

Comptonization

Two versions, which are really the same process seen from different frames:

"classical" Compton scattering, in which electron gains energy from photon;

"inverse" Compton scattering, where photon energy boosted by electron.

Different from Thomson scattering, where photon energy is unchanged,

Cross-section for Compton scattering can also be different,

QED calculations show the (differential) cross-section for Compton scattering is

ϵ_i = initial photon energy

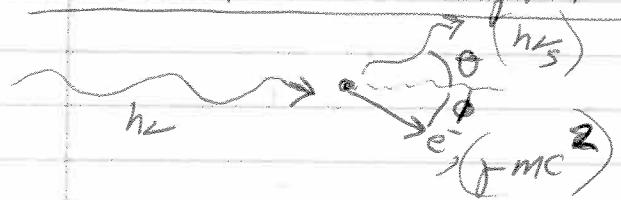
$$\frac{d\sigma}{d\Omega} = \frac{r_0^2}{2} \frac{\epsilon_f^2}{\epsilon_i^2} \left(\frac{\epsilon_i}{\epsilon_f} + \frac{\epsilon_f - \sin^2\alpha}{\epsilon_i} \right)$$

ϵ_f = final photon energy

For $\epsilon_f \approx \epsilon_i$, this reduces

to the Thomson expression; otherwise it reduces the cross-section. (Called Klein-Nishina cross-section)

Kinematics of Compton Scattering



Photon scatters off stationary electron.

Energy of scattered photon depends on scattering angle θ .

Experimentally found that photons scattering off electrons show wavelength shift off $\lambda_s - \lambda = \frac{h}{mc}(1 - \cos \theta)$.

For visible light, this shift is negligible, but is large for X-rays, where photon E approaches electron rest energy.

Derive Compton wavelength shift.

Use energy, momentum conservation, in both longitudinal & transverse components of \vec{p} .

Initial e^- energy = mc^2 ; recoil e^- energy $\gamma m_e c^2$.
 e^- momentum = $\vec{p} = \gamma m_e v$, where $\gamma = (1 - \beta^2)^{-1/2}$,
 $= \gamma \beta m_e c$

Initial photon energy = $h\nu$, initial photon $\vec{p} = h\nu/c$,
scattered photon $E = h\nu_s$, scattered $\vec{p} = h\nu_s/c$.

From energy conservation, $h\nu + mc^2 = h\nu_s + \gamma m_e c^2$.

Longitudinal momentum: $\frac{h\nu}{c} = \frac{h\nu_s}{c} \cos\theta + \gamma \beta m_e c \cos\phi$

Transverse momentum: $0 = \frac{h\nu_s}{c} \sin\theta - \gamma \beta m_e c \sin\phi$

First eliminate ϕ ; start by solving for ϕ ,

$$\cos\phi = \left(\frac{h\nu}{c} - \frac{h\nu_s}{c} \cos\theta \right) / \gamma \beta m_e c$$

$$\sin\phi = \left(\frac{h\nu_s}{c} \sin\theta \right) / \gamma \beta m_e c$$

Plug these into $\cos^2\phi + \sin^2\phi = 1$; start with

$$\left(\frac{h\nu}{c} - \frac{h\nu_s}{c} \cos\theta \right)^2 = \frac{h^2 \nu^2}{c^2} - 2 \frac{h^2 \nu \nu_s}{c^2} \cos\theta + \frac{h^2 \nu_s^2}{c^2} \cos^2\theta$$

$$\text{So } \cos^2\phi = \frac{h^2}{\gamma^2 \beta^2 m_e^2 c^4} \left(\nu^2 - 2 \nu \nu_s \cos\theta + \nu_s^2 \cos^2\theta \right)$$

while $\sin^2\phi = \frac{h^2}{\gamma^2 \beta^2 m_e^2 c^4} \left(\nu_s^2 \sin^2\theta \right)$. Putting these into $\cos^2\phi + \sin^2\phi = 1$ gives

$$\frac{h^2}{c^2} \left(\nu^2 - 2 \nu \nu_s \cos\theta + \frac{\nu_s^2 \cos^2\theta + \nu_s^2 \sin^2\theta}{\nu_s^2} \right) = (\gamma \beta m_e c)^2$$

Next, square the E equation, keeping γmc^2 isolated;

$$h^2(r^2 - 2rv_s + v_s^2) + 2hv_mc^2(r - v_s) + m_e^2c^4 = (\gamma mc^2)^2$$

$$-h^2r^2 + h^22rv_s \cos\theta - h^2v_s^2 - \delta(\beta m_e c^2)^2$$

$$-2hv_s(1-\cos\theta) + 2hv_mc^2(r - v_s) + m_e^2c^4 = \gamma^2 m_e^2 c^4 (1 - \beta^2)$$

But $\gamma^2 = 1/(1-\beta^2)$, so cancel the γ 's, β 's, & $m_e^2 c^4$ terms.

$$2hv_s(1-\cos\theta) = 2hv_mc^2(r - v_s)$$

$$\frac{(1-\cos\theta)}{m_e c^2} = \frac{(r - v_s)}{hv_s}$$

$$\frac{1-\cos\theta}{m_e c^2} = \frac{1}{hv_s} - \frac{1}{hr} \quad \text{Use } r = c/\lambda$$

$$\boxed{\frac{h}{m_e c}(1-\cos\theta) = \lambda_s - \lambda} \quad \text{Compton effect.}$$

$h/m_e c \equiv \lambda_c$ is the Compton wavelength:

$\lambda_c = 2.4 \times 10^{-12} \text{ m}$, which comes (with $E = hv$) to give an energy of 0.511 MeV, the electron rest energy.

$$\text{We can also rearrange terms to get } hv_s = \frac{hr}{1 + \frac{hr}{m_e c^2}(1-\cos\theta)}$$

This shows how the scattered photon energy changes as hr approaches mc^2 .

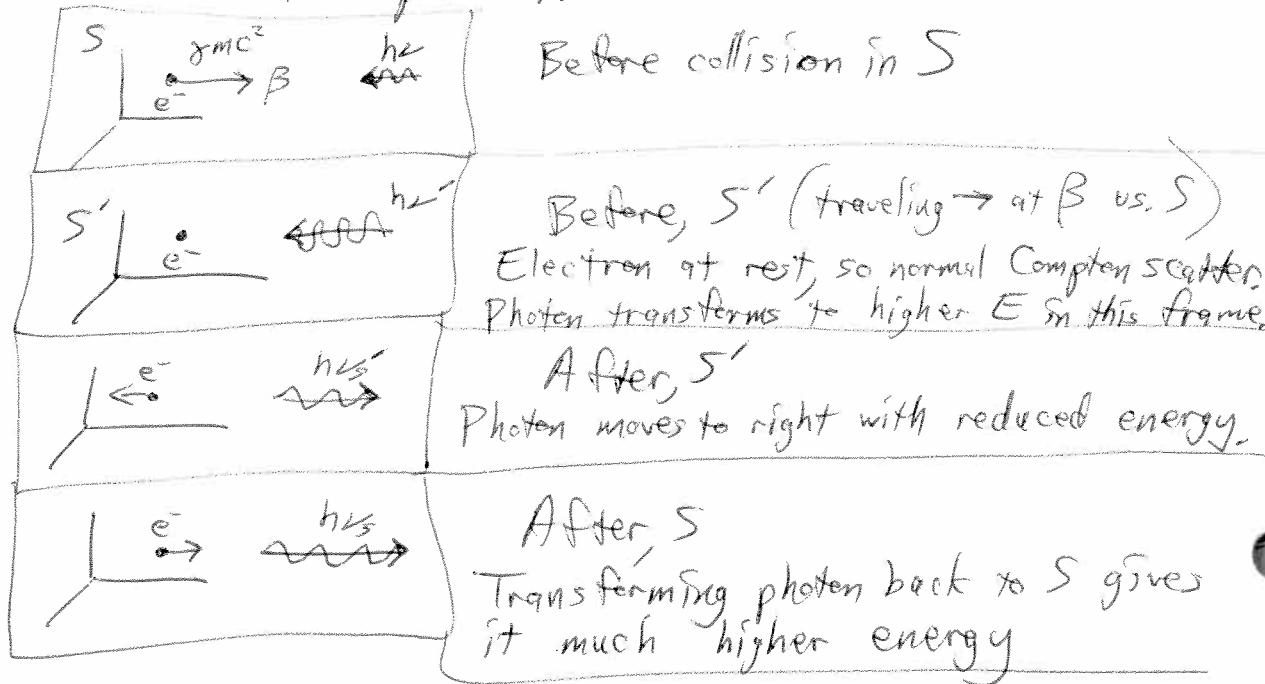
The decrease in E is largest when $\theta = \pi$ (backscattering).

At very high photon energies ($hr \gg m_e c^2$), the backscattered photon has energy $m_e c^2$, & the electron takes the photon's initial energy.

Note that in the frame of the stationary electron, the photon always loses energy. But we can switch frames,

To deal with moving electrons & photon energy gain, we transform to the electron-at-rest frame, do calculation as above, then transform back.

- Start with head-on collision of relativistic electron & photon.



To find photon energy in S' before collision, use relativistic Doppler transform:

$$h\nu' = \left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}} h\nu \quad (\text{These } \beta\text{s are for the electron before the collision.})$$

Now we do the Compton scatter, for 180° back scatter:

$$h\nu'_S = \frac{h\nu'}{1 + \left(\frac{2h\nu'}{mc^2}\right)} \quad \text{Photon energy is reduced in } S'.$$

Now transform back to S; this transform boosts the photon energy, since it's going in the same direction,

$$h\nu_S = \left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}} h\nu'_S \quad \text{Both Doppler shifts are upwards in frequency.}$$

Now substitute, first to get $h\nu_s'$, then $h\nu_s$.
 We use a trick; if $\beta \sim 1$, we can simplify $(\frac{1+\beta}{1-\beta})^{1/2}$.

$$\frac{(1+\beta)^{1/2}}{(1-\beta)} \frac{(1+\beta)^{1/2}}{(1+\beta)^{1/2}} = \frac{1+\beta}{(1-\beta^2)^{1/2}}$$

But $(1-\beta^2)^{1/2} = \gamma$
 & $1+\beta \sim 2$, so

$\sim 2\gamma$. (for $\beta \sim 1$). So get $h\nu_s'$,

$$h\nu_s' = \frac{h\nu'}{1 + \left(\frac{2h\nu'}{m_e c^2}\right)} = \frac{2\gamma h\nu}{1 + \left(\frac{4\gamma h\nu}{m_e c^2}\right)}$$

$$h\nu_s = 2\gamma h\nu' = \frac{4\gamma^2 h\nu}{1 + \left(\frac{4\gamma h\nu}{m_e c^2}\right)}$$

Single inverse Compton
backscatter.

Note that γ is the original Lorentz factor of the electron.

Often $4\gamma h\nu \ll m_e c^2$ (the incoming photon's energy is much less than the electron rest energy, even after transforming), so the Compton-scatter reduction in photon energy is very small. (Essentially, in electron frame, this is Thomson scattering.)

In that case, $h\nu_s \sim 4\gamma^2 h\nu$ Thomson approximation
of IC backscatter.

However, most collisions occur at some angle, not head-on. We can calculate these shifts, but they are complicated functions of the angle in both frames,

$$h\nu_s = h\nu \gamma^2 (1 - \beta \cos \theta + \beta \cos \theta' - \beta^2 \cos \theta \cos \theta'),$$

of order a γ^2 increase.

The average energy boost, for photons at all angles, is

$$\langle h\nu_s \rangle = \frac{4}{3} \gamma^2 h\nu$$

Isotropic average IC boost, for Thomson scattering in electron frame.