunctions ( $V$ ).
(b) Each of the disjuncts is a conjunction, so their main connectives are $\wedge$
(c) Each of the items thus conjoined is a literal - either a sentence letter or the negation of a sentence letter.

Every formula can be put into DNF (and there is a corresponding CNF). This shows that every one of the infinite number of truth functions can be written using only the unary $\neg$ and the binary $\wedge$ and $V$.

Alternatively put: The set of connectives, $\{\neg, \wedge, \vee\}$ is functionally complete.

Who remembers the DeMorgan Laws??

So this means that each of $\{\neg, \wedge\}$ and $\{\neg, \vee\}$ are functionally complete, since using just the connectives in one of these sets, one can define everything in an already-known-to-be-functionally-complete set (namely $\{\neg, \wedge, \vee\}$ )

And we all know the definition of $p \supset q$ as either $\neg(p \wedge \neg q)$ or $\neg p \vee q$. So $\{\neg, \supset\}$ is also functionally complete.

Maybe you also know that the connectives $\uparrow$ and $\downarrow$ are each functionally complete by themselves? Their truth tables are:

| $\varphi_{1}$ | $\varphi_{2}$ | $\left(\varphi_{1} \downarrow \varphi_{2}\right)$ | $\left(\varphi_{1} \uparrow \varphi_{2}\right)$ |
| :---: | :---: | :---: | :---: |
| $T$ | $T$ | $F$ | $F$ |
| $T$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $F$ | $T$ |
| $F$ | $F$ | $T$ | $T$ |

Can you show how to use one of these to define the connectives of an already-known-to-be-functionally-complete set of connectives?

Exercise: Show that $\{\supset, \oplus\}$ is functionally complete. Show that $\{*, \top, \perp\}$ is functionally complete.

Ok-that's all well and good. But WHY:
How come $\{\neg, \wedge\},\{\neg, \vee\},\{\neg, \supset\},\{*, \top, \perp\},\{\supset, \oplus\},\{\uparrow\}$, and $\{\downarrow\}$ are functionally complete, but nonetheless
$\{\neg, \equiv\}$ is not functionally complete??
$\{*, \oplus, \perp\}$ is not functionally complete??
The answer is given in Post's Functional Completeness Theorem (Emil Post, 1941).

The most AMAZING and AWE-INSPIRING thing ever proved about two-valued truth-functional logic!!

But first some preliminaries. We need to define five abstract properties of truth functions:

1. Closed under T
2. Closed under $F$
3. Self-Dual
4. Monotonic (requires preliminary notion of truth-vector order)
5. Counting (requires preliminary notions of truth-vector order and of dummy variable)

Easy definitions:
A truth-function is closed under $T$ iff it has a $T$ in its first row (i.e., if all its arguments are T then so is its resulting value)

A truth-function is closed under $F$ iff it has a $F$ in its last row (i.e., if all its arguments are $F$ then so is its resulting value)

So: $\supset, \wedge, \vee, \equiv, *, \top$ are all closed under $\top . \wedge, \vee, \oplus, *, \perp$ are all closed under F .

## SELF-DUALITY:

Let $\bar{p}$ be the name of the truth value that is the opposite of the value of $p$.

A truth function $f$ is self-dual iff
for each input vector (line of truth table) $<p_{1}, \cdots, p_{n}>$,

$$
f\left(<p_{1}, \cdots, p_{n}>\right)=\overline{f\left(<\overline{p_{1}}, \cdot \bar{\cdots}, \overline{p_{n}}>\right)}
$$

So therefore a function $f$ is not self-dual iff there is a row of the truth table, $<q_{1}, \cdots, q_{n}>$ such that

$$
f\left(<q_{1}, \cdots, q_{n}>\right)=f\left(<\overline{q_{1}}, \cdots, \bar{q}_{n}>\right)
$$

For example, $\neg, \top, \perp$ are self-dual. $\wedge, \supset, \equiv, \vee, \oplus, *$ are not.

Background notion: truth-vector ordering.
This is an ordering of the input rows of a truth table (technically, it's a partial order - not every pair of rows can be compared). Maybe clearest with an example. Suppose a truth vector of three elements. Here's an ordering:

$$
\begin{array}{lll} 
& <T, T, T> & \\
<F, T, T> & <T, F, T> & <T, T, F> \\
<F, F, T> & <F, T, F> & <T, F, F> \\
& <F, F, F> &
\end{array}
$$

$$
<F, F, T>
$$

T is considered "greater" than F. And so each member of each level is "greater" than any of the ones in the next level lower if they differ by only one position change of truth value. Being "greater than" is transitive, so that following a pathway downward always will yield cases of "greater than".

## MONOTONIC TRUTH FUNCTION:

The $n$-ary truth function $f^{n}$ is monotonic iff whenever $n$-ary truth vectors are such that $V_{1} \geq V_{2}$,

$$
f^{n}\left(V_{1}\right) \geq f^{n}\left(V_{2}\right)
$$

So, a (two-valued) truth function $f$ is not monotonic iff there is a row of the truth table where the value of $f$ at that row is F while the value of $f$ at some row that is less than the first is T .

For example: $\vee, \wedge, \top, \perp$ are monotonic. $\supset, \oplus, \equiv, *, \neg$ are not.

DUMMY POSITION (for a truth function $f$ ):
A position in a truth function's input values is dummy iff it "never makes a difference" in evaluating the truth function.

That is: The $i^{\text {th }}$ position of $\left\langle x_{1}, \cdots, x_{n}\right\rangle$ is a dummy position iff $f^{n}\left(x_{1}, \cdots, x_{i-1}, F, x_{i+1}, \cdots, x_{n}\right)=$ $f^{n}\left(x_{1}, \cdots, x_{i-1}, T, x_{i+1}, \cdots, x_{n}\right)$
for every combination of values of all the other positions.
Consider this imaginary binary connective $(p \cdot q)$ :

| $\varphi_{1}$ | $\varphi_{2}$ | $\left(\varphi_{1} \cdot \varphi_{2}\right)$ |
| :---: | :---: | :---: |
| $T$ | $T$ | $F$ |
| $T$ | $F$ | $T$ |
| $F$ | $T$ | $F$ |
| $F$ | $F$ | $T$ |

Obviously, this is really just $\neg \varphi_{2} . \varphi_{1}$ never makes a difference.

## COUNTING FUNCTIONS:

A function $f$ is a counting function iff
(i) No position is dummy.
(ii) if $V_{i}$ and $V_{j}$ are immediately adjacent in the truth-vector ordering, then $f\left(V_{i}\right) \neq f\left(V_{j}\right)$

Or in other words, if two input vectors differ in only one value, but the values of the function applied to those two rows are the same, then the function is not counting (well, unless that position is dummy).

| $\varphi_{1}$ | $\neg \varphi_{1}$ |
| :---: | :---: |
| $T$ | $F$ |
| $F$ | $T$ |

Negation is self-dual and counting (but not the other properties)

| $\varphi_{1}$ | $\varphi_{2}$ | $\wedge$ | $\vee$ | $\supset$ | $\equiv$ | $\oplus$ | $T$ | $\perp$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $F$ | $T$ | $F$ |
| $T$ | $F$ | $F$ | $T$ | $F$ | $F$ | $T$ | $T$ | $F$ |
| $F$ | $T$ | $F$ | $T$ | $T$ | $F$ | $T$ | $T$ | $F$ |
| $F$ | $F$ | $F$ | $F$ | $T$ | $T$ | $F$ | $T$ | $F$ |

$\wedge, \vee$ are T-preserving, F-preserving, monotonic
$\supset$ is T -preserving
$\equiv$ is T-preserving and counting
$\oplus$ is F-preserving and counting
$\top$ is $T$-preserving, monotonic and counting
$\perp$ is F-preserving, monotonic and counting
And of course, atomic sentences themselves (or, if you prefer, the identity function on truth values) are T-preserving, F-preserving, self-dual, monotonic, and counting.

# POST'S FUNCTIONAL COMPLETENESS THEOREM 

The most amazing theorem about two-valued truth functions ever!!

A set of (2-valued) truth functions, X , is functionally complete iff
for each of:
closed under T
closed under F
monotonic
self-dual
counting,
there is a member of $X$ which does not manifest that property.

## HOW TO PROVE THIS BEAUTIFUL THEOREM??

Well, it's an "iff" theorem, so let's try proving each of the two directions separately:
(i) If $X$ is a functionally complete set, then for each of the properties, some member of $X$ does not manifest it.
(ii) Given any set of connectives $X$ such that, for each of the properties, there is at least one connective in X lacks that property, then X is functionally complete.

Let's prove (ii) first. We'll do it by construction - that is, I will assume I have connectives that lack the five properties, and will then show you how to define a set of connectives that you already know to be functionally complete (such as $\{\neg, \wedge\}$ or $\{\neg, \supset\}$, etc.)

Afterwards I'll show you how a proof of (i) proceeds, but won't actually present the detailed proof.

| $\varphi_{1}$ | $\varphi_{2}$ | $\varphi_{3}$ | $f_{1}$ | $f_{2}$ | $f_{3}$ | $f_{4}$ | $f_{5}$ | $f_{6}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $T$ | $T$ | $T$ | $F$ | $T$ | $T$ | $F$ | $T$ | $F$ |
| $T$ | $T$ | $F$ | $T$ | $F$ | $F^{1}$ | $T^{1}$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $T$ | $F$ | $F$ | $T^{1}$ | $F$ | $F$ |
| $T$ | $F$ | $F$ | $T$ | $T$ | $F$ | $T$ | $T^{1}$ | $F$ |
| $F$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T^{1}$ | $F$ |
| $F$ | $T$ | $F$ | $F$ | $T$ | $T^{1}$ | $F$ | $F$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $T$ | $F$ | $F$ | $T$ |
| $F$ | $F$ | $F$ | $F$ | $T$ | $F$ | $T$ | $F$ | $T$ |

$f_{1}$ and $f_{2}$ are non-T- and non-F-preserving (respectively).
$f_{3}$ is non-monotonic (as proved by the 1 's).
$f_{4}$ is non-counting [no dummies: rows $1 \& 5$ for 1st position, $6 \& 8$ for 2 nd position, $1 \& 2$ for 3 rd position]. The 1's show where it isn't counting. $f_{5}$ is non-self-dual, as shown by the 1 's.
$f_{6}$ is a function that is non-T-and-non-F-preserving, non-monotonic, non-self-dual, non-counting.

