

Thermal bremsstrahlung

Requires electron to change direction due to E field of ion. (Why not another electron?) Key cause of emission is second derivative of dipole moment of two particles,

Define dipole moment ($\vec{d} = \sum q_i \vec{r}_i$)

$$\text{Then rewrite } P = \frac{q^2 a^2}{6\pi\epsilon_0 c^5} = \frac{\vec{d}^2}{6\pi\epsilon_0 c^3}, \text{ dipole approximation}$$

Consider an interaction between 2 electrons, or 2 protons.

$$\vec{d} = \sum q_i \vec{r}_i = \sum_i \frac{q_i}{m_i} m_i \vec{r}_i = \frac{q}{m} \sum m_i \vec{r}_i$$

$$\vec{d} = \frac{q}{m} \sum \vec{p} = \vec{0} \quad \text{since } \vec{F}_{12} = -\vec{F}_{21}.$$

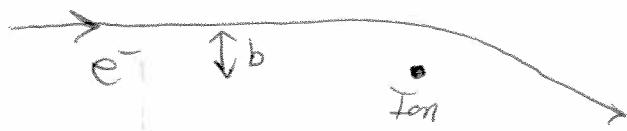
Thus the particles must have different q/m in order to radiate.

Treat the ion as fixed, electron accelerating.

$$\vec{F} = \frac{q_1 q_2 e}{4\pi\epsilon_0 r^2} \hat{r} \quad \text{Coulomb force } (\hat{r} \text{ is unit vector})$$

$$\text{Acceleration of electron } \vec{a} = \frac{\vec{F}}{m_e} = -\frac{1}{4\pi\epsilon_0} \frac{Z e^2}{r^2 m} \hat{r}$$

where ion has charge Ze .



Impact parameter b is distance of closest approach.

Assume electron is deflected only slightly, ion is infinitesimally small.

Insert b to find acceleration at closest approach,

$$\vec{q}_{\max} = \frac{1}{4\pi\epsilon_0} \frac{Ze^2}{mb^2} \hat{r}$$

To estimate radiated power, we'll say that the time interval the electron is near the ion is $\tau_b \sim b/v$. Properly, the total energy emitted is

$$\langle \Delta E \rangle = Q(b, v) = \int_{-\infty}^{\infty} P(t) dt = \frac{1}{6\pi\epsilon_0} \frac{e^2}{c^3} \int_{-\infty}^{\infty} a(t)^2 dt$$

where we

integrate over the changing acceleration.

To simplify, assume a is a_{\max} for τ_b , zero at other t .

Then

$$Q(b, v) \sim \frac{1}{6\pi\epsilon_0} \frac{e^2}{c^3} a_{\max}^2 \tau_b$$

(This is wrong by about a factor of two. See

Rybicki & Lightman for the correct geometric integration.)

(We'll follow Bradt and not put in the factor Z until the final spectrum, his p. 194.)

Put in the acceleration & duration to get

$$Q(b, v) \sim \frac{1}{(4\pi\epsilon_0)^3} \frac{(2)}{(3)} \frac{Z^2 e^6}{c^3 m_e^2 b^3 v}$$

the total energy radiated by 1 electron of speed v passing an ion of charge Ze at impact parameter b .

Obviously, Q increases strongly with smaller b .

Faster electrons radiate less.

Frequency of radiation The acceleration is always in roughly one direction.

Thus the \vec{E}_{rad} vector points (for a negative charge) in the same direction as \vec{a} , & builds to a max, then decreases once. This gives a single pulse of \vec{E} vectors to the observer.

The frequency of the radiation is set by the timescale on which the electron moves. The timescale τ_b gives roughly the max ~~freq~~

$$\omega_{fb} \approx \frac{1}{\tau_b} = \frac{V}{b}, \quad V = \frac{\omega_{fb}}{2\pi} = \frac{V}{2\pi b}$$

Lower frequencies correspond to longer timescales on which the electron experiences less acceleration, so we anticipate the bulk of the power to come out around ω_{fb} . Relating b & ν , we get

$$db = -\frac{V}{2\pi\nu^2} d\nu.$$

Single-speed electron beam

Next, consider many electrons at one speed. First compute power from one annulus, at

average distance b from ion, & width db ,

Take electron density n_e , so electron flux $n_e V$.

Area of annulus is $2\pi b db$, so # electrons/s is just $n_e V 2\pi b db$.

So the energy emitted per s is $P(b, \nu) db = Q(b, \nu) n_e V 2\pi b db$.

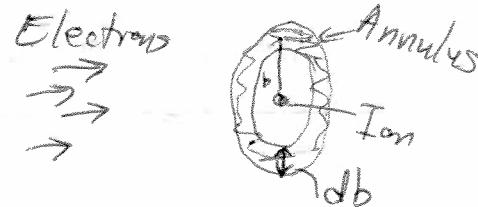
If we consider electrons coming from all directions, but n_e is the same, we get same answer.

Now b and ν are related, so we can estimate the frequency dependence.

Bradt's argument: Convert $P(b, \nu)$ to $P(\nu, \nu)$ by integrating over appropriate ranges,

$$\int_{b_1}^{b_2} P_b(b, \nu) db = - \int_{\nu_1}^{\nu_2} P_\nu(\nu, \nu) d\nu$$

where the minus sign comes from $db \propto -d\nu$.



The total power thus must be equal, and the integrands must be equal (to allow any choice of b_1, b_2 , or ν_1, ν_2), so $P_b(b, \nu)db = -P_\nu(\nu, v)d\nu$,

$$P_\nu(\nu, v) = P_b(b, v) \frac{db}{d\nu}$$

We already have these, so

$$P_\nu(\nu, v) = Q(b, v) n_e v \frac{2\pi b v}{2\pi v^2}$$

Plug in $Q(b, v)$

$$P_\nu(\nu, v) = \frac{1}{(4\pi\epsilon_0)^3} \left(\frac{2}{3}\right) \frac{Z^2 e^6}{c^3 m_e^2 b^{3/2}} \frac{n_e v^{2/3}}{v^2}$$

Finally, use $b = \frac{v}{2\pi\nu}$

$$P_\nu(\nu, v) = \frac{1}{(4\pi\epsilon_0)^3} \frac{8\pi^2}{3} \frac{n_e Z^2 e^6}{c^3 m_e^2 v}$$

Add $d\nu$ to both sides, & can integrate to get power in any free. range.

Note that this is independent of frequency, since the power from close-approaching electrons is at high frequencies, & there are fewer close approaches, but each provides more emitted energy.

You should think — wait a minute, this is the ultraviolet catastrophe again! Quite right, this classical approach obviously must go wrong at high frequencies.

Two ways to see the limit. First, even classically the electron is not a point, so can't approach arbitrarily close. $\Delta x \propto p \approx h$, so for $\Delta p \sim p$, $\Delta x \approx \frac{h}{mv}$, so get rough answer from $b_{\min} \sim \frac{h}{meV}$, $\nu_{\max} \sim \frac{meV^2}{2\pi h}$.

Alternatively consider that an electron can not radiate more energy than it has in KE. So ν_{\max} cannot be larger than $\frac{mv^2}{2}$, so $\nu_{\max} \sim \frac{meV^2}{2h}$. These values are similar!

Electrons of various speeds.

Next consider a range of electron speeds, since $P(\nu, v)$ and we want $P(\nu)$ for, say given temperature.

We'll use $P(v)dv$ to be the probability of an electron having speed v in interval dv .

For a (nondegenerate) gas, the velocity distribution of particles is

$$P(\vec{v}) = \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mv^2}{2kT}\right)$$

To find speeds from this

3D velocity expression, we need to account for the possible directions. Consider 3D-velocity space; we want all vectors lying in a spherical shell, thus with the total velocity magnitude of interest,



Can write
 $P(v)dv = P(\vec{v}) 4\pi v^2 dv$.

Sum of all probabilities is unity, $\int_0^\infty P(v)dv = 1$.

Thus, to get the power radiated per ion, we take our previous result (still a classical expression) and multiply by the probability of each v , & integrate over all $P(v)$.

$$\langle P(\nu) \rangle_{\text{ion}} = \int_{v_{\min}}^{\infty} P(\nu, v) P(\vec{v}) 4\pi v^2 dv \quad (\text{W/m}^{-1}\text{Hz}^{-1})$$

This is the power from one ion at freq. ν , in a Maxwell-Boltzmann distribution of electron speeds.

The lower limit v_{\min} defines the smallest velocity an electron can have, while still emitting h.c.

$$v_{\min} = \sqrt{\frac{2h\nu}{m_e}} \quad \text{will be this (quantum) limit.}$$