

## Aberration of light

The propagation direction of EM waves will seem to differ between different frames

R&L  
p. 109-110

How do angles transform in relativity?

Lorentz transforms for differentials are the same as for variables

$$\begin{aligned} dx &= \gamma(dx' + v dt'), & dy = dy', & dz = dz', \\ dt &= \gamma(dt' + \frac{v}{c^2} dx') \end{aligned} \quad \text{where } v \text{ is the frame speed}$$

which we can put together to make (for a velocity  $\vec{u}'$  in frame K')

$$u_x = \frac{dx}{dt} = \frac{\gamma(dx' + v dt')}{\gamma(dt' + \frac{v}{c^2} dx')} = \frac{u'_x + v}{1 + \frac{vu'_x}{c^2}}$$

$$u_y = \frac{dy}{dt}, \quad u_z = \frac{dz}{dt}$$

These can be generalized to arbitrary velocities  $\vec{v}$ , not having to be along x-axis, in terms of parallel & perpendicular components of  $\vec{u}$  (relative to  $\vec{v}$ ),

$$u_{||} = \frac{u'_{||} + v}{(1 + vu'_{||}/c^2)}, \quad u_{\perp} = \frac{u'_{\perp}}{\gamma(1 + vu'_{||}/c^2)}$$

The angle  $\Theta$  between  $\vec{v}$  and  $\vec{u}$  transforms, as we see from comparing  $u_{||}$  and  $u_{\perp}$ , where  $u'_{\perp} = u' \sin \theta'$ ,  $u'_{||} = u' \cos \theta'$

$$\tan \theta = \frac{u_{\perp}}{u_{||}} = \frac{u_{\perp}}{\frac{u'_{||} + v}{(1 + vu'_{||}/c^2)}} = \frac{u'_{\perp} (1 + vu'_{||}/c^2)}{\gamma(1 + vu'_{||}/c^2)(u'_{||} + v)} = \frac{u' \sin \theta'}{\gamma(u' \cos \theta' + v)}$$

If we take  $u' = c$ , appropriate for photons,

$$\tan \theta = \frac{\sin \theta'}{\gamma(\cos \theta' + v/c)}$$

(note  $\cos \theta = \frac{u_{||}}{c}$ )

$$\cos \theta = \frac{\cos \theta' + v/c}{1 + (\frac{v}{c}) \cos \theta'}$$

Consider some limiting cases for the last expression; if  $v=0$ ,  $\cos\theta = \cos\theta'$ . (Good.)

If  $\theta' = 90^\circ$ ,  $\cos\theta' = 0$ ,  $\cos\theta = v/c$ ; radiation angles transform to beam radiation forward.

If  $\beta \rightarrow 1$ , ( $v \approx c$ ),  $\cos\theta \rightarrow 1$ , even if  $\cos\theta'$  is negative.

We also see that  $\sin\theta = \frac{v}{c} = \frac{\sin\theta'}{\sqrt{1 + \frac{v}{c}\cos\theta'}}$ . For  $\theta = \pi/2$ ,

$$\sin\theta = \frac{1}{\gamma(1+0)} = \frac{1}{\gamma}, \quad \text{and for small angles}$$

$\sin\theta \approx \theta = \frac{1}{\gamma}$  So half the (isotropically emitted) radiation from a relativistic particle is beamed within a small angle  $\sim 1/\gamma$  ahead.

## Power emitted by relativistic particles

ReL 4.8

To calculate the power emitted by a relativistic particle, we use an instantaneous rest frame  $K'$ , so at any moment the particle has zero velocity in that frame.

This allows us to use the dipole (Larmor) formula to find the emitted power, and transform it into our frame. We know energy transforms as  $dE = \gamma dE'$  (for the particle moving at velocity  $v$ ), but time also,  $dt = \gamma dt'$ , so

$$\frac{dE}{dt} = \frac{dE'}{dt'}, \text{ so total emitted power is a Lorentz invariant.}$$

Next we have to deal with acceleration. First start with the difference of spatial 4-vectors,  $[d\vec{x}, c dE]$ .

Dividing by  $d\tau$  (proper time) gives ~~something~~

$$\left[ \frac{d\vec{x}}{d\tau}, \frac{cdt}{d\tau} \right]$$

We can divide by proper time since it's a Lorentz invariant.

As  $d\tau = dt'$  for its frame,

$$= [\gamma \vec{u}, \gamma c] \quad \text{the four-velocity.}$$