

Propagation of EM waves in a plasma:

Dispersion

Consider EM waves passing through ~~unmagnetized~~ plasma. Both the free electrons & the B-fields will alter these EM waves. If we observe polarized and/or pulsed EM waves (especially at low freq), we can measure the charge density & B-fields in interstellar space.

Much of the Galaxy is filled with ionized gas at low density, so such studies are best way to probe this gas.

Maxwell's equations in medium: (isotropic, well-behaved)

$$\nabla \cdot (K_e \vec{E}) = \frac{\rho}{\epsilon_0} \quad (\text{Gauss' Law})$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (\text{Faraday's Law})$$

$$\nabla \times \left(\frac{\vec{B}}{K_m} \right) = \mu_0 \epsilon_0 K_e \frac{\partial \vec{E}}{\partial t} + \mu_0 \sigma \vec{E} \quad (\text{Ampere's Law})$$

$$\nabla \cdot \vec{B} = 0 \quad (\text{Gauss' Law for magnetism})$$

where ρ = local charge density (C/m^3)
 σ = conductivity ($ohm^{-1} m^{-1}$), ($\vec{J} = \sigma \vec{E}$)
 & susceptibility to electric & magnetic fields is given by dimensionless quantities,

(Magnetic) relative permeability $K_m = \frac{\mu}{\mu_0}$ \leftarrow permeability

(Electric) dielectric constant $K_e = \frac{\epsilon}{\epsilon_0}$ \leftarrow permittivity

In vacuum, $K_m = K_e = 1$, $\sigma = \rho = 0$.

In vacuum, we can manipulate the Maxwell equations to get, (using $\partial/\partial z$ on Faraday, & $\partial/\partial t$ on Ampere), starting with E_x alone,

$$\frac{\partial^2 E_x(z,t)}{\partial z^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_x(z,t)}{\partial t^2}$$

$$\& \frac{\partial^2 B_y(z,t)}{\partial z^2} = \mu_0 \epsilon_0 \frac{\partial^2 B_y(z,t)}{\partial t^2}$$

which are wave equations,

They have solutions of form $E_x = f(\omega t \pm kz)$, if $\omega/k = (\mu_0 \epsilon_0)^{-1/2}$.

$\omega = 2\pi\nu$, $k = 2\pi/\lambda$. We can choose

$$E_x = E_0 \cos(\omega t - kz) \leftarrow \text{traveling wave.}$$

$$B_y = (E_0/c) \cos(\omega t - kz).$$

The energy in the B field is equal to that in E field; in cgs units, $E_0 = B_0$.

EM wave

Phase velocity of wave $v_p = \frac{\Delta z}{\Delta t}$ is speed at which a ~~feature~~ ^{feature} (e.g. a wave peak) moves in propagation direction.

Set $(\omega t - kz)$ - the argument - to 'constant'. So $\omega t_1 - kz_1 = \omega t_2 - kz_2$, $\omega \Delta t - k \Delta z = 0$.

$$v_p = \frac{\Delta z}{\Delta t} = \frac{\omega}{k} = \lambda \nu.$$

$$\text{In vacuum, } \frac{\omega}{k} = (\mu_0 \epsilon_0)^{-1/2} = c.$$

In plasma, radiation pulls the electrons around in the time-varying electric field.

For high-freq. radiation, the electrons don't have substantial currents so $\vec{J} = \sigma \vec{E}$ is small.

~~typical~~ (More accurately, \vec{J} and \vec{E} are 90° out of phase but in any case no work is done.)

Typical plasmas have $K_m \sim 1$ (this is different mainly in ferromagnetic material).

Since the plasma is neutral, $\rho = 0$.
We rewrite Maxwell with these notes,

$$\begin{aligned}\vec{\nabla} \cdot (K_e \vec{E}) &= 0 \\ \vec{\nabla} \times \vec{E} &= -\partial \vec{B} / \partial t \\ \vec{\nabla} \times \vec{B} &= \mu_0 \epsilon_0 K_e \partial \vec{E} / \partial t \\ \vec{\nabla} \cdot \vec{B} &= 0\end{aligned}$$

Again $\vec{E} = E_0 \cos(\omega t - kz) \hat{x}$ works,
but now ~~it is~~ $\epsilon_0 \rightarrow \epsilon_0 K_e$, so

$$\frac{\omega}{k} = (\mu_0 \epsilon_0 K_e)^{-1/2} = \frac{c}{K_e^{1/2}} \quad \text{Phase velocity for plasma.}$$

K_e may vary with frequency.
The index of refraction n relates up to c ,
 $v_p = c/n(\omega)$. So $n(\omega) = K_e^{1/2}$.

Light waves travel more slowly in glass or water than vacuum; for glass $n(\text{optical } \lambda) \sim 1.5$, for water, $n(\text{opt}) \sim 1.3$.

Since n is a function of ω prisms have different refraction indices for diff colors.
This changed speed of light is due to the radiation forcing electrons to oscillate; they radiate (re-radiating what radiation they've absorbed), slightly out of phase with the incoming wave.
Thus the net radiation is shifted in phase.

The dielectric constant, K_e , depends on how easily electrons can be displaced by a field.
Polarization vector \vec{P} is the dipole moment per unit volume,

(not the permeability μ)

If one electron is displaced by \vec{r} , the dipole moment $\vec{\mu} = -e\vec{r}$ points in direction of + charge displacement.

Bradt
dipoles

Large medium will show polarization from charges on surfaces in electric field.

We define permittivity ϵ by \vec{P} ,

$$\vec{P} = (\epsilon - \epsilon_0)\vec{E}, \text{ or } \vec{P} = \epsilon_0(K_e - 1)\vec{E}$$

dielectric constant.

Can describe polarization by movement of charges;

if we have base area A , height h , & n_e electrons/m³, end with 2 layers of charge, area A separated by h . Total charge Q is charge density ($n_e e$) times area, times separation distance h , so

$$\vec{\mu} = -Qh\hat{r} = -(n_e e A h)h\hat{r}$$

\hat{r} unit vector

$$\vec{P} = -n_e e \vec{r}$$

In a static \vec{E} field, \vec{P} is parallel to \vec{E} , so $K_e > 1$ and thus $n > 1$, so phase velocity $< c$.

We'll see that an oscillating EM field gives \vec{P} antiparallel to \vec{E} , so $K_e < 1$, $n < 1$, phase velocity $> c$.

(However, information travels at group velocities, so Einstein is safe.)

Electrons in plasma respond to electric part of EM wave

$$\vec{F}_E = q\vec{E} = -e\vec{E}_0 \cos(\omega t - k\vec{z})\hat{x}$$

$$F_x = m a_x, \text{ so } qE_x(z, t) = m_e \ddot{x}, \quad x = x_0 \cos(\omega t), \quad \ddot{x} = -x_0 \omega^2 \cos(\omega t) = -\omega^2 x$$

$$(-e)E_x = -m\omega^2 x, \text{ so } x = \frac{eE_x}{m\omega^2}$$

(Damn Franklin!)

x is in phase with E_x , but out of phase with $F_x = -eE_x$.

Polarization in plasma, high freq:

$$\vec{P} = -\frac{n_e e^2}{m\omega^2} \vec{E} \quad (\text{from } \vec{P} = -n_e e \vec{r})$$

This analysis applies for high frequencies -- we'll see the limit to this, at low freq.

From $\vec{D} = \epsilon_0 (K_e - 1) \vec{E}$, we see that

$$K_e = 1 - \frac{n_e e^2}{\epsilon_0 m_e \omega^2}, \text{ and since } \sqrt{K_e} = \eta,$$

$$\eta(\omega) = \left[1 - \frac{n_e e^2}{\epsilon_0 m_e \omega^2} \right]^{1/2}$$

If we define a plasma frequency $\omega_p = \left(\frac{n_e e^2}{\epsilon_0 m_e} \right)^{1/2}$,

$$\eta(\omega) = \left[1 - \frac{\omega_p^2}{\omega^2} \right]^{1/2}$$

$$\text{Note } v_p = \frac{c}{\eta(\omega)} > c.$$

The plasma frequency is the resonant freq. of the plasma. If $\omega < \omega_p$, $\eta(\omega)$ is imaginary, & the waves are reflected.

In SI units, changing to $\nu_p = \omega_p / 2\pi$,

$$\nu_p = 9.0 \left(n_e \right)^{1/2}.$$

The Earth's ionosphere, with $n_e \sim 10^{12} \text{ m}^{-3}$, gives $\nu_p \sim 9.0 \text{ MHz}$.

So we cannot study radio emission from space at very low freq., but can bounce low-freq. AM waves off ionosphere.

Metals can be considered essentially as plasmas (since electrons are free), of $\rho \sim 10^3 \text{ kg/m}^3$, $n_e \sim (6 \times 10^{26}) \times 1000 \text{ m}^{-3}$, so $\nu_p \sim 7 \times 10^{15} \text{ Hz}$.

So EM waves below this will be reflected from metals (can describe by Thomson scattering), while higher frequencies (above mid-UV) pass through.

Group velocity is the speed with which a pulse of radiation travels.

Wave packet

Information moves with wave packet, while individual peaks move at phase velocity.

Phase vs. group velocities

Consider two waves in a group, with slightly diff. λ s, $\lambda_1 > \lambda_2$, ν is function of freq, so phase velocities

v_1, v_2 differ;

$$v_1 = c/n_1, \quad v_2 = c/n_2$$

Center of wave packet (or group) is where waves of diff freq. interfere constructively.

Group velocity is how fast this center travels,

$$v_g = \frac{\Delta z_g}{\Delta t} \quad (\text{or } \frac{z_g}{t} \text{ here}).$$

If the waves have solutions: $E_1 = E_0 \cos(\omega_1 t - k_1 z)$
 $E_2 = E_0 \cos(\omega_2 t - k_2 z)$

& the group center is when the arguments are equal, then $\omega_1 t - k_1 z_g = \omega_2 t - k_2 z_g$,

$$\frac{z_g}{t} = \frac{\omega_2 - \omega_1}{k_2 - k_1},$$

$$v_g = \frac{d\omega}{dk}$$

$$(vs. v_p = \frac{\omega}{k})$$

It's easier to calculate as

$$\frac{1}{v_g} = \frac{dk}{d\omega}, \quad \omega = \frac{c}{\eta(\omega)} k, \quad \text{or } k = \omega \eta(\omega)/c.$$

$$\text{So } \frac{1}{v_g} = \frac{d(\omega \eta(\omega)/c)}{d\omega} = \frac{\eta}{c} + \frac{\omega}{c} \frac{d\eta}{d\omega} = \frac{1}{v_g}$$

Now if we use our plasma index,

$$\frac{1}{v_g} = \frac{1}{c} \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{-1/2}, \quad \text{or } v_g = c \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2}$$

We see v_g is less than c .

As for the phase velocity, this only works above ω_p ; we see that approaching ω_p the v_g goes to zero, while at higher freq. v_g approaches c .

Application to radio pulsars - pulsars produce individual clear pulses, which are delayed diff. amounts at diff. frequencies.

Consider pulse from distance D ; this will take time $t = \frac{D}{v_g} \approx \frac{D}{c} \left(1 + \frac{1}{2} \frac{\omega_p^2}{\omega^2}\right)$

expanding $(1+x)^n \approx 1+nx$ for small x ,
For two different freqs, using $\omega_p = \left(\frac{n_e e^2}{\epsilon_0 m_e}\right)^{1/2}$

$$t_2 - t_1 = \frac{1}{8\pi^2} \frac{D}{c} \frac{n_e e^2}{\epsilon_0 m_e} \left(\frac{1}{\omega_2^2} - \frac{1}{\omega_1^2}\right)$$

If n_e isn't constant, then substitute $\int_0^D n_e ds$ for $D n_e$,

$$t_2 - t_1 = 1.345 \times 10^{-7} \left(\frac{1}{\omega_2^2} - \frac{1}{\omega_1^2}\right) \int_0^D n_e ds \quad \text{Time delay.}$$

So measuring the time delay at diff. freq. gives the $\int n_e ds$, called dispersion measure, DM, & directly probes ionized interstellar medium.

Pulsar dispersion

PSR J0437-4715 is closest millisecond pulsar shows clear dispersion. Pulsar searches must first "de-disperse" (add delays by freq.) before searching data, to find pulsars.

DM map

If distances to some pulsars are known, can make map of DM in galaxy, & thus