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Gentzen and Jaśkowski Natural Deduction: Fundamentally Similar but Importantly Different

Abstract. Gentzen's and Jaśkowski's formulations of natural deduction are logically equivalent in the normal sense of those words. However, Gentzen's formulation more straightforwardly lends itself both to a normalization theorem and to a theory of "meaning" for connectives (which leads to a view of semantics called 'inferentialism'). The present paper investigates cases where Jaskowski's formulation seems better suited. These cases range from the phenomenology and epistemology of proof construction to the ways to incorporate novel logical connectives into the language. We close with a demonstration of this latter aspect by considering a Sheffer function for intuitionistic logic.

Keywords: Jaśkowski, Gentzen, Natural deduction, Classical logic, Intuitionistic logic, Inferential semantics, Generalized natural deduction, Sheffer functions.

1. Historical Remarks and Background

Two banalities that we often hear are (a) that it often happens that two apparently radically different theories or approaches in some field of study turn out to share fundamental underlying similarities that were hidden by the differences in approach; and the inverse of that observation, (b) that it can happen that two apparently similar approaches or theories can in fact conceal some important differences when they are applied to areas that the approaches weren't originally designed to consider.

As indicated in its title, the present paper is a study of this second banality as it has occurred in the natural deduction approach to the field of logic. It seems apposite to pursue this study now, at the 80^{th} anniversary of the first publications of the natural deduction approach to logic. Astonishing as it now seems, there were two totally independent strands of research that each resulted in the publication of two approaches to natural deduction, and

Special Issue: Gentzen's and Jaśkowski's Heritage
80 Years of Natural Deduction and Sequent Calculi
Edited by **Andrzej Indrzejczak**

these were published (amazingly) in the same year, with (amazingly!) no interaction between the two authors nor between the two publishing venues.

The thrust of this paper is that, although it is commonly thought that the two papers (Jaśkowski, 1934; Gentzen, 1934) were "merely two different approaches to the same topic", and that the differences in their approaches were minor, we find that these differences are in fact quite important in the further development of natural deduction. This is what we will set out to demonstrate in the following pages.[1]

1.1. Jaśkowski on Natural Deduction

According to Jaśkowski (1934), Jan Łukasiewicz had raised the issue in his 1926 seminars that mathematicians do not construct their proofs by means of an axiomatic theory (the systems of logic that had been developed at the time) but rather made use of other reasoning methods; especially they allow themselves to make "arbitrary assumptions" and see where they lead. Łukasiewicz wondered whether there could be a logical theory that embodied this insight but which yielded the same set of theorems as the axiomatic systems then in existence. Again according to Jaśkowski (1934), he (Jaśkowski) developed such a system and presented it to the First Polish Mathematical Congress in 1927 at Lvov, and it was mentioned (by title) in their published proceedings of 1929. There seems to be no copies of Jaśkowski's original paper in circulation, and our knowledge of the system derives from a lengthy footnote in Jaśkowski, 1934. (This is also where he said that it was presented and an abstract published in the Proceedings. Jan Woleński, in personal communication, tells us that in his copy of the *Proceedings*, Jaśkowski's work (Jaśkowski, 1929) was reported only by title.) Although the footnote describes the earlier use of (what we call) a graphical method to represent these proofs, the main method described in Jaśkowski (1934) is rather different—what we in earlier works called a bookkeeping method. Celluci (1995) recounts Quine's visit to Warsaw in 1933, and his meeting with Jaśkowski. Perhaps the change in representational method might be due to a suggestion of Quine (who also used a version of this bookkeeping method in his own later system, Quine, 1950).

In the present paper we will concentrate on Jaśkowski's graphical method, especially as it was later developed and refined by Fitch (1952). We will call this representational method "Fitch-style", although it should be

[1]Much of the material in the upcoming historical introduction is available in a sequence of papers by the present authors: Pelletier (1999, 2000); Pelletier and Hazen (2012). Some related material is surveyed in (Indrzejczak, 2010, Chapt. 2).

kept in mind that it was originally developed by Jaśkowski. (And so we often call it 'Jaśkowski-Fitch'). As Jaśkowski originally described his earlier method, it consisted in drawing boxes or rectangles around portions of a proof. The restrictions on completion of subproofs (as we now call them) are enforced by restrictions on how the boxes can be drawn. We would now say that Jaśkowski's system had two subproof methods: conditional-proof (conditional-introduction)[2] and reductio ad absurdum (negation-elimination). It also had rules for the direct manipulation of formulas (e.g., Modus Ponens).

Proofs in this system look like this[3]:

1.	$((p \to q) \land (\neg r \to \neg q))$	Supposition
2.	p	Supposition
3.	$((p \to q) \land (\neg r \to \neg q))$	1 Reiterate
4.	$(p \to q)$	3 Simplification
5.	q	2, 4 Modus Ponens
6.	$(\neg r \to \neg q)$	3 Simplification
7.	$\neg r$	Supposition
8.	$(\neg r \to \neg q)$	6 Reiterate
9.	$\neg q$	7, 8 Modus Ponens
10.	q	5 Reiterate
11.	r	7–10 Reductio ad Absurdum
12.	$p \to r$	2-11 Conditionalization
13.	$(((p \to q) \land (\neg r \to \neg q)) \to (p \to r))$	1–12 Conditionalization

[2]Obviously, this rule of Conditional-Introduction is closely related to the deduction theorem, that from the fact that $\Gamma, \varphi \vdash \psi$ it follows that $\Gamma \vdash (\varphi \to \psi)$. The difference is primarily that Conditional-Introduction is a rule of inference in the object language, whereas the deduction theorem is a metalinguistic theorem that guarantees that proofs of one sort could be converted into proofs of the other sort. According to (Kleene, 1967, p. 39fn33) "The deduction theorem as an informal theorem proved about particular systems like the propositional calculus and the predicate calculus... first appears explicitly in Herbrand (1930) (and without proof in Herbrand, 1928); and as a general methodological principle for axiomatic-deductive systems in Tarski (1930). According to (Tarski, 1956, p. 32fn), it was known and applied by Tarski since 1921."

[3]Jaśkowski had different primitive connectives and rules of inference, but it is clear how this proof would be represented if he did have $\land$ in his language.

In this graphical method, each time an assumption is made it starts a new portion of the proof which is to be enclosed with a rectangle (a "subproof"). The first line of this subproof is the assumption… in the case of trying to apply conditional introduction, the assumption will be the antecedent of the conditional to be proved and the remainder of this subproof will be an attempt to generate the consequent of that conditional. If this can be done, then Jaśkowski's rule CONDITIONALIZATION says that the conditional can be asserted as proved *in the subproof level of the box that surrounds the one just completed.* So the present proof will assume the antecedent, $((p \supset q) \wedge (\neg r \supset \neg q))$, thereby starting a subproof trying to generate the consequent, $(p \supset r)$. But this consequent itself has a conditional as main connective, and so it too should be proved by conditionalization with a yet-further-embedded subproof that assumes its antecedent, p, and tries to generate its consequent, r. As it turns out, this subproof calls for a yet further embedded subproof, which is completed using Jaśkowski's REDUCTIO AD ABSURDUM.

The main difference between Jaśkowski's graphical method and Fitch's is that Fitch does not completely draw the whole rectangle around the embedded subproof (but only the left side of the rectangle), and he partially underlines the assumption. The same proof displayed above using Jaśkowski's graphical method is done like the following in Fitch's representation (with a little laxness on identifying the exact rules Fitch employs).

1	$\vert\ \vert\ ((p \supset q) \wedge (\neg r \supset \neg q))$ (assumption)	
2	$\vert\ \vert\ \vert\ p$ (assumption)	
3	$\vert\ \vert\ \vert\ ((p \supset q) \wedge (\neg r \supset \neg q))$	1, Reiteration
4	$\vert\ \vert\ \vert\ (p \supset q)$	3, $\wedge$E
5	$\vert\ \vert\ \vert\ q$	2,4 $\supset$E
6	$\vert\ \vert\ \vert\ (\neg r \supset \neg q))$	3, $\wedge$E
7	$\vert\ \vert\ \vert\ \vert\ \neg r$ (assumption)	
8	$\vert\ \vert\ \vert\ \vert\ (\neg r \supset \neg q))$	6, Reiteration
9	$\vert\ \vert\ \vert\ \vert\ \neg q$	7,8 $\supset$E
10	$\vert\ \vert\ \vert\ \vert\ q$	5, Reiteration
11	$\vert\ \vert\ \vert\ r$	7–10, $\neg$E
12	$\vert\ \vert\ (p \supset r)$	2–11, $\supset$I
13	$\vert\ (((p \supset q) \wedge (\neg r \supset \neg q)))$	1–12, $\supset$I

This Fitch-style of natural deduction proof-representation is pretty well-known, since it was followed by a large number of elementary textbooks and is employed by a large number of writers when they display their own proofs.[4] We will be comparing this style of natural deduction proof-representation with another, that of Gentzen.

1.2. Gentzen on Natural Deduction

The method of presenting natural deduction proofs in Gentzen (1934) was by means of trees. The leaves of a tree were assumed formulas, and each interior node was the result of applying one of the rules of inference to parent nodes. The root of the tree is the formula to be proved. The following is a tree corresponding to the example we have been looking at, although it should be mentioned that Gentzen's main rule for indirect proofs first generated $\bot$ ("the absurd proposition") from the two parts of a contradiction, and then generated the negation of the relevant assumption.[5]

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{3}{\neg r} \quad \dfrac{\overset{1}{((p \to q) \land (\neg r \to \neg q))}}{(\neg r \to \neg q)}\,\land\text{-E}}{\neg q}\,\to\text{-E} \qquad \dfrac{\dfrac{\overset{1}{((p \to q) \land (\neg r \to \neg q))}}{(p \to q)}\,\land\text{-E} \quad \overset{2}{p}}{q}\,\to\text{-E}}{\bot}\,\bot\text{-I}}{r}\,\bot\text{-E (3)}}{(p \to r)}\,\to\text{-I (2)}}{(((p \to q) \land (\neg r \to \neg q)) \to (p \to r))}\,\to\text{-I (1)}$$

The lines indicate a transition from the upper formula(s) to the one just beneath the line, using the rule of inference indicated on the right edge of the line. (We might replace these horizontal lines with vertical or splitting lines to more clearly indicate tree-branches, and label these branches with the rule of inference responsible, and the result would look even more tree-like). Gentzen uses the numerals on the leaves as a way to keep track of subproofs. Here the main antecedent of the conditional to be proved is entered (twice, since there are two separate things to do with it) with the numeral '1', the antecedent of the consequent of the main theorem is entered with numeral '2', and the formula $\neg r$ (to be used in the reductio part of the proof) is entered with numeral '3'. When the relevant "scope changing" rule

[4] See Pelletier (1999, 2000) for details of the elementary textbooks that use this method.

[5] He also considered a double negation rule, for classical logic.

is applied (indicated by citing the numeral of that branch as part of the citation of the rule of inference, in parentheses) this numeral gets "crossed out", indicating that this subproof is finished.

Gentzen (unlike Jaśkowski) considered both classical logic and intuitionist logic, and many have claimed that his characterization of "the proper form of rules of logic" in terms of introduction and elimination rules for each of the logical connectives was the key to describing not only what was "meant" by a logical connective, but also what The One True Logic must be — namely, a logic in which all connectives obeyed this property. As shown in Gentzen (1934), it is intuitionist logic that would be The One True Logic if this criterion were correct. And that one would need to add some further rules which were *not* in the introduction-and-elimination (Int-Elim) paradigm in order to get classical logic.

Although this was described in Gentzen's work and employed using Gentzen's tree-format representation of proofs, it is pretty straightforward to adopt the same Int-Elim restrictions on the "correct" rules and require that proofs in the Jaśkowski (or Fitch) graphical representations implement them. Thus one can straightforwardly have graphical style proofs for intuitionist logic as well as for classical logic.

2. Some Heuristic Advantages of Jaśkowski over Gentzen

Pretty clearly, in any of the standard senses of the word, the systems generated using the Gentzen representation of proofs are *equivalent* to those generated by employing the Jaśkowski-Fitch representational method. Their respective intuitionist logic systems are equivalent, and their respective classical logic systems are equivalent. The particular *styles of representing proofs* do not affect these equivalences, in any of the standard meanings of this term.[6]

In this section we will be arguing that despite this "fundamental equivalence" of Gentzen and Jaśkowski-Fitch systems, we think that there are some important differences lurking; and we turn our attention to describing them.

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[6]The same can perhaps not be said of the other natural deduction methods, such as sequent natural deduction (the first description of which is in Gentzen, 1936), which was popularized by Suppes (1957); Mates (1965); Lemmon (1965), since there are various differences of a meta-theoretic nature between natural deduction and sequent natural deduction. And the notion of "generalized natural deduction" that we discuss later in this paper is also importantly different; we reserve our comments about this method until a later section.

2.1. Phenomenology and Epistemology

Though the Jaśkowski and Gentzen systems have the same fundamental mathematical properties, the notion of a subproof in Jaśkowski-inspired systems is suggestive for phenomenology and epistemology. From a purely formal standpoint, the Gentzen and Jaśkowski-Fitch presentations of natural deduction are both workable. If one is interested in theoretical proof theory—in proving that derivations can be converted into a normal form, etc.—Gentzen's trees are perhaps more convenient: a single occurrence of a formula in a Jaśkowski-Fitch derivation can be cited as a premise for multiple inferences lower down in the proof, and this leads to complications which the Gentzen presentation avoids by putting multiple copies of the premise on different branches of the tree. By the same token, if one engaged in the construction of many, large, formal derivations—if, for example, one were trying to rewrite *Principia Mathematica*, or were trying to construct a formal verification of a complex computer program—the Jaśkowski-Fitch "linearization" of natural deduction could provide significant economies: the increase in the size (number of symbols) of a formal derivation when it is converted from a linear form to a tree-form can be exponential.

But there are also non-formal standpoints. Logic is a branch of philosophy as well as a branch of mathematics, and one reason for studying formal systems of logic is the hope that, as simplified models of informal proof, they will yield insights into the *epistemology* or the *phenomenology* of deductive reasoning. (This may have been a major part of Łukasiewicz's motivation in posing the question that inspired Jaśkowski. Gentzen, in contrast, was a member of the Hilbert school, and—though he clearly valued the way in which his N-systems represent the methodology of informal mathematical proof—was more interested in formal aspects: his primary hope was that the normalization, or, with the L-systems, cut-elimination properties of his systems would lead to consistency proofs for mathematical axiomatic systems.) Here, it seems to us, the Jaśkowski-Fitch presentation is more perspicuous.[7]

The nodes in a Gentzen-style natural deduction tree are occupied by formulas. Those at the leaves of the tree are termed hypotheses, and the others are said to be inferred, by one or another rule, from the formulas standing immediately above it. Speaking of *inference*, however, though appropriate for rules like *modus ponens* or the conjunction rules, is lunatic for other rules. In Disjunction Elimination ($\vee$**E**), a formula γ is inferred from three formulas: a "major premise" $\alpha \vee \beta$, a "minor premise" γ (derived under the hypothesis

[7]The interested reader can consult a fuller discussion in Hazen (1999).

α), and a "minor premise" γ (derived under the hypothesis β): this sounds like a logically trivial rule of inferring γ from two copies of itself (together with a redundant major premise)! $\forall$**I** sounds even worse: infer $\forall x Fx$ from an instance, Fa: this sounds simply invalid. (As does the comparable rule of $\Box$**I** in a formulation of modal logic: infer $\Box\alpha$ from α.) Obviously more is going on: the key logical property of the rules is that in applying them hypotheses can be "discharged", and that, with the quantifier rule, there is a restriction on the form of undischarged hypotheses standing above the "inference". But this notion of "discharging" a hypothesis is—if the formal system is to be thought of as somehow representing some aspects of natural reasoning—in need of analysis. One can perhaps characterize it roughly (how many of us have done this when trying to explain natural deduction to students in elementary courses?) in terms of *pretending*: parts of a formal derivation are written under the pretense that we believe a hypothesis and consist of sentences we would be willing to assert if that pretense were true, and other parts—those coming after the hypothesis has been "discharged"—are written honestly, without pretense. But what does this psychological or theatrical business of pretending have to do with deductive reason?

Things look different in a Jaśkowski-Fitch version of natural deduction. Formulas standing as conclusions of rules like *modus ponens* or the Conjunction rules are still naturally described as inferred, or at least inferable, from the formulas standing as premises for the applications of these rules: at least in the main proof (i.e., not in a subproof, not under a subsequently discharged hypothesis) an application of one of these rules can be thought of as representing a step in a possible episode of reasoning, in which the reasoner infers a conclusion from premises which are already believed. In the other rules, however, the conclusion is not presented as being inferred (solely) from other formulas at all. In $\vee$**E** the conclusion γ is justified by reference to (in the terminology of Fitch, 1952: *is a direct consequence of*) two subproofs containing γ, one with the hypothesis α, one with the hypothesis β, along with the formula $\alpha \vee \beta$. In $\forall$**I**, the assertion of $\forall x Fx$ is justified by reference, not to the formula Fa, but to a subproof ("general with respect to" the *eigenvariable* a) containing it.

This, it seems to us, is more faithful to the phenomenology of reasoning. If one draws a conclusion from two already believed premisses by *inferring* it from them, one is thinking about the subject matter of the three propositions, not about their logical relations. You believe α, you believe $\alpha \rightarrow \beta$. At some stage, while thinking about their common subject matter, you call both to conscious consideration at the same time, and at that stage you come to believe β: that is what inference is.[8] In contrast, when drawing a conclusion in a context where the formalized representation of one's reasoning would involve $\vee\mathbf{E}$, one must be aware of logical relations: one cannot think solely about the subject matter of the propositions involved. You believe $\alpha \vee \beta$. You don't (initially) believe γ, but you are aware that the *argument* from α to γ is valid, and likewise that the argument from β to γ is valid. Your awareness of these validities, as well as your (disjunctive) belief about the subject matter, plays an essential role in your coming to believe γ. If there is inference, it is not simply inference from premisses you already believe about the subject matter: it is from the (subject matter relevant) $\alpha \vee \beta$ together with two metalogical premisses, α *validly implies* γ, and β *validly implies* γ, premisses you come to believe not by inference, but by *inspection* of deductions from hypotheses you don't believe.[9]

There is another issue belonging to the philosophy of language (or to epistemology broadly construed) connected with natural deduction: the relation between logical rules and the meanings of logical operators. Here Gentzen had something but Jaśkowski apparently nothing to say. Gentzen, 1934, p. 295 remarks, almost as an aside, that in his N-systems the Introduction rules can be seen as *defining* the operators, with the Elimination rules giving

[8]This is, of course, an oversimplification. Sometimes, after all, what happens when you simultaneously consider α (which you believe), $\alpha \to \beta$ (which you also believe) and β (which you *dis*believe) is not that you come to believe β, but rather that ...your confidence in one or the other or both of α and $\alpha \to \beta$ is shaken. *Modus ponens* and *modus tollens* are both logical rules, but more is needed to determine which rule will guide your inference, your change of belief. For further discussion of the complex relation between logical implication and inference, see Harman (1973).

[9]Alonzo Church also saw the difference in the structure of the reasonings represented by natural deduction and axiomatic formulations of logic, but for some purposes preferred the axiomatic! Cf. (Church, 1956, pp. 164–165):

> The idea of using the deduction theorem as a primitive rule of inference in formulations of the propositional calculus or functional calculus is due independently to Jaśkowski and Gentzen. Such a primitive rule of inference has a less elementary character than is otherwise usual [...], and indeed it would not be admissible for a logistic system according to the definition we actually gave of it [...]. But this disadvantage may be thought to be partly offset by a certain naturalness of the method; indeed to take the deduction theorem as a primitive rule is just to recognize formally the usual informal procedure (common especially in mathematical reasoning) of proving an implication by making an assumption and drawing a conclusion.

What gives "the deduction theorem" (i.e., $\to\mathbf{I}$) its "less elementary character" is precisely that in it a conclusion is drawn, not from one or a fixed number of formulas, but from a more complex structure.

consequences valid in virtue of these definitions. Subsequent writers have expanded on this (e.g., Belnap, 1962; Prawitz, 1979; Dummett, 1978, 1993; Brandom, 1994, 2000; Peregrin, 2008; Read, 2010), and the idea (which can be made fairly precise for logical operators and seems temptingly workable at least in the context of Intuitionistic logic) has been extended into a general program in the philosophy of language: Inferentialism, the attempt to explain the meaning of vocabulary items as determined by, even in some sense supervenient on, inferential patterns connected with those items. This *aperçu* of Gentzen, to which nothing corresponds in Jaśkowski's paper, depends on a formal feature of his N-systems: each operator has two rules, an Introduction and an Elimination, governing it, these rules being so related as to allow derivations to be normalized: converted into derivations in which nothing is inferred by an Elimination rule from a (major) premise itself inferred by an Introduction rule. Jaśkowski's aims in formulating his systems did not require this feature, and, indeed, some of his systems lack it. In what follows we will see a number of examples of applications of the natural deduction idea in which Gentzen's pairing of Introduction and Elimination rules is not achieved. In this respect these systems can be seen as being in the tradition of Jaśkowski, but not that of Gentzen. In some cases we will also find that they are easier to formulate using the Jaśkowski-Fitch technique of subproofs than they would be to put in Gentzen's format of trees of formulas.

2.2. Jaśkowski-Friendly Logical Systems

We offer in this subsection a series of examples of logical systems that, we feel, are more in keeping with Jaśkowski's vision than Gentzen's. We think of two dimensions along which this is true: the picture in which rules for the connectives need not (versus must) follow the Int-Elim rules paradigm, and advantages of the subproof-and-reiteration picture of the development of a proof (versus proofs as tree structures).

2.2.1. Subminimal Negation

One prominent feature of Gentzen's work on natural deduction is the systematic pairing of Introduction and Elimination rules: each logical operator is governed by one rule of each kind (counting doublets like, e.g., the left and right versions of disjunction introduction or conjunction elimination as single rules) which are—in a sense made precise in the proof of the normalization theorem (cf. Prawitz, 1965)—inverses to one another. Reflecting on this pattern, he made the remark about how the operators could be thought of as defined by introduction rules, with the elimination rules simply drawing out consequences of the definitions: a remark that has since inspired the major philosophical project of "inferential semantics." This pairing of rules is far less prominent in Jaśkowski's work: he did not use the terminology of 'Introduction' and 'Elimination', and, indeed, was happy to consider systems with unpaired rules. Particularly in the light of the intense current philosophical interest in inferential semantics, it is worth noting that much later work has been done on systems of Jaśkowski's more relaxed style. Such systems, it seems fair to say, don't *define* the operators governed by the rules in the sense in which Gentzen-style rules can be said to, but they should not be scorned on that account! In at least some cases, moreover, the formulation of the rules of these systems seems simpler when when subproofs are used than it would be in a more Gentzen-ish presentation.

At least in the most familiar formulations, rules for Intuitionistic Negation don't seem to define it in a way completely parallel to that in which Gentzen's rules for positive connectives define them. Gentzen's own preference, followed by many writers, seems to have been to treat negation as a defined connective:

$$\neg\alpha =_{df} (\alpha \to \bot)$$

where $\bot$ is a propositional *Falsum* or *Absurdity* constant. And, contrasting strangely with the idea that logical operators are defined by Introduction rules, there is no Introduction rule for $\bot$[10]: its meaning is embodied in an Elimination rule, *Ex falso quodlibet* or Explosion, providing that any formula whatever may be inferred from $\bot$. But rules can also be given for negation as a primitive connective:

Negation Introduction: $\neg\alpha$ may be inferred from a subproof in which, for some formula β, both β and $\neg\beta$ are derived from the hypothesis α

Negation Elimination: Any formula whatever may be inferred from the two premises α and $\neg\alpha$.

Now these rules (which are derivable from the usual rules for implication and the Explosion rule for $\bot$ when negation is defined in terms of implication and $\bot$) don't quite fit the standard formula for Introduction and Elimination rules—the mention of negation in describing the subproof required

[10]The apparent use of an $\bot$ introduction rule in the example proof done in the Gentzen style in Section 1.2 is in fact a derived rule: the negation there is more properly $q \to \bot$, and the rule that is appealed to is in fact $\to$E.

for the Introduction rule is nonstandard—but they have as good a claim as any Gentzen-ish pair to specify uniquely the meaning of the connective they govern. Certainly they pass the standard test: if you incorporated two connectives, $\neg_1$ and $\neg_2$, into a formal language, governed by copies of these rules, $\neg_1$ and $\neg_2$ would be interderivable. But this is so only because of the assumption that absurdity implies everything! (Note that if we had two propositional constants, $\bot_1$ and $\bot_2$, each governed by an Explosion rule, they would be interderivable.)

Which leads us to a consideration of a weaker logical system. It is now standardly called Minimal Logic, after the title of Johansson's 1936 paper, in which it was proposed as a response to C.I. Lewis's worries about the "paradoxes of implication": it avoids the (counterintuitive?) principle that a contradiction implies everything. The same logic (or at least its implication-negation fragment) had, however, been proposed by Kolmogorov (1925)—as a formalization of the logic of Brouwerian Intuitionism—over a decade before Johansson's paper; and Jaśkowski, in his natural deduction paper, chose Kolmogorov's system (rather than the now-standard logic of Heyting) as his intuitionistic system. (In the terminologic of Jaśkowski's paper it is ITD, the "intutitionistic theory of deduction.") For this logic, we have the Negation Introduction rule as stated above... but *no* Negation Elimination rule. This represents a major departure from Gentzen's paradigm. It is perhaps not as often remarked as it should be that, in whatever sense Gentzen's rules define the operators they govern, the single negation rule of Minimal Logic does not define its negation operator. Negation Introduction by itself is not enough to make $\neg_1\alpha$ and $\neg_2\alpha$ equivalent when both are governed by duplicates of it. One can have a formal language, in which the positive connectives are governed by the standard intuitionistic rules, with two distinct, non-equivalent, negation operators, each governed by the rules of Minimal Logic. (This can perhaps be seen most clearly by noting that Minimal Logic is what you get if you define negation in terms of $\to$ and $\bot$, as above, but do not assume Explosion—or, indeed, anything else—about what $\bot$ means. Two propositional constants, $\bot_1$ and $\bot_2$, aren't interderivable if we don't make any assumptions about them at all!)

It is perhaps insufficiently appreciated just how radically this diverges from the Gentzenian paradigm! The single rule governing the negation operator in Minimal Logic is formally the same as the $\neg$-Introduction operator of Intuititionistic logic, so perhaps it is assumed that there can be nothing very novel about the system. A still weaker logic of negation will perhaps dramatize the point: the system, again coinciding with standard intuitionistic logic in its treatment of the positive connectives, known as *Sub*minimal

Negation.[11] Semantically, this can be thought of as defining negation, not as implication of some unique "absurd" proposition, but by reference to a *class* of absurd propositions, with a negation being counted as true if the negated formula implies *some* absurdity. On this interpretation, a number of principles holding in Intuitionistic and Minimal Logic fail. For example, $(\neg\alpha \wedge \neg\beta) \rightarrow \neg(\alpha \vee \beta)$: this holds in the stronger logics, since if α and β both imply "the" absurd, their disjunction implies it, but it is not valid in Subminimal Logic, since α and β might imply different absurd propositions, neither of which is implied by their disjunction.

Subminimal Logic can be axiomatized over Positive Intuitionisitic Logic by adding a single axiom scheme, the two-negations-added form of contraposition. Equivalently, a Jaśkowski-Fitch form of natural deduction could put the rule as:

> **Contraposition**: $\neg\alpha$ may be inferred from $\neg\beta$ together with a subproof in which β is derived from the hypothesis α.

We have stated the rule in a Jaśkowski-Fitch style. Of course, it could be given in a Gentzen style, but we find it to be considerably less simple:

> **Contraposition**$_{\mathrm{G}}$: $\neg\alpha$ may be inferred from the two formulas β and $\neg\beta$; occurrences of α as hypotheses above β may be discharged.

2.2.2. Modal Logic

The issues arising in connection with negation re-occur in considering natural deduction formulations of modal logic. The idea that a connective is somehow *defined* by its introduction rule (or by its introduction and elimination rules together) seems thoroughly implausible for modal operators: if nothing else, the existence of many different interpretations (logical, physical, temporal, epistemic...) for modal logics such as S_4 and S_5 implies that

[11] Cf. Hazen (1992, 1995); Dunn (1993). A semantic description can be found in (Curry, 1963, pp. 255, 262), where it is proposed that a formalized theory might include, in addition to its axioms, some set of specified counteraxioms, with the negation of a given formula being counted as a theorem if one or another of the counteraxioms is derivable from it in the theory. Curry did not, however, name or specify rules for this sort of negation: the weakest logic of negation treated in detail is Minimal Logic, obtained by adding a new Falsum constant that is assumed to follow from any counteraxiom. The logic was considered, apparently without reference to Curry, around 1990 by I.L. Humberstone, who conjectured its axiomatization and proposed the proof of its completeness as a problem to Hazen. Hazen (1992) gives two completeness proofs, one by a suitably modified canonical model and the other proof-theoretic, obtained by embedding the logic in a quantified Positive Logic, quantified variables being thought of as ranging over "counteraxioms."

the meanings of the modal operators are not determined by the logical rules governing them. (Which is not to say that the search for an "inferential" semantics of some more general kind for modal notions is hopeless: just that the simple approach suggested by Gentzen is insufficient.) The demand for "neatly balanced" introduction and elimination rules, therefore, seems to lose much of its philosophical force in connection with them. It is perhaps unsurprising that the first writer on natural deduction systems for modal logics, Fitch, claimed to be inspired by *both* Gentzen and Jaśkowski; certainly he was happy to stray from Gentzen's strict path of Int-Elim balance.[12]

As has often been noted, the logical behavior of modal operators shows analogies with that of quantifiers. The rules for a necessity operator (putting aside questions of what the "squares" of logical, physical, temporal, epistemic, deontic, doxastic . . . logics have in common!) will resemble those for a universal quantifier. We have (subscripted for the Jaśkowski/Fitch systems and the Gentzen systems)

> **Necessity Introduction$_{\mathrm{J-F}}$**: $\Box\alpha$ can be inferred from a subproof having no hypothesis, but containing α as a line (with some restrictions on what can be "reiterated" into the subproof, see discussion in the next subsection).

or

> **Necessity Introduction$_{\mathrm{G}}$**: $\Box\alpha$ can be inferred from α (with some restrictions on the undischarged hypotheses allowed above α).

For an elimination rule, we take the principle *ab necesse ad esse valet consequentia*:

> **Necessity Elimination**: α may be inferred from $\Box\alpha$.

So far, so neat. But suppose we generalize from the alethic modal logics to such weaker ones as deontic logics. Here necessity is thought of as obligation or requiredness, and we can't assume that what ought to be is (Adam and Eve *shouldn't have eaten the apple*. . .). We must (as in Fitch, 1966) replace the elimination rule with the weaker

> **Deontic Necessity Elimination**: $\Diamond\alpha$ may be inferred from $\Box\alpha$.

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[12]Starting in the 1960s, many natural deduction systems for many modal logics were published, largely by logicians based in philosophy departments. For an encyclopedic survey, see Fitting (1983). A nice classification of approaches to natural deduction in modal logics can be found in (Indrzejczak, 2010, Chapters 6–10). This book contains much else that is of interest about natural deduction.

Going further afield, a natural deduction system for the weak modal logic K will (like Minimal Logic's version of negation) have an introduction rule but no elimination rule at all for the modal operator!

Possibility is, in the same way, analogous to existential quantification. Due to the lesser expressive power of the modal language, however, strictly analogous rules are harder to find, and the obvious ones have (less obvious!) weaknesses. Possibility Introduction (for alethic modalities) is simple enough:

> **(Alethic) Possibility Introduction**: $\Diamond\alpha$ may be inferred from α.

Possibility Elimination is more problematic. The guiding idea is that whatever follows logically from a possible proposition must itself be possible, and this can be embodied in the rule

> **Possibility Elimination**$_{\mathrm{J-F}}$: $\Diamond\beta$ may be inferred from $\Diamond\alpha$ together with a subproof (with the same restrictions on reiteration as for Necessity Introduction subproofs) having α as a hypothesis and β as a line.

(Note that this is perhaps not strictly speaking an elimination rule, as the conclusion as well as the premise has to be governed by a possibility operator: it is analogous to the rule for Subminimal negation.) This pair of rules for possibility is pleasingly similar to the rules for the existential quantifier[13], but fails to yield a number of desirable derivabilities:

i) It does not, in combination with standard classical rules for non-modal logic, allow the derivation of $\neg\Diamond\bot$. (This is easy to see: the rules are validated by an interpretation on which every proposition is considered possible!)

ii) As a consequence, the combination of these rules, classical rules for negation, and the definition $\Box\alpha =_{df} \neg\Diamond\neg\alpha$ does not allow us to derive the rule of Necessity Introduction. (This contrasts with the situation

[13]In fact, the Possibility Elimination rule is more nearly parallel, not to the standard Existential Quantifier Elimination rule, but to a simplified version (mistakenly believed to be "the" rule by some elementary students): $\exists xF(x)$ may be inferred from $\exists yG(y)$ together with a subproof (with *eigenvariable* a) in which $F(a)$ is derived from the hypothesis $G(a)$., This rule can perhaps be seen as a version of the method of *ecthesis* from Aristotle's *Prior Analytics*. When categorical sentences are formalized, in the usual way, in First Order Logic, this rule can replace Existential Quantifier Introduction in the proof of the validity of valid syllogisms, but outside syllogistic it has weaknesses paralleling those of Possibility Elimination.

when necessity is taken as primitive: the necessity rules, classical negation rules, and the definition $\Diamond\alpha =_{df} \neg\Box\neg\alpha$ yield both possibility rules as derived rules.)

iii) Similarly, when both modal operators are taken as primitive, the four rules do not, in combination with classical negation rules, suffice to prove the equivalences $\Box\alpha \dashv\vdash \neg\Diamond\neg\alpha$ and $\Diamond\alpha \dashv\vdash \neg\Box\neg\alpha$.[14]

iv) Even in a modal logic based on intuitionistic, rather than classical, logic, it seems desirable to have possibility distributing over disjunction: $\Diamond(\alpha \vee \beta)$ ought to be equivalent to $\Diamond\alpha \vee \Diamond\beta$, but the possibility rules (together with standard disjunction rules) don't allow us to infer the latter from the former (as noted on Fitch, 1952, p. 73).

Natural deduction systems of modal logic based on these rules do not, therefore, have all the nice metatheoretic properties Gentzen found in nonmodal natural deduction. Supplemented by some rule or rules to compensate for the weaknesses enumerated (Fitch, 1952 adds the equivalences of (iii) as additional primitive rules), however, they provide a simple and efficient way to formalize intuitively plausible reasoning: Jaśkowski's more modest goal for natural deduction.

2.2.3. Advantages of Subproofs with Restrictions on Reiteration

In the discussion just concluded, we argued that in modal logics we should not (in general: there's an exception coming up!) expect Introduction and Elimination rules to pair off as Gentzen's do, but didn't give any positive reason for preferring the Jaśkowski-Fitch subproof-and-reiteration format for natural deduction rules. This format does, however, have distinct advantages over Gentzen's for modal logics. The rule of "Reiteration", in its simplest form, permits a formula, occurring as a line of a (sub)proof above a given (sub)subproof, to be repeated within the given (sub)subproof. This version of the rule suffices for the formulation of rules for the standard, nonmodal, operators of propositional logic. More complicated versions of Reiteration are used elsewhere. For example, the rules for $\forall$**I** and $\exists$**E** make use of subproofs that are (in Fitch's terminology) *general with respect to* a parameter or *Eigenvariable*, and for such subproofs Reiteration is restricted: a formula

[14] These weaknesses of the possibility elimination rule (and of analogous sequent-calculus rules) were overlooked by some early writers on natural deduction for modal logics: cf. Routley, 1975).

may be reiterated into a general subproof only if it does not contain the Eigenvariable.[15]

A fairly obvious way to do this would be to have a restriction that a formula may only be reiterated into a strict subproof if it begins with a necessity operator. (This version, assuming the "alethic" rule of $\Box$**E**, gets you S_4.) The corresponding proviso in a Gentzen-style natural deduction system would be that all undischarged hypotheses above a $\Box$**I** inference must begin with $\Box$ operators: there isn't too much to choose between them here, though proof checking might be a bit easier with the Fitch version, since in the Gentzen version you have to check which hypotheses are discharged. But a wide variety of modal logics can be given Fitch-style formulations simply by modifying the restriction on Reiteration into strict subproofs. For S_5, allow formulas beginning with $\Diamond$ operators to be reiterated. For T, allow only formulas starting with a $\Box$ operator, but delete the first necessity operator from the reiterated copy. For B, allow any formula to be reiterated, but prefix the reiterated copy with a $\Diamond$ operator.[16]

In at least some cases, the flexibility of Reiteration-with-restrictions seems to allow more straightforward derivations. In S_5, the formula

$$\alpha \rightarrow \Box\Diamond\alpha$$

is valid. Its proof in a Gentzen-style system, with restrictions on the form of undischarged hypotheses allowed above $\Box$**I** inferences, however, is . . . roundabout. One cannot simply use $\rightarrow$**I** with the hypothesis α: the restriction on hypotheses would prevent the derivation of $\Box\Diamond\alpha$. One must, instead, derive two conditionals, $\alpha \rightarrow \Diamond\alpha$ and $\Diamond\alpha \rightarrow \Box\Diamond\alpha$, and then derive the desired conclusion from them by the non-modal rules supporting transitivity of implication: the available proof, then, is not a "normal" derivation.[17] In

[15]The analogous rules ($\Box$**I** and $\Diamond$**E**) in modal logic similarly employ *strict* subproofs. Thinking of the modal operators as generalizing over "possible worlds" motivates corresponding restrictions on Reiteration into strict subproofs: the restriction on general subproofs prevents the *Eigenvariable* from being confused with a name for any particular object, and we want, similarly, to prevent assertions about the actual world from being treated as holding about arbitrary worlds.

[16]The versions giving T and S_4 can be found in Fitch (1952); they and the version for S_5 are stated and proved correct in an Appendix to Hughes and Cresswell (1968). The version giving B is in Fitting (1983) and Bonevac (1987).

[17]Corcoran and Weaver (1969) can be taken as presenting such a Gentzen-style system, though it is formally a purely metalinguistic study and does not explicitly formulate a natural deduction system. (Prawitz, 1965, p. 60) contains a compressed discussion of the relations between natural deduction systems for S_5 analogous to that in Corcoran and Weaver and to the one described in our text.

a Fitch-style system, we can use α as the hypothesis of a $\rightarrow$**I** subproof, infer $\Diamond\alpha$ by $\Diamond$**I**, and then reiterate this into an inner, strict, subproof to obtain $\Box\Diamond\alpha$ by $\Box$**I**. To obtain the same advantage with a Gentzen-style system one would need some a restriction along the lines of

> Any path leading up from a $\Box$**I** inference to an undischarged and non-modalized hypothesis must pass through some modalized formula

But this is surely less elegant![18]

2.2.4. Natural Deduction for Other Intensional Logics

Reiteration with restrictions seems to have similar advantages in connection with intensional logics other than the modal ones. We do not here survey more than just one, but it is suggestive of the fact that the linear method of Jaśkowski-Fitch has advantages over the tree method of Gentzen.

Thomason (1970) presents a natural-deduction formulation of the Stalnaker-Thomason logic of conditionals. The standard $\rightarrow$**E** rule for conditionals —modus ponens—is postulated, along with a $\rightarrow$**I** rule using special subproofs: A conditional $\alpha > \beta$ may be inferred from a (special) subproof with α as hypothesis and β as a line. There are four conditions under which a formula may be reiterated into a special subproof[19]; the basic one (which would be applicable to a wide range of conditional logics, e.g. logics of conditional obligation) is that γ may appear as a reiterated line of a special

[18] In fact, a further liberalization of the restriction on reiteration into strict subproofs allows a very elegant formulation of propositional S_5. Call a formula *fully modalized* if every occurrence of a propositional letter in it is in the scope of some modal operator. Now allow all and only fully modalized formulas to be reiterated into strict subproofs, and strengthen the Possibility-elimination rule to

> A fully modalized formula β maybe inferred from $\Diamond\alpha$ and a strict subproof with α as its hypothesis containing β as a line.

This formulation maximizes the formal parallel between modal and quantifier rules; it also allows, e.g., the derivation of the $\Box$ rules from the $\Diamond$ and $\neg$ rules when $\Box$ is taken as a defined operator. It doesn't *quite* allow a full normalization theorem: we may still sometimes have to use $\Diamond$**I**, on a non-fully-modalized formula to allow its reiteration into a strict subproof and then use $\Diamond$**E**, inside that subproof. It can be shown, however, that only this limited "abnormality" is required: derivations can be put into a normal form with the weak subformula property that every formula occurring in them is either (i) a subformula of the conclusion or of one of the premises, or (ii) the negation of such a formula, or (iii) the result of prefixing a single possibility operator to a formula of one of the first two sorts.

[19] One of them yields the principle of "conditional excluded middle." Alas, it is needed for other things as well, so a system for the logic of Lewis, 1973, which does not contain this principle, cannot be obtained simply by dropping it without replacement.

subproof with hypothesis α if $\alpha > \gamma$ occurs as an earlier line of the containing proof. (Another has the effect of guaranteeing that equivalent formulas are substitutable as antecedents, another that "necessary" formulas may be reiterated.)

A corresponding Gentzen-style system is doubtless formulable, but we doubt it would be as simple or intuitive.

2.2.5. Natural Deduction for Free Logic

Jaśkowski first introduces quantifiers binding propositional variables, yielding what he calls the *extended theory of deduction* (= what Church, 1956 calls the *extended propositional calculus*). $\forall$**E** (Jaśkowski's Rule V) allows any instance of a universal quantification to be inferred from it. His Rule VI is essentially the same as Gentzen's formulation of $\forall$**I**: an instance, $F(q)$, may be followed by its universal quantification, $\forall q F(q)$, provided that the variable q is not one occurring freely in any hypothesis in effect at that point in the deduction. He does not give rules for the existential quantifier, treating it as defined. Fitch (1952), however, does give rules for the existential quantifier. $\exists$**I** allows, simply, the inference of an existential quantification from one of its instances, but $\exists$**E** involves reasoning from a hypothesis, and so—in what we have been calling the Jaśkowski-Fitch form of natural deduction—the erection of a new subproof:

> $\exists$**E**: a formula, α, may be inferred from a pair of items, the existentially quantified formula $\exists q F(q)$ and a subproof having the instance $F(s)$ as hypothesis and α as an item, provided that the free propositional variable s (the *eigenvariable* of the subproof) does not occur free in α or in any formula from outside the subproof appealed to within it: any formula, that is, which is reiterated into the subproof.

Given this formulation of the $\exists$**E** rule, it was natural for him[20] to reformulate $\forall$**I** in a way that makes use of a similar subproof:

> $\forall$**I**: a universal quantification $\forall q F(q)$ may be inferred from a *categorical* (i.e. hypothesis-less) subproof containing its instance $F(q)$ as an item, provided that the free variable q (the *Eigenvariable* of the subproof) does not occur free in any formula from outside the subproof

[20] Though not strictly forced on him: various elementary textbooks, for example Bergmann et al. (2008) and others that are displayed in Pelletier and Hazen (2012), combine a Fitch-like $\exists$**E** rule, using a subproof, with a Gentzen or Jaśkowski-like formulation of $\forall$**I**, with no subproof. The interderivability of the rules for the two quantifiers, in classical logic, is certainly easier to see, however, when they are given parallel formulations.

appealed to within it: any formula, that is, which is *reiterated into* the subproof.

In what follows we will consider this, rather than the more Gentzen-like formulation of Jaśkowski's original paper, as the Jaśkowski-Fitch rule: we claim that various modified forms of the quantifier rules are simpler and more perspicuous when we use it than when we use the Gentzen formulation.[21]

Having proved basic properties of the system with propositional quantification, Jaśkowski, in the final section of his paper, introduces individual variables to give a formulation of the *calculus of functions* (= First Order Logic). He notes that adopting the rules without change would give "a system. . . differing from those of *Principia Mathematica* and of Hilbert only in" its different set of well-formed formulas. He then complains that in this system if would be possible to prove

$$\forall x F(x) \rightarrow \neg\forall x \neg F(x),$$

with the meaning "If for every x, $F(x)$, then for some x, $F(x)$." But this, he says, fails in the empty domain (Jaśkowski: "the null field of individuals"): "under the supposition that no individual exists in the world, this proposition is false." And he thinks it better to have the existence of individuals settled by non-logical theories rather than written into the rules of logic itself.[22] And so he goes on to give modified rules for what is now called

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[21] *Obiter dictum*, an unusual feature of Fitch (1952) is that it treats modal logic before quantifiers. Were modal logic a more important or interesting part of the logic curriculum, this might have pedagogical value: students can get used to the idea of special subproofs with restrictions on reiteration before they have to master the complexities of substitution for variables.

[22] It took a surprisingly long time for the logical community to come to terms with this issue. As early as 1919, [Russell, 1919, p. 203] remarked that he had come to regard it as a "defect in logical purity" that the axioms of *Principia Mathematica* allowed the proof that at least one individual exists. Most logicians were willing to tolerate the defect: if one is interested in the metamathematics of a formalized theory that requires a nonempty domain, it is hardly a major worry if the existence of an object can be proven without appeal to the non-logical axioms! By the early 1950s, however, several logicians turned their attention to axiomatizing First Order Logic in a way that did not require nonemptiness of the domain. The best-known effort in this direction is Quine (1954). Quine cites the earlier Church (1951); Mostowski (1951); Hailperin (1953), but not Jaśkowski's much earlier formulation, though Mostowski (1951) does. (The philosophical community outside mathematical logic was even slower. In 1953 the British philosophical journal *Analysis* proposed an essay competition on the topic of whether the logical validity of $\forall x(F(x) \vee \neg F(x))$ entails the existence of at least one individual: the winning responses, Black (1953); Kapp (1953); Cooper (1953), discuss the distinction between natural language and the formalism of logic, but show no awareness that the problem can be avoided by making a minor change in that formalism!)

inclusive First Order Logic: First Order Logic with validity defined as truth in all domains, empty as well as non-empty.

The revised formulation of $\forall$**I** makes use of special subproofs. These subproofs start with a declaration that a certain variable—in effect the *Eigenvariable* of the subproof—is to be treated as if it were a constant in that subproof: in particular, it may be substituted for the universally quantified variable in an inference by $\forall$**E**. $\forall$**I** allows a universal quantification, $\forall xFx$, to be written, not *in*, but *after* a subproof with *Eigenvariable* a containing $F(a)$ as a line. In Fitch's terminology: the universal quantification, rather than being a direct consequence of a formula, is a direct consequence *of the subproof.* Given that formulas containing free variables are of use only as part of the machinery of quantifier rules, we can simplify the statement of the rules of the system a bit:

(i) The actual Introduction and Elimination rules for the Universal Quantifier (and also for the Existential Quantifier if we want to include it as a primitive of the system) are *exactly* as they are in ordinary, non-inclusive, First Order Logic, but

(ii) Formulas in which variables occur free may *only* occur within $\forall$**I** (or $\exists$**E**) subproofs of which they are *eigenvariables*, either as lines of these subproofs or as lines of further subproofs subordinate to them.

The derivation of the suspect $\forall xF(x) \rightarrow \exists xF(x)$:

1	$\quad\vert\ \forall xFx$	hypothesis
2	$\quad\vert\ F(a)$	$\forall$**E**
3	$\quad\vert\ \exists x(F(x))$	$\exists$**I**
4	$\forall xF(x) \rightarrow \exists xF(x)$	1-3, $\rightarrow$**I**

violates the restriction: the subproof 1-3 is an *ordinary* subproof, with no *eigenvariable*, but contains a formula, 2, in which the variable a occurs free. It is easy to see that the restricted system is sound for the inclusive interpretation: the only way to get something out of an $\forall$**I** subproof is to make an inference from it by the $\forall$**I** rule, and this will yield a formula beginning with a (perhaps vacuous!) universal quantifier.[23] But all universal quantifications

[23] There is an annoying technical issue with vacuous quantification and inclusive logic. If the well-formedness definition permits vacuous quantifiers—the simplest option—then

are trivially true in the empty domain.[24] It is, of course, possible to formulate a corresponding restriction on Gentzen-style proofs: any path through the tree leading up from the root to a hypothesis containing a free variable must pass through a formula inferred by an $\forall$**I** (or $\exists$**E**) inference having that variable as *eigenvariable.* But it seems to us that the formulation—due essentially to Jaśkowski—in terms of subproofs is more perspicuous (and perhaps, graphically, makes proofs easier to check).

Note, however, that the restriction applies only to free *variables*: if the system is used with a language containing individual constants, it once again becomes possible to prove the existence of at least one individual, as constants are assumed to denote objects in the domain over which the quantified variables range. In the late 1950s and 1960s, several logicians[25] developed systems avoiding *this* defect in logical purity. Versions of First Order Logic in which constants are allowed not to denote are called *free logics* (logics, that is, that are free of existential presuppositions on their constants). As we have seen, *inclusive* logic doesn't have to be *free*, and free logics do not always allow the empty domain, but the two modifications to standard First Order logic seem to be in a similar spirit, and the most natural systems seem to be those which are both inclusive and free: what are called *universally free logics.* Here it seems simplest[26] to enrich the language with an *existence predicate*: for any term, t, whether constant, variable[27], or a complex term built up from these by using function symbols[28], $E!(t)$ is interpreted

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the most natural interpretation, as argued by Quine (1954), is to count all formulas beginning with universal quantifiers as true and all beginning with existential quantifiers as false. But then the elimination of a vacuous universal quantifier can lead from truth to falsity: $\forall x \exists y(Fy \rightarrow Fy)$ is true and $\exists y(Fy \rightarrow Fy)$ false in the empty domain. The necessary restriction on the inference rules can be brought under the letter of the statement in the text by saying that universal quantifier elimination is *always* instantiation to a term, and that the term involved is deemed to "occur" in a subproof even if it has no occurrence in the conclusion of the $\forall$-elimination inference (and similarly for $\exists$-introduction).

[24]Systems of this sort are simple, and apparently natural, as witness the fact that they have been repeatedly re-invented by different authors: cf., e.g., Wu (1979).

[25]A representative few sources—we make no claims of completeness for the list—would include Leonard (1957); Hintikka (1959); Leblanc and Hailperin (1959); Rescher (1959); Lambert (1963).

[26]There are alternatives. In First Order Logic with Identity we could *define* $E!(t)$ as $\exists x(t = x)$.

[27]Some authors have treated free variables and individual constants differently, but—at least if the free logic is to be incorporated as part of a modal logic with quantification over contingent existents—it seems better to give them a common treatment, as here.

[28]Logicians concerned with metaphysical applications speak of an *existence* predicate. Those concerned with formalizing a theory of partial functions for use in computer science

as meaning that t denotes some object in the domain of quantification. All the quantifier rules now get minor amendments involving this. $\forall$**E** and $\exists$**I** require an extra premiss, stating that the term replacing the quantified variable denotes. $\forall$**I** subproofs now have a hypothesis, and $\exists$**E** subproofs a second hypothesis, saying that the *eigenvariable* denotes. A system of this sort (as part of a modal system) is presented in Hazen (1990). Our earlier formulation of inclusive logic can be seen as an abbreviated special case of this system: if all constants are assumed to denote, existence premises for constants can be left tacit, and since all *eigenvariable*-possessing subproofs have existential hypotheses for their *eigenvariables*, we can save lines by leaving them tacit as well.

Mainstream mathematical logicians have tended not to feel the need for free logic, preferring to consider formalizations of mathematical theories in languages defined so that every term denotes an object in the structure described. Still, there are applications, such as modal logic and the theory of partially computable functions, in which the logical flexibility of free logic is convenient. There are also contexts in which *many-sorted* logics are useful, and the machinery of free logic can be used for them: use multiple "existence" predicates, each saying that a term denotes an object of one of the sorts.[29] And, though it is surely possible to design a Gentzen-style system for universally free logic, it seems to us that the Jaśkowski-Fitch-style described above is more perspicuous and more convenient in use.[30]

2.3. Sheffer Stroke Functions in Classical Logic

Sheffer (1913) is usually credited with "discovering" the truth-functional connective now popularly known as NAND, and remarked in a footnote there there was also a function we now call NOR. Of course, both these functions had already made their appearances in earlier logicians, but the name 'Sheffer' has stuck.

Price (1961) gives these three rules for NAND, symbolized as | (their names subscripted with 'P' to indicate Price):

$|\mathbf{I}_P$: From a subproof $[\alpha] \cdots (\beta|\beta)$, infer $(\alpha|\beta)$

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might prefer to speak of a *definedness* or even a *convergence* predicate. The logics can be the same.

[29] Hailperin's formulation of inclusive logic, in Hailperin (1953) similarly gains importance as a preliminary to Hailperin (1957).

[30] Garson (2006) contains a number of natural deduction systems (and also tableaux methods) for both "normal" modal logics and also free logics, done in a student-oriented manner.

$|\mathbf{E}_P$: From the two formulas α and $(\beta|\alpha)$, infer $(\beta|\beta)$

$||\mathbf{E}_P$: From the two formulas α and $((\beta|\beta)|\alpha)$, infer β

Price shows that these three rules are independent of one another, and are "complete" for the classical propositional logic. Note that the third rule clearly violates the Int-Elim picture. This of course is to be expected, since we are dealing with classical logic and *any* complete set of rules will somewhere involve a violation of this ideal. But a consequence of this is that the goal of having all rules matched as Int-Elim rules cannot be given. The Jaśkowski-Fitch method, however, has no such difficulties; this seems to be yet another place where the formalism of Jaśkowski-Fitch is superior to that of Gentzen.

In the classical propositional logic, where $\uparrow$ (NAND) and $\downarrow$ (NOR) were introduced, either of these functions could be employed as a complete foundation for the logic. But it is easy to see that there can be no pure Int-Elim rules for either one of them: there will need to be *some* rule like Price's $||\mathrm{E}_P$ rule that does not merely eliminate the main occurrence of $|$. In turn, this suggests that the picture offered by inferentialism falls short.

Here's another set of rules in the same vein, again illustrating the issue that using the usual types of natural deduction rules is going to involve us in something that violates the inferentialist's desired form of rules. We start by defining *explicit contradiction*, **e.c.**, as a three (or two) member set of formulas containing, for some α and β, (i) α, (ii) β, and (iii) $\alpha|\beta$[31] An **e.c.** is derivable from given hypotheses off all its members are. (We subscript these rules with '1'):

$|\mathbf{I}_1$: From a subproof with two hypotheses, α and β, that contains an **e.c.**, infer $(\alpha|\beta)$ in the superordinate proof.[32]

$|\mathbf{E}_1$: From the formula $(\alpha|\beta)|(\alpha|\beta)$, both α and β can be inferred.[33]

$||\mathbf{Transfer}_1$: From the two formulas α and $(\alpha|\beta)$, infer $(\beta|\beta)$[34]

As can be easily seen, both the $|\mathrm{E}_1$ and $||\mathrm{Transfer}_1$ rules violate the form of rule that inferentialism requires. Is it at all possible for there to be a set of

[31] With, for any α, the two-member set $\{\alpha, (\alpha|\alpha)\}$ as a special case.

[32] In the special case where $\alpha = \beta$, this is just the standard Negation Introduction (Reductio) rule.

[33] Given the definition of $\wedge$, this is just Conjunction Elimination. For the special case where $\alpha = \beta$, this is just Double Negation Elimination.

[34] A logically equivalent (in the context of the other rules) version of $||\mathbf{Transfer}_1$, but which may be more easy to employ in proofs, would conclude $(\beta|\gamma)$ from the same premises.

rules for | that are inferentialism-acceptable? Before answering this question, and before moving to the case of Sheffer strokes in intuitionist logic, we pause for a parenthetical piece of background.

3. Generalized Natural Deduction: Case Studies

3.1. Generalized Natural Deduction

Schroeder-Heister (1984a,b) gave a reformulation of natural deduction rules where, besides allowing single formulas to be hypotheses of a subproof, statements that some non-logical inference held, or was valid, were also allowed to be hypotheses. This employs a structural generalization of ordinary natural deduction: we can have subproofs in which, instead of a formula, a "rule" is hypothesized. Fredric Fitch, in his 1966 paper on natural deduction rules for obligation, introduces notational conventions about "columns" that have much the same effect (see footnote 38 below). Schroeder-Heister calls this method of representing natural deduction "generalized natural deduction". Notationally, the Jaśkowski-Fitch format lends itself to this generalization more readily than Gentzen's: his notation for generalized Gentzen-style natural deduction is very awkward, and it is hard to avoid the suspicion that most of the time he worked with the analogous "generalized" Sequent Calculus (with "higher order" sequents, having sequents as well as formulas in the antecedent), which he also defines. In contrast, the notation of Fitch (1966) is easy to read and use in the actual construction of formal derivations.

A simple example of the idea might be the following intuitive argument. Suppose we are given: β follows from α. We certainly seem justified in concluding: from α or γ we can conclude either β or γ. Yet the standard way of expressing this would be the proof

1	$(\alpha \to \beta)$	premise
2	$\alpha \vee \gamma$	premise
3	$\quad \alpha$	hypothesis
4	$\quad (\alpha \to \beta)$	1, Reiteration
5	$\quad \beta$	4,3, $\to E$
6	$\quad \beta \vee \gamma$	5, $\vee I$
7	$\quad \gamma$	hypothesis
8	$\quad \beta \vee \gamma$	7, $\vee I$
9	$\beta \vee \gamma$	2, 3-6, 7-8 $\vee E$

But this proof supposes that $(\alpha \rightarrow \beta)$ is the regimentation of the argument "β follows from α"—and we have been long taught that this is a bad identification! A better regimentation would allow that "β follows from α" is an *argument* (a non-logical one, being assumed by the larger argument), so that a more appropriate regimentation of the reasoning would be:

1	α	(Hypothesized
2	β	inference)
3	$\alpha \vee \gamma$	premise
4	α	hypothesis (for $\vee E$)
5	α	1-2, hypothesized argument
6	β	Reiterated)
7	β	4, 5-6, "column elimination"
8	$\beta \vee \gamma$	7, $\vee I$
9	γ	hypothesis (for $\vee E$)
10	$\beta \vee \gamma$	9, $\vee I$
11	$\beta \vee \gamma$	4, 5-8, 9-10, $\vee E$

Here we see just what is in the informal presentation. We hypothesize (lines 1 and 2) some non-logical inference; we correctly infer $\beta \vee \gamma$ from that inference's conclusion; and so we have that the argument from α to $\beta \vee \gamma$ is shown.

We will give more details concerning this generalized natural deduction in our remarks about the issue of definability of connectives in intuitionist logic, followed by a discussion the Sheffer stroke… first in classical logic and then in intuitionist logic.

3.2. Humberstone's Umlaut Function[35]

The possibility of characterizing a logical operator in terms of its Introduction and Elimination rules has made possible a precise formulation of an interesting question. One of the properties of classical logic that elementary students are often told about is *functional completeness*: every possible

[35] This section closely follows a discussion that appears in Pelletier and Hazen (2012).

truth-functional connective (of any arity) is explicitly definable in terms of the standard ones. The question *should* present itself of whether there is any comparable result for intuitionistic logic. But this can't be addressed until we have some definite idea of what counts as a possible intuitionistic connective. We now have a proposal: a possible intuitionistic connective is one that can be added (conservatively) to a formulation of intuitionistic logic by giving an introduction rule (and an appropriately matched, not too strong and not too weak elimination rule) for it. Appealing to this concept of a possible connective, Zucker and Tragesser (1978) prove a kind of functional completeness theorem. They give a general format for stating introduction rules, and show that any operator that can be added to intuitionistic logic by a rule fitting this format can be defined in terms of the usual intuitionistic operators. Unexpectedly, the converse seems not to hold: there are operators, explicitly definable from standard intuitionistic ones, which do *not* have natural deduction rules of the usual sort. For a simple example, consider the connective $\ddot{\vee}$ defined in intuitionistic logic by the equivalence:[36]

$$(\alpha \mathbin{\ddot{\vee}} \beta) =_{df} ((\alpha \to \beta) \to \beta).$$

(In classical logic, this equivalence is a well-known possible definition for disjunction, but intuitionistically $(\alpha \mathbin{\ddot{\vee}} \beta)$ is much weaker than $(\alpha \vee \beta)$.) The introduction and elimination rules for the standard operators of intuitionistic logic are pure, in the sense that no operator appears in the schematic presentation of the rule other than the one the rules are for, and $\ddot{\vee}$ has no pure introduction and elimination rules.[37] (Obviously, a system in which every connective has pure rules will have the separation property: in deriving a conclusion from a set of premisses, no rule for a connective not actually occurring in the premisses or conclusion need be used.) To get around this problem, Schroeder-Heister (1984b) can use his generalized natural deduction: subproofs may have inferences instead of (or in addition to) formulas as hypotheses.[38] In this framework we can have rules of $\ddot{\vee}I$ allowing the infer-

[36]This connective was suggested to Allen Hazen by Lloyd Humberstone.

[37]Trivially, it has *impure* rules: an introduction rule allowing $(\alpha \mathbin{\ddot{\vee}} \beta)$ to be inferred from its definiens and a converse elimination rule.

[38] Fitch (1966) had proposed a similar generalization, but used it only for abbreviative purposes. He represents an inference from $H_1, \cdots, H_n$ to C by using a notation similar to that for a subproof, but with no intermediate steps between the hypotheses and the conclusion. (Rather than using logically loaded words like *rule* or *inference*, he calls such things simply *columns*.) These diagrams can occur in a proof in any way a formula can: they can be used as hypotheses of subproofs, they may be reiterated, etc. There are two rules for their manipulation: by Column Introduction, an abbreviated column without intermediate steps can be inferred from a real subproof with the same hypotheses and last

ence of $(\alpha \ddot{\vee} \beta)$ from a subproof in which β is derived on the hypothesis that $\alpha \vdash \beta$ is valid, and $\ddot{\vee}E$ allowing β to be inferred from $(\alpha \ddot{\vee} \beta)$ and a subproof in which β is derived on the hypothesis α. Schroeder-Heister proves that any connective characterized by natural deduction rules of this generalized sort is explicitly definable in terms of the standard intuitionistic connectives, and that any connective so definable is characterized by generalized Introduction and Elimination rules of a tightly constrained form, with at most a controlled bit of impurity (see §3.4 below for details). Schroeder-Heister (1984a) proves a similar result for intuitionistic logic with quantifiers.

3.3. Generalized Natural Deduction and the Sheffer Stroke in Classical Logic

We return now to the topic of whether it is possible to give a set of rules for | that are inferentialism-acceptable. Recalling that **e.c.** means "explicit contradiction" as defined earlier (and importantly, will always contain at least one formula with | as its main connective), we start by considering a possible set of pure Int-Elim rules for an "ambiguous" (see below) | (we use the subscript '2' here).

> $|\mathbf{I}_2$: Given a subproof having either α or β as its only hypothesis and which contains an **e.c.**, infer $\alpha|\beta$.
>
> $|\mathbf{E}_2$: Given an **e.c.**, infer any formula α

It is obvious that the system defined by these rules (and standard general framework stuff about, e.g., reiteration into subproofs) is consistent, and that they can be used to add a | connective conservatively to classical or intuitionistic natural deduction systems: the rules allow normalization (in the sense of Prawitz, 1965). It should be almost as immediately obvious that they aren't complete: the elimination rule demands auxiliary premisses, so you can't always apply it to derive things that would then allow you to reintroduce the | by the introduction rule. Put another way: the rules are valid *both* for $|_1$ (defined by $(\alpha|_1\beta) =_{df} \neg(\alpha \wedge \beta)$) and for $|_2$ (defined by $(\alpha|_2\beta) =_{df} (\neg\alpha \vee \neg\beta)$), and these are not intuitionistically equivalent. As a result, the system defined by these rules does not characterize a unique connective, but rather an operator that is *ambiguous* between (at least) these two readings. See, for discussion, (Humberstone, 2011, pp. 605–628).

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line, and by Column Elimination the conclusion of a column may be inferred from the column together with its hypothesis or hypotheses. The reader is referred to Fitch's paper for further discussion and examples.

Still, the distinction between the two readings is collapsed in classical logic, so there might be a lingering hope that the rules might be, in some sense, sufficient in a classical context. They certainly don't, all by themselves, give a complete classical logic of $|$. After all, they are sound for the intuitionistic $|_1$ connective, and there are classical principles concerning $|$ which do not hold for this intuitionistic connective. (For example: if θ is derivable from α and also derivable from $\alpha|\alpha$, then θ may be asserted—a stroke-analogue of an excluded middle rule.) So the hope, if there is one, must be for using these $|$-rules in a system which is, by other means, forced to be classical. Here's an example: Suppose we add the $|$, governed by these rules, to a natural deduction system for the classical logic of, say, $\wedge$ and $\neg$. We would then have a complete classical system for $\wedge, \neg$, and $|$. (Proof: the system is strong enough to prove the equivalence of $(\alpha|\beta)$ with $\neg(\alpha \wedge \beta)$.)

But this result—call it *parasitic completeness*—isn't very exciting. It would be nicer to have a result that didn't depend on using other connectives. Well, one way would be to add a *third* rule for $|$. There are various natural deduction systems for $|$ in the literature (two were described above in §2.3), and they all have more than two rules. For instance, rules similar to these two, plus another rule. But we think this isn't the most satisfying way to go: for one thing, it sacrifices one of the nice features of (Gentzen-style) natural deduction, the pairing of Int-Elim rules.

Starting with a system that is sound on an intuitionistic interpretation, there can be several different ways of strengthening it into a classical system, adding new rules for any of a variety of operators.

Here are five such rules (convention: enclosing a formula in square brackets indicates that it is an assumption, and the vertical dots beneath an assumption show a proof that leads to some formula. The horizontal line then says that if there is such a subproof, one can infer the formula below the line.)

$$\frac{\begin{matrix}[\alpha] & & [\neg\alpha] \\ \vdots & & \vdots \\ \beta & & \beta\end{matrix}}{\beta}\ (\text{LEM}) \qquad \frac{\neg\neg\alpha}{\alpha}\ (\neg\neg E) \qquad \frac{\begin{matrix}[\neg\alpha] \\ \vdots \\ \beta \\ \vdots \\ \neg\beta\end{matrix}}{\alpha}\ (\text{Indr.Pr.}) \qquad \frac{\begin{matrix}[\alpha \to \beta] \\ \vdots \\ \alpha\end{matrix}}{\alpha}\ (\text{Peirce})$$

$$\frac{\begin{matrix}[\neg\alpha] \\ \vdots \\ \neg\beta\end{matrix}}{\beta \to \alpha}\ (\text{Contrapose})$$

Gentzen himself employed the *Law of the Excluded Middle*, LEM, which is the left-most of these five rules, although he also mentioned that double-negation elimination ($\neg\neg E$) could also be used. The negation-eliminating version of a reductio proof ($\neg E$) is a popular addition to the pure Int-Elim rules in many elementary logic textbooks. Peirce's Law and the displayed version of a contraposition law can also yield classical logic. Any of these rules could easily be (and have been) added to either Gentzen's or Fitch's formulations to describe classical logic.

Maybe one could argue that in some way a formulation that plays with rules for $\neg$ is more fundamental than others, but it is hard to see how. Anyway, in order to get separation, the rule that is added to | can't involve either $\rightarrow$ or $\neg$. Furthermore, in a sequent calculus, classicality can be achieved without any change to the rules for any connective, by a structural change: allowing multiple succedent formulas. So we would like to try to find a way to get classical logic by a rule that doesn't involve any particular connective!

Here's a possibility: For any formulas α, β, and γ: γ may be asserted if it is both derivable from the formula α and also derivable, with no particular formula as extra hypothesis, if we allow the (non-logical) inference of β from α. Or, to put it into a more Jaśkowski-Fitch style of exposition:

> **Rule $\mathbf{B}_2$**[39]: γ is a consequence of two subproofs, each containing γ as an item, one with α as hypothesis and the other with no single formula as hypothesis, but within which an additional, non-logical, rule of inferring β from α may be used.

This is a connective-free classicizing rule: added to an intuitionistic system it gives us classical logic. It is analogous to the move, in sequent calculus, to sequents with multiple succedent formulas in classicizing without postulating anything new about any particular connective. (Schroeder-Heister, 1984b showed that adding a new connective to intuitionistic logic by proper generalized Int-Elim rules would always yield a definitional extension of intuitionistic logic. Rule $\mathbf{B}_2$ is a generalized rule which is neither an Introduction nor an Elimination rule for any connective: we have obtained classical logic by a non-Schroeder-Heisterian application of the Schroeder-Heister framework.) To illustrate how the rule works, consider the following derivation of α from $\neg\neg\alpha$, using Rule $\mathbf{B}_2$ but only intuitionistically acceptable rules for the connectives:

[39]'B' for Bivalence, or for Boolean perhaps.

1	$\neg\neg\alpha$	premise
2	α	assume
3	α	2, Repetition
4	α	(Hypothesized
5	$\neg\alpha$	inference)
6	α	assume
7	α	4–5, Reiteration of
8	$\neg\alpha$	hypothesized inference
9	$\neg\alpha$	6, 7–8, $\vdash E$
10	$\neg\alpha$	6–9, $\neg I$
11	$\neg\neg\alpha$	1, Reiteration
12	α	8, 9, explosion
13	α	2–3, 4–12, $\mathbf{B}_2$
14	$(\neg\neg\alpha \rightarrow \alpha)$	1–13, Conditional proof

A system having the rules $|\mathbf{I}_2$, $|\mathbf{E}_2$ and $\mathbf{B}_2$ is sound and complete for the classical logic of NAND.[40]

Of course, $\mathbf{B}_2$ is not the only way to introduce classical logic in this manner. We could instead formulate a rule related to Peirce's Law as follows:

Rule $\mathbf{P}_2$ α is a consequence of a subproof containing α as an item and having no single formula as a hypothesis, but within which an additional, non-logical, rule of inferring β from α may be used.

3.4. Generalized Natural Deduction and Sheffer Strokes in Intuitionistic Logic

Our success in giving pure rules for $\ddot{\vee}$ might lead us to conjecture that generalized natural deduction can provide pure introduction and elimination

[40] For simplicity of exposition, we forego the proofs. Soundness should be obvious. The completeness proof uses a Henkin construction and involves some subtleties—note that multiple nested applications of $\mathbf{B}_2$ allows us to encode truth tables in the derivation.

rules for all (intuitionistic) connectives. This, alas, would be a mistake. (Schroeder-Heister, 1984b instead considers connectives introduced in order, with the conventional $\{\rightarrow, \wedge, \vee, \bot\}$ at the head of the list, each connective being characterized by Int-Elim rules in whose schemata only earlier connectives may appear: he proves that these are precisely the connectives definable from the conventional ones.) If a connective is definable using only $\rightarrow$, as $\ddot{\vee}$ is, pure rules are easy to provide: the introduction rules will require subproofs like those for $\rightarrow$**I**, with hypothesized inferences when the "implication" to be introduced has another "implication" as antecedent, and the elimination rules will be similar to modus ponens, but with columns instead of minor premises if the connective is defined by a conditional with a conditional antecedent. Allowing $\wedge$ as well to occur in the definiens presents no essential difficulty: for a connective defined by a conjunction, the introduction rule will require whatever would be required by each conjunct, and the elimination rule will permit any inference that would be permitted by either conjunct.

Both the difficulties encountered in going further, and a further generalization of generalized natural deduction, can be illustrated by using the "Sheffer-connective" given in Došen (1985) as an example. (In popular usage a Sheffer Stroke for a logic is any function that will generate all the truth functions for the logic. But to also include logics that are not directly amenable to this "truth conditional" characterization, a different account can be given by means of what Hendry and Massey (1969) calls an "indigenous Sheffer function". If a function f can define some other set of connectives in a given logical system, and in turn they can define f... in whichever way is appropriate for the logical system in question, f is then said to be an indigenous Sheffer function for the logic defined by the initial set of functions.)

Došen (1985) shows that

$$\bigstar(\alpha, \beta, \gamma) =_{df} ((\alpha \vee \beta) \leftrightarrow (\gamma \leftrightarrow \neg\beta))^{41}$$

is an indigenous Sheffer-connective for intuitionistic propositional logic defined by $\{\vee, \wedge, \rightarrow, \neg\}$. Since a biconditional is equivalent to a conjunction of conditionals, a suitable introduction rule will allow $\bigstar(\alpha, \beta, \gamma)$ to be inferred from a collection of subproofs sufficient to establish the implications from

[41] We prefer to use this and similar functions, even though they employ the defined symbol $\leftrightarrow$, because (as reported in Došen, 1985) Kuznetsov (1965) has shown that there are no indigenous Sheffer functions that have less than five occurrences of variables when they employ only the symbols $\{\vee, \wedge, \rightarrow, \neg\}$ in their definitions. Using $\leftrightarrow$ we need employ only four occurrences.

left to right and from right to left. Consider first the left to right part of this. A conditional with a disjunction as antecedent is equivalent to a conjunction of two conditionals, and the right-hand side is itself equivalent to a conjunction of conditionals, so, in order to establish

$$((\alpha \vee \beta) \rightarrow (\gamma \leftrightarrow \neg\beta))$$

it would suffice to have four subproofs, which (if we were allowed to use the negation operator!) could have the forms

Preliminary I-1*: A subproof with α and γ as hypotheses containing $\neg\beta$ as a line,

Preliminary I-1: A subproof with α and $\neg\beta$ as hypotheses containing γ as a line,

Preliminary I-2: A subproof with β and γ as hypotheses containing $\neg\beta$ as a line.

Preliminary I-2*: A subproof with β and $\neg\beta$ as hypotheses containing γ as a line

Two of these, however, can be omitted. A contradiction implies anything, so **I-2*** would be trivial, and in effect **I-2** subsumes **I-1***: an **I-2** subproof in effect derives absurdity from β and γ, and **I-1*** does the same thing with an extra hypothesis, α.

So, first problem: how does one formulate these without the using $\neg$ symbol? A solution is possible because the meaning of $\neg$ can be specified logically: $\neg\phi$ means that ϕ implies any proposition whatsoever! Where, as in our **Preliminary I-1**, a negation is being used as a premise, this means it can be replaced by a collection of inferences in which whatever we need is inferred from the negated formula. So we can have

★I-1: A subproof containing γ as a line, and with, as hypotheses, the formula α and some number of inferences of other formulas from β.[42]

But what sort of $\neg$-free subproofs, in the other direction, would be equivalent to a subproof in which a negation is derived? In an infinitary logic one might have a rule requiring an infinite subproof in which every formula whatsoever is derived from the one we want to negate, but we want a rule that can be used in a real formal system! Here a further generalization of generalized

[42] In Fitch's (1966) terminology: hypothesized "columns," each having β as a hypothesis and some other formula as a line.

natural deduction is needed. Recall that, in a system with propositional quantifiers, $\neg\phi$ can be defined as $\forall p(\phi \to p)$. So, let us allow free propositional variables in our language, and allow subproofs (of the sort that would be used in the $\forall$**I** rule for propositional quantification) *general with respect to* a propositional variable: a subproof, that is, into which neither the given propositional variable nor any formula containing it as a subformula may be reiterated. Then we may have:

> ★**I-2**: A subproof, general with respect to some propositional variable p not occurring in β or γ, with β and γ as hypotheses and p as a line.[43]

(Our final ★**I** rule will be: $\bigstar(\alpha, \beta, \gamma)$ may be inferred from three subproofs, of the forms ★**I-1** and ★**I-2** described here and a ★**I-3** establishing the right to left implication. ★**I-3** is described below, after we discuss the ★**E** rules.)

The elimination rule for a connective defined as a biconditional will have multiple forms, corresponding to modus ponens for the left to right and the right to left conditionals. Let us, again, start by considering the left to right forms. Since a conditional with a disjunctive antecedent is equivalent to a conjunction of conditionals, we distinguish "subforms" for the two disjuncts of $(\alpha \vee \beta)$. So the first two forms of ★**E** can be taken as

> ★**E-1**: from $\bigstar(\alpha, \beta, \gamma)$, α and a subproof, general with respect to a propositional variable p not occurring in any of α, β or γ, in which p is derived from the hypothesis β, to infer γ, and
>
> ★**E-2**: from $\bigstar(\alpha, \beta, \gamma)$, β and γ to infer any formula whatever.

Right to left, $\bigstar(\alpha, \beta, \gamma)$ tells us that if γ is equivalent to $\neg\beta$ we may infer $\alpha \vee \beta$. Avoiding the symbol $\vee$ in a form of the elimination rule is fairly easy: we simply take the equivalence of γ and $\neg\beta$ to license any inference we could make by disjunction elimination if we had the premise $\alpha \vee \beta$. So our third form of ★**E** is

> ★**E-3**: any formula, ϕ, may be inferred from $\bigstar(\alpha, \beta, \gamma)$ together with four subproofs:
>
> — one, general with respect to a variable p, in which p is derived from β and γ,

[43] One of us recalls seeing a suggestion by Fitch of an alternative introduction rule for negation: $\neg\alpha$ is a direct consequence of a subproof, general with respect to p, in which the propositional variable p (not occurring in α) is derived from the hypothesis α. We have been unable to locate it in his publications.

— one in which γ is derived from some number of hypothesized "columns", in each of which some formula is inferred from β,

— one in which ϕ is derived from the hypothesis α, and

— one in which ϕ is derived from the hypothesis β.

Returning now to the final introduction rule for ★: in the subproof needed for the right to left case of the **★I** rule, it is not a matter of *using* $\alpha \vee \beta$, as in **E-3**, but of *establishing* it. Here again we need[44] our generalization of generalized natural deduction. (Prawitz, 1965, p. 67) notes that disjunction can be defined, in intuitionistic logic with propositional quantification, as

$$(\alpha \vee \beta) =_{\mathrm{df}} \forall p((\alpha \to p) \to ((\beta \to p) \to p))^{45}$$

Making use of this idea, we can specify the third subproof needed for **★I**:

★I-3: Subproof, general with respect to p, in which a propositional variable p, not occurring in α, β or γ, is derived from

— the hypothesized inference ("column") from α to p,

— the hypothesized inference from β to p,

— a hypothesized inference of γ from some number of hypothesized inferences of other formulas from β,

— some number of hypothesized inferences of other formulas from β and γ.

The system with just the ★-connective, governed by these **★I** and **★E** rules, is sound and complete for intuitionistic logic. It has the Gentzen-Prawitz normalizability property: if $\bigstar(\alpha, \beta, \gamma)$ is derivable by the **★I** rule, then anything inferable from it by **★E** is derivable without making the detour through the "maximum" formula. The rules specify the meaning of the connective uniquely: if we have two connectives, $\bigstar_1$ and $\bigstar_2$, governed by "copies" of the same pair of rules, $\bigstar_2(\alpha, \beta, \gamma)$ is derivable from $\bigstar_1(\alpha, \beta, \gamma)$.[46] (Hint: the three forms of the elimination rule correspond roughly to the three subproofs needed for the introduction rule, though both $\bigstar_1$**E-2** and $\bigstar_1$**E-1** are used in constructing the $\bigstar_2$**I-1** subproof.)

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[44](Schroeder-Heister, 1984b, p. 1296) notes that no set of connective-free rules of ordinary generalized natural deduction can replace the $\vee$.

[45]In the context of classical rather than intuitionistic logic, Russell (1906) states this as an equivalence (though not adopting it as a definition), as his Proposition 7.5.

[46]For discussion of the philosophical significance of these properties, see Belnap (1962).

Although some readers will like our "informal" presentation of the rules for ★, certainly other readers would prefer to see a more "formal" presentation of them. Such readers are directed to Schroeder-Heister (2014), where a formal version is given in his Section 5.

We note in passing that $\bot$ is something of an anomaly: it has an elimination rule (anything whatever can be inferred from $\bot$), but no introduction rule. We can't think of any real use for it, but those who love symmetry can use this further generalization to give one: $\bot$ may be inferred from a categorical subproof, general with respect to a propositional variable p, in which p occurs as a line.

4. Conclusion

Mathematically, Gentzen's natural deduction and Jaśkowski's "suppositional" system are essentially the same thing, and proof theorists find Gentzen's presentation more elegant. But, we hope to have convinced you, the Jaśkowski-Fitch version has certain advantages, and probably helped some later logicians discover their modifications and extensions of the natural deduction method.

Acknowledgements. We gratefully acknowledge the assistance of Andrzej Indrzejczak in carefully proofreading this paper as well as making several important suggestions. We also give thanks to Peter Schroeder-Heister for his comments on our paper. We direct the reader's attention to his paper (Schroeder-Heister, 2014) in this special issue for further related aspects of this topic.

References

[1] Belnap, N. (1962) Tonk, plonk and plink, *Analysis* 22:130–134.

[2] Bergmann, M., J. Moor, and J. Nelson (2008), *The Logic Book, Fifth Edition*, Random House, New York.

[3] Black, M. (1953) Does the logical truth that $\exists x(Fx \vee \neg Fx)$ entail that at least one individual exists?, *Analysis* 14:1–2.

[4] Bonevac, D. (1987) *Deduction*, Mayfield Press, Mountain View, CA.

[5] Brandom, R. (1994) *Making it Explicit*, Harvard University Press, Cambridge, MA.

[6] Brandom, R. (2000) *Articulating Reasons*, Harvard UP, Cambridge, MA.

[7] Cellucci, C. (1995) On Quine's approach to natural deduction, in P. Leonardi, and M. Santambrogio (eds.), *On Quine: New Essays*, Cambridge UP, Cambridge, pp. 314–335.

[8] Church, A., A formulation of the logic of sense and denotation, in P. Henle (ed.), *Structure, Method and Meaning: Essays in Honor of H.M. Sheffer*, LiberalArts Press, NY, 1951.

[9] Church, A., *Introduction to Mathematical Logic*, Princeton UP, Princeton, 1956.

[10] Cooper, N. (1953) Does the logical truth that $\exists x(Fx \vee \neg Fx)$ entail that at least one individual exists?, *Analysis* 14:3–5.

[11] Corcoran, J., and G. Weaver (1969) Logical consequence in modal logic: Natural deduction in S5, *Notre Dame Journal of Formal Logic* 10:370–384.

[12] Curry, H. (1963) *Foundations of Mathematical Logic*, McGraw-Hill, New York.

[13] Došen, K. (1985) An intuitionistic Sheffer function, *Notre Dame Journal of Formal Logic* 26:479–482.

[14] Dummett, M. (1978) The philosophical basis of intuitionistic logic, in *Truth and Other Enigmas*, Duckworth, London, pp. 215–247.

[15] Dummett, M. (1993) Language and truth, in *The Seas of Language*, Clarendon, Oxford, pp. 117–165.

[16] Dunn, Michael (1993) Star and perp, *Philosophical Perspectives: Language and Logic* 7:331–358.

[17] Fitch, F. (1952) *Symbolic Logic: An Introduction*, Ronald Press, NY.

[18] Fitch, F. (1966) Natural deduction rules for obligation, *American Philosophical Quarterly* 3:27–28.

[19] Fitting, M. (1983) *Proof Methods for Modal and Intuitionistic Logics*, Reidel, Dordrecht.

[20] Garson, J. (2006) *Modal Logic for Philosophers*, Cambridge Univ. Press, Cambridge.

[21] Gentzen, G. (1934) Untersuchungen über das logische Schließen, I and II, *Mathematische Zeitschrift* 39:176–210, 405–431. English translation "Investigations into Logical Deduction", published in *American Philosophical Quarterly* 1:288–306, 1964, and 2:204–218, 1965. Reprinted in M. E. Szabo (ed.) (1969) *The Collected Papers of Gerhard Gentzen*, North-Holland, Amsterdam, pp. 68–131. Page references to the *APQ* version.

[22] Gentzen, G. (1936) Die Widerspruchsfreiheit der reinen Zahlentheorie, *Mathematische Annalen* 112:493–565. English translation "The Consistency of Elementary Number Theory" published in M. E. Szabo (ed.) (1969) *The Collected Papers of Gerhard Gentzen*, North-Holland, Amsterdam, pp. 132–213.

[23] Hailperin, T. (1953) Quantification theory and empty individual domains, *Journal of Symbolic Logic* 18:197–200.

[24] Hailperin, T. (1957) A theory of restricted quantification, I and II, *Journal of Symbolic Logic* 22:19–35 and 113–129. Correction in *Journal of Symbolic Logic* 25:54–56, (1960).

[25] Harman, G. (1973) *Thought*, Princeton UP, Princeton.

[26] Hazen, A. P. (1990) Actuality and quantification, *Notre Dame Journal of Formal Logic* 41:498–508.

[27] Hazen, A. P. (1992) Subminimal negation, Tech. rep., University of Melbourne. University of Melbourne Philosophy Department Preprint 1/92.

[28] Hazen, A. P. (1995) Is even minimal negation constructive?, *Analysis* 55:105–107.

[29] Hazen, A. P. (1999) Logic and analyticity, *European Review of Philosophy* 4:79–110.

Special issue on "The Nature of Logic", A. Varzi (ed.). This special issue is sometimes characterized as a separate book under that title and editor.

[30] Hendry, H., and G. Massey (1969) On the concepts of Sheffer functions, in K. Lambert (ed.), *The Logical Way of Doing Things*, Yale UP, New Haven, CT, pp. 279–293.

[31] Herbrand, J. (1928) Sur la théorie de la démonstration, *Comptes rendus hebdomadaires des séances de l'Académie des Sciences* (Paris) 186:1274–1276.

[32] Herbrand, J. (1930) *Recherches sur la théorie de la démonstration*, Ph.D. thesis, University of Paris. Reprinted in W. Goldfarb (ed. & trans.) (1971) *Logical Writings*, D. Reidel, Dordrecht.

[33] Hintikka, J. (1959) Existential presuppositions and existential commitments, *Journal of Philosophy* 56:125–137.

[34] Hughes, G., and M. Cresswell (1968) *An Introduction to Modal Logic*, Methuen, London.

[35] Humberstone, L. (2011) *The Connectives*, MIT Press, Cambridge, MA.

[36] Indrzejczak, A. (2010) *Natural Deduction, Hybrid Systems and Modal Logics*, Springer, Berlin.

[37] Jaśkowski, S. (1929) Teoria dedukcji oparta na regułach założeniowych (Theory of deduction based on suppositional rules), in *Księga pamiątkowa pierwszego polskiego zjazdu matematycznego* (Proceedings of the First Polish Mathematical Congress), *1927*, Polish Mathematical Society, Kraków, p. 36.

[38] Jaśkowski, S. (1934) On the rules of suppositions in formal logic, *Studia Logica* 1:5–32. Reprinted in S. McCall (1967) *Polish Logic 1920–1939* Oxford UP, pp. 232–258.

[39] Johansson, I. (1936) The minimal calculus, a reduced intuitionistic formalism, *Compositio Mathematica* 4:119–136. Original title Der Minimalkalkul, ein reduzierter intuitionistischer Formalismus.

[40] Kapp, A. (1953) Does the logical truth that $\exists x(Fx \vee \neg Fx)$ entail that at least one individual exists?, *Analysis* 14:2–3.

[41] Kleene, S. (1967) *Elementary Logic*, Wiley, NY.

[42] Kolmogorov, A. (1925) On the principle of the excluded middle, in J. van Heijenoort (ed.), *From Frege to Gödel: A Sourcebook in Mathematical Logic, 1879–1931*, Harvard UP, Cambridge, MA, pp. 414–437. Originally published as "O principe tertium non datur", *Matematičeskij Sbornik* 32:646–667.

[43] Kuznetsov, A. (1965) Analogi 'shtrikha sheffera' v konstruktivnoĭ logike, *Doklady Akademii Nauk SSSR* 160:274–277.

[44] Lambert, K. (1963) Existential import revisited, *Notre Dame Journal of Formal Logic* 4:288–292.

[45] Leblanc, H. and T. Hailperin (1959) Nondesignating singular terms, *Philosophical Review* 68:129–136.

[46] Lemmon, E. J. (1965) *Beginning Logic*, Nelson, London.

[47] Leonard, H. S. (1957) The logic of existence, *Philosophical Studies* 7:49–64.

[48] Lewis, D. (1973) *Counterfactuals*, Blackwell, Oxford.

[49] Mates, B. (1965) *Elementary Logic*, Oxford UP, NY.

[50] Mostowski, A. (1951) On the rules of proof in the pure functional calculus of the first order, *Journal of Symbolic Logic* 16:107–111.

[51] Pelletier, F. J. (1999) A brief history of natural deduction, *History and Philosophy of Logic* 20:1–31.

[52] Pelletier, F. J. (2000) A history of natural deduction and elementary logic textbooks, in J. Woods, and B. Brown (eds.), *Logical Consequence: Rival Approaches, Vol. 1*, Hermes Science Pubs., Oxford, pp. 105–138.

[53] Pelletier, F. J., and A. P. Hazen (2012) A brief history of natural deduction, in D. Gabbay, F. J. Pelletier, and J. Woods (eds.), *Handbook of the History of Logic; Vol. 11: A History of Logic's Central Concepts*, Elsevier, Amsterdam, pp. 341–414.

[54] Peregrin, J. (2008) What is *the* logic of inference?, *Studia Logica* 88:263–294.

[55] Prawitz, D. (1965) *Natural Deduction: A Proof-theoretical Study*, Almqvist & Wicksell, Stockholm.

[56] Prawitz, D. (1979) Proofs and the meaning and completeness of the logical constants, in J. Hintikka, I. Niiniluoto, and E. Saarinen (eds.), *Essays on Mathematical and Philosophical Logic*, Reidel, Dordrecht, pp. 25–40.

[57] Price, R. (1961) The stroke function and natural deduction, *Zeitschrift für mathematische Logik und Grundlagen der Mathematik* 7:117–123.

[58] Quine, W. V. (1950) *Methods of Logic*, Henry Holt & Co., New York.

[59] Quine, W. V. (1954) Quantification and the empty domain, *Journal of Symbolic Logic* 19:177–179. Reprinted, with correction, in Quine (1995).

[60] Quine, W. V. (1995) *Selected Logic Papers: Enlarged Edition*, Harvard UP, Cambridge, MA.

[61] Read, S. (2010) General-elimination harmony and the meaning of the logical constants, *Journal of Philosophical Logic* 39:557–576.

[62] Rescher, N. (1959) On the logic of existence and denotation, *Philosophical Review* 69:157–180.

[63] Routley, R. (1975) Review of Ohnishi & Matsumoto, *Journal of Symbolic Logic* 40:466–467.

[64] Russell, B. (1906) The theory of implication, *American Journal of Mathematics* 28:159–202.

[65] Russell, B. (1919) *Introduction to Mathematical Philosophy*, Allen and Unwin, London.

[66] Schroeder-Heister, P. (1984a) Generalized rules for quantifiers and the completeness of the intuitionistic operators $\wedge, \vee, \supset, \perp, \forall, \exists$, in M. Richter, E. Börger, W. Oberschelp, B. Schinzel, and W. Thomas (eds.), *Computation and Proof Theory: Proceedings of the Logic Colloquium Held in Aachen, July 18–23, 1983, Part II*, Springer-Verlag, Berlin, pp. 399–426. Volume 1104 of *Lecture Notes in Mathematics*.

[67] Schroeder-Heister, P. (1984b) A natural extension of natural deduction, *Journal of Symbolic Logic* 49:1284–1300.

[68] Schroeder-Heister, P. (2014) The calculus of higher-level rules and the foundational approach to proof-theoretic harmony, *Studia Logica*.

[69] Sheffer, H. (1913) A set of five independent postulates for Boolean algebras, with application to logical constants, *Trans. of the American Mathematical Society* 14:481–488.

[70] Suppes, P. (1957) *Introduction to Logic*, Van Nostrand/Reinhold Press, Princeton.

[71] Tarski, A. (1930) Über einige fundamentalen Begriffe der Metamathematik, *Comptes rendus des séances de la Société des Sciences et Lettres de Varsovie* (Classe III) 23:22–29. English translation "On Some Fundamental Concepts of Metamathematics" in Tarski, 1956, pp. 30–37.

[72] Tarski, A. (1956) *Logic, Semantics, Metamathematics*, Clarendon, Oxford.

[73] Thomason, R. (1970) A Fitch-style formulation of conditional logic, *Logique et Analyse* 52:397–412.

[74] Wu, K. Johnson (1979) Natural deduction for free logic, *Logique et Analyse* 88:435–445.

[75] Zucker, J., and R. Tragesser (1978) The adequacy problem for inferential logic, *Journal of Philosophical Logic* 7:501–516.

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