

Radiation of accelerated charges

Bradt

Bremsstrahlung ("braking radiation" in German) is radiation from electrons accelerating in electric fields of atomic nuclei.

Equivalent absorption process, free-free absorption.

Start by deriving the radiation field of accelerated charge, in SI, Classical derivation.

Lorentz force: $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

Maxwell's equations,

$$\begin{cases} \nabla \cdot \vec{E} = \rho / \epsilon_0 \\ \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} = -\partial \vec{B} / \partial t \\ \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \end{cases}$$

Charge conservation: $\nabla \cdot \vec{J} + \partial \rho / \partial t = 0$

The EM flux vector, or Poynting vector, \vec{S} (of the S ; in Bradt, \vec{J}_p)

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0}$$

as ~~Bradt~~ Bradt always uses \vec{S} for energy flux density, or flux.

\vec{E} and \vec{B} can be defined via their potentials, which aren't uniquely defined.

(you choose ϵ_0 conveniently, $\nabla \cdot \vec{A} - \epsilon \frac{\partial \phi}{\partial t} = 0$ Lorentz gauge)

$$\vec{B} = \nabla \times \vec{A}, \quad \vec{E} = -\nabla \phi - \frac{\partial \vec{A}}{\partial t}$$

Putting these into Maxwell's equations gives

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{A} - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

inhomogeneous wave equations.

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \phi - \nabla^2 \phi = \frac{\rho}{\epsilon_0}$$

Since only the curl of \vec{A} gives \vec{B} , we can choose the Lorentz gauge.

To find solutions, introduce "retarded time",

$$t_r = t - \frac{|\vec{r} - \vec{r}'|}{c}$$

Here \vec{r} is the current location of the particle, & \vec{r}' is the point where the field was produced.

The key is that EM waves take time to propagate, so the particle must produce a field a time $|\vec{r} - \vec{r}'|/c$ before it is felt at \vec{r} .

Then it is possible (see Jackson) to find solutions for ϕ, \vec{A} :

$$\phi(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} d^3\vec{r}'$$

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} d^3\vec{r}'$$

These integrate over all space to find charges & current densities which, at any t_r , were $|\vec{r} - \vec{r}'|$ away, & thus contributing to the field at point \vec{r} .

Then write charge & current densities for a particle moving along a path $\vec{r}_0(t)$,

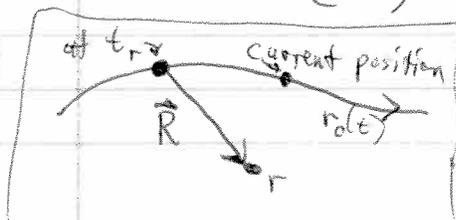
$$\rho(\vec{r}, t) = q \delta(\vec{r} - \vec{r}_0(t))$$

$$\vec{J}(\vec{r}, t) = q \vec{v} \delta(\vec{r} - \vec{r}_0(t))$$

where $\vec{v}(t) = \dot{\vec{r}}_0(t)$ & the δ -function localizes the charge & current.

Let's calculate the potential at another point, off the particle's path, r .

Define $|\vec{r} - \vec{r}_0(t)| = R$, $\vec{r} - \vec{r}_0(t) = \vec{R}$,
 & $K = 1 - \frac{\hat{R} \cdot \vec{v}}{c}$, where $\hat{R} = \vec{R}/R$ (a unit vector from the particle position at the retarded time to the point we're evaluating).



Then the potentials are (at retarded times) ^{evaluate}

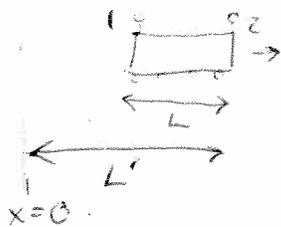
$$\phi = \left[\frac{1}{4\pi\epsilon_0} \frac{q}{KR} \right]_{\text{retarded}}, \quad \vec{A} = \left[\frac{\mu_0}{4\pi} \frac{q\vec{v}}{KR} \right]_{\text{retarded}}$$

Does this make sense? The q/R for ϕ makes sense, & $q\vec{v}/R$ for \vec{A} , what about K ?

When $\vec{v} \Rightarrow 0$, $K \Rightarrow 1$; it's an effect of finite velocity on the apparent amount of charge emitting.

If v/c is large, K concentrates the potentials into the direction of motion, because we see photons "piled up" in this direction.

Consider a moving train,



observer

If a photon from the back is emitted at $x=0$, then the photon from the front $x=L$ emitted at $x=L$ will arrive to the observer at the same time if $L' = L/(1-v/c)$.

So L' will be the apparent length of the train. The v here is the velocity component coming towards the observer, $v = \hat{k} \cdot \vec{v}$ in our formulation. So the charge element suffers the same distortion, $\rho' = \rho/(1 - \hat{k} \cdot \vec{v}/c)$, or ρ/K . This is how we get K .

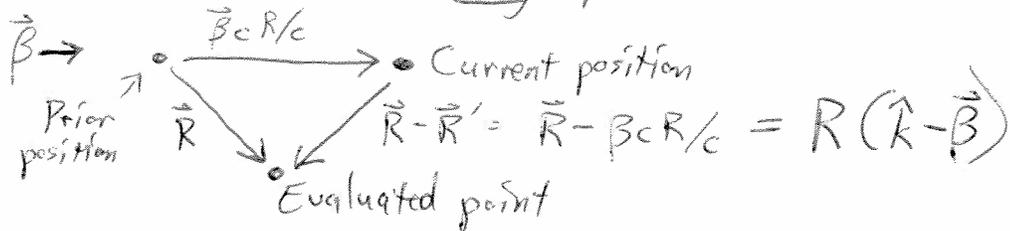
Ok, next step is to differentiate & get the fields. We skip algebra, & use $\vec{\beta} = \vec{v}/c$,

$$\vec{E}(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \left[\frac{(\hat{R} - \vec{\beta})(1 - \beta^2)^{3/2}}{K^3 R^2} \right]_{\text{ret}} + \frac{q}{4\pi\epsilon_0 c} \left[\frac{\hat{R}}{K^3 R} \times \left[(\hat{R} - \vec{\beta}) \times \dot{\vec{\beta}} \right] \right]_{\text{ret}}$$

$$\vec{B}(\vec{r}, t) = \left[\frac{\hat{R}}{c} \times \vec{E}(\vec{r}, t) \right]_{\text{ret}}$$

The first term of \vec{E} is dep. on velocity & position, but reduces to Coulomb's law for $\beta \rightarrow 0$.

If the particle has constant velocity, this term is the only part that contributes.



Note that $\hat{k} - \vec{\beta}$ points to the current particle position, though the field is set by the retarded position.

So at constant velocity, E-field lines continue to point to the particle's current position all along their length.

The second term depends on $\dot{\vec{\beta}}$; this is the acceleration field. It falls off as $1/R$, is proportional to the acceleration, & is perpendicular to \hat{k} , so it's orthogonal to the direction of the retarded particle.

\vec{B} is \perp to both \vec{E} and \hat{k} , so they form a triad of \perp vectors. This will remind you, correctly, of the radiation solutions to Maxwell's equations!

Fig. 3.2 of R&L, [Fig. 9.1 of Longair]

The ~~second~~ \vec{E} term, & its \vec{B} counterpart are the radiation field, & dominate at large R .

Now consider the total radiation from non-relativistic particles being accelerated. Assume $\beta \ll 1$, so $K \sim 1$ & we won't worry about retarded times.

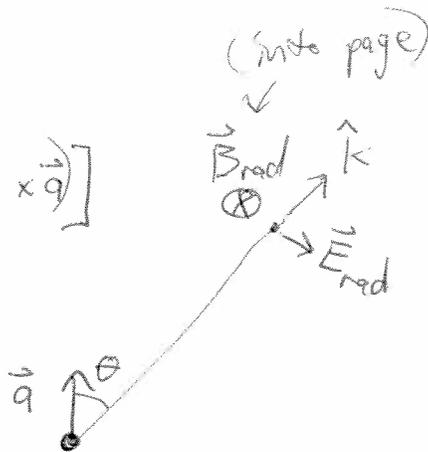
This simplification gives

$$\vec{E}_{\text{rad}} = \left[\left(\frac{q}{4\pi\epsilon_0 c^2 R} \right) \hat{k} \times (\hat{k} \times \vec{a}) \right]$$

$$\vec{B}_{\text{rad}} = \left[\frac{\hat{k} \times \vec{E}_{\text{rad}}}{c} \right]$$

$\hat{k} \times \vec{a}$ gives $\sin\theta$, so the magnitude of (\vec{E}_{rad}) is

$$\frac{q a \sin\theta}{4\pi\epsilon_0 R c^2}, \text{ which also equals } c |\vec{B}_{\text{rad}}|.$$



We can also see this result geometrically.

Fig. 9.1
of Longair

Accelerate particle for Δt by Δv .

At time t later, a sphere at radius ct has E -field lines pointing to the current particle position, while more distant space points to the previous position.

In the transition zone, the E field shifts in the θ direction by $\Delta v t \sin\theta$, while travelling radially by $c \Delta t$. Thus

$$\frac{E_{\theta}}{E_r} = \frac{(\Delta v) t \sin\theta}{c \Delta t}$$

But from Coulomb's Law, we know $E_r = \frac{q}{4\pi\epsilon_0 r^2} = \frac{q}{(ct)^2 (4\pi\epsilon_0)}$

$$\text{so } E_{\theta} = \frac{q (\frac{\Delta v}{\Delta t}) \sin\theta}{4\pi\epsilon_0 c^3 t} = \frac{q a \sin\theta}{4\pi\epsilon_0 c^2 r}$$

which we saw earlier.

The Poynting flux is $\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0}$. Since $\vec{B} = \frac{\hat{k} \times \vec{E}}{c}$,

$$\vec{S} = \frac{1}{\mu_0 c} (\vec{E} \times (\hat{k} \times \vec{E})) = \frac{1}{\mu_0 c} \hat{k} |\vec{E}|^2 = \epsilon_0 c \hat{k} |\vec{E}|^2$$

$$\vec{S} = \frac{q^2 a^2 \sin^2\theta \hat{k}}{16\pi^2 \epsilon_0 R^2 c^3}$$

Note the $\sin^2\theta$ term, always non-negative. Shows that radiation is emitted preferentially perpendicular to the acceleration.

Bradt figure of radiation pattern

So what's the total power radiated?
 \mathcal{F} is $\frac{dE}{dt dA}$, & $dA = R^2 d\Omega$, so

$$\frac{dE}{dt d\Omega} = \frac{q^2 a^2 \sin^2\theta}{(4\pi)^2 \epsilon_0 c^3} \quad \text{power radiated per unit solid angle.}$$

Integrate over all solid angles, getting power,

$$P = \frac{dE}{dt} = \frac{q^2 a^2}{(4\pi)^2 \epsilon_0 c^3} \int \sin^2\theta d\Omega$$

$$P = \frac{q^2 a^2}{6\pi \epsilon_0 c^3}$$

Larmor's formula for instantaneous emitted power.

Note that the \vec{E}_{rad} formula gives direction for \vec{E} . Thus, the emitted radiation is polarized in one plane, if the acceleration is linear. Bremsstrahlung is typically not directional, but we will see examples of directed acceleration later.

Spectra of emitted radiation of accelerated electron:

Emitted spectrum depends on how \vec{E} varies. Must be measured over a time interval Δt ; thus, can only measure spectrum with frequency resolution $\Delta\omega > 1/\Delta t$.

A short burst of radiation gives a broad spectrum in frequency (up to $h\nu \sim kT$), while an extended sine wave gives a narrow freq. range.

Hilborn