

Relativistic covariance

Bremsstrahlung & blackbody radiation are typically nonrelativistic, but the next key radiation mechanisms are fundamentally relativistic, motivating application of SR.

We first review special relativity, choosing Bradt's notation.

Postulates:

- Laws of nature same in any 2 nonrotating, unaccelerated reference frames,
- Speed of light is c in all such frames,

Take frames S, S' , where frame S' moves $+x$ direction at velocity v . Origins coincide at $t = t' = 0$.

Light pulse emitted at $t, x, y, z = 0$, so both frames see expanding light wave centered at origin.

Both frames measure $x^2 + y^2 + z^2 = c^2 t^2$, so in S'

$$x'^2 + y'^2 + z'^2 = c^2 t'^2$$

Lorentz transformation for this problem, relating primed, unprimed space & time:

$$\boxed{\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{vx}{c^2}\right) \end{aligned}}$$

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}, \text{ or } \gamma = (1 - \beta^2)^{-1/2}$$

where $\beta = v/c$

$$\boxed{\begin{aligned} x &= \gamma(x' + vt') \\ t &= \gamma\left(t' + \frac{vx'}{c^2}\right) \end{aligned}}$$

Inverse transformation gives

Time dilation

A clock at rest in frame S' , at x'_1 , measures time interval $\Delta t' = t'_2 - t'_1$.

We transform the times of these events, getting
 $t_1 = \gamma(t'_1 + \beta x'_1/c)$
 $t_2 = \gamma(t'_2 + \beta x'_2/c)$ In subtraction, the x'_1 terms cancel,

$\boxed{\Delta t = \gamma \Delta t'}$ Relativistic time dilation.

We directly measure time dilation in the increasing half-lives of radioactive particles produced in accelerators & in cosmic rays. E.g., muons shouldn't be able to penetrate the atmosphere to our detectors before decaying, if time dilation didn't exist.

Length contraction A stick lies on x' -axis in S' . Ends at x'_1 and x'_2 , so length $\Delta x' = x'_2 - x'_1$.

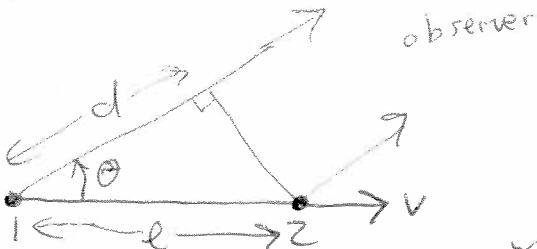
In S , at an agreed time, observers mark positions of x'_1 & x'_2 , giving coordinates x_1, t_1 & x_2, t_2 , where $t_2 = t_1$.

Use first Lorentz transform, $x' = \gamma(x - vt)$ to eliminate t , get $x_2 - x_1 = \frac{x'_2 - x'_1}{\gamma}$, or generally

$$\boxed{\Delta x = \frac{\Delta x'}{\gamma}}$$

Doppler shift Any periodic phenomenon will show time dilation - in addition, the classical light-travel time changes the period (this 2nd effect is called the classical Doppler effect).

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Consider moving source
(in our ~~S~~ S' frame) that
emits one period of radiation
while moving from 1 to 2 at speed v .

Take frequency in ~~S'~~ S' to be ω' . Now, ωt
between points 1, 2 in S is

$$\Delta t = \Delta t' = \frac{2\pi}{\omega'} \text{ from time dilation.}$$

Now set $d = v\Delta t$; see $d = v\Delta t \cos\theta$.

Difference in arrival times at the observer
is Δt plus the time to cross d , d/c . So

$$\Delta t_A = \Delta t - \frac{d}{c} = \Delta t \left(1 - \frac{v}{c} \cos\theta\right)$$

So the observed angular frequency $\omega = \frac{2\pi}{\Delta t_A} = \frac{\omega'}{\Delta t_A \left(1 - \beta \cos\theta\right)}$

$$\omega' = \omega_j \left(1 - \frac{v}{c} \cos\theta\right)$$

$$\omega = \omega_j \left(1 + \frac{v}{c} \cos\theta'\right) \quad \text{Note that } \theta \text{ also varies,}$$

If the source approaches head-on, $\theta = 0$, so

$$\boxed{\omega = \left(\frac{1+\beta}{1-\beta}\right)^{1/2} \omega'}$$

However, if $\cos\theta = 0$, then $\boxed{\omega = \omega'/j}$, a lower freq.
This is purely time dilation.

Lorentz invariants = Quantities that are the
same in all Lorentz (nonaccelerating) frames. ~~not~~
Examples: from emission event to arrival event, a pulse
of light covers a spacetime interval,

For any transformation,
 $x^2 + y^2 + z^2 - c^2 t^2 = -s^2$, a spacetime invariant.

Consider 2 events in frames S, S'; events 1 & 2,
 with differing x, t .

$$(\Delta x)^2 - c^2 (\Delta t)^2 = -(\Delta s)^2 \text{ is an invariant.}$$

We define 3 cases;

- $(\Delta s)^2 = 0$ A light signal can connect these, so it is a lightlike interval.

- $(\Delta s)^2 < 0$ $|\Delta x| > c|\Delta t|$ so light can't cross it in time.
 Event 2 can't be caused by event 1.

This is a spacelike interval.

- $(\Delta s)^2 > 0$ $|\Delta x| < c|\Delta t|$ Light signal, plus time, can connect these; a timelike interval.
 For these, $\Delta s/c$ is the proper time,

$$(\Delta \tau)^2 = (\Delta t)^2 - \frac{(\Delta x)^2}{c^2}$$

This proper time is an invariant & is the time measured in a frame where both events have the same position; e.g. a traveller at constant velocity.

Similarly, if $(\Delta s)^2 < 0$, a proper distance interval
 $(\Delta \sigma)^2 = (\Delta x)^2 - (c\Delta t)^2$ is an invariant.

This is the distance in a frame where both events are simultaneous.

Four-vectors Describe events as 4-vectors $[x, y, z, ct]$, or $[\vec{x}, ct]$. These have invariants $x^2 + y^2 + z^2 - c^2 t^2$, the "length squared" of the 4-vector.

It's basically a dot product with a minus for 4th component; some texts define "ict" instead of "ct", to emphasize this.

Difference of 2 4-vectors is also a 4-vector $[\vec{dx}, ct]$, & transforms by Lorentz transform rules. Can rewrite as $[\vec{dx}, cd\tau]$.

Can multiply a 4-vector by a scalar to get a new ~~4-vector~~ 4-vector.

The ratio $(m/\Delta\tau)$ is a scalar, as both m and $\Delta\tau$ are invariant.

$$\gamma = m/\Delta\tau (= \gamma m/dt)$$

Multiply $[\vec{dx}, cd\tau]$ by this scalar, get

$$[\gamma m \vec{dx}/dt, \gamma mc^2/c] = [\vec{p}, u/c]$$

which is a momentum-energy 4-vector.

$$\gamma m \vec{v} = \vec{p} = \gamma m \vec{v} c.$$

$$U = \gamma mc^2 = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} mc^2 \xrightarrow[\text{velocities}]{\text{law}} \left(1 + \frac{1}{2} \frac{v^2}{c^2}\right) mc^2 = mc^2 + \frac{1}{2} mv^2$$

Clearly $\gamma = \frac{U}{mc^2} = \frac{\text{total energy}}{\text{rest energy}} \rightarrow \frac{\text{rest energy}}{\text{kinetic energy}}$

Invariant quantity of a 4-vector is its dot product with itself. For the $[\vec{p}, u/c]$ 4-vector, the last term is negative:

$$[\vec{p}, u/c] \cdot [\vec{p}, u/c] \rightarrow p_x^2 + p_y^2 + p_z^2 - \frac{U^2}{c^2} = p^2 - \frac{U^2}{c^2}$$

For any single particle or system of particles this is an invariant.

In the frame where the particles at rest, $p^2 = 0 \Rightarrow \gamma = 1$, so $p^2 - U^2/c^2 = -m^2 c^2$.

If one multiplies through by c^2 , get $-(mc^2)^2$, the rest energy squared, so $U^2 - (pc)^2 = (mc^2)^2$.

Photons

Set $m=0$, get $\boxed{U=pc}$, or $p = \frac{h\nu}{c}$.

From our previous transforms we get transformations of the $[\vec{p}, U/c]$ 4-vector:

$$\begin{aligned} p_x' &= \gamma(p_x - \beta U/c) & p_x &= \gamma(p_x' + \beta U/c) \\ p_y' &= p_y & & \vdots \\ p_z' &= p_z & & \\ U' &= \gamma(U - \beta c p_x) & U &= \gamma(U' + \beta c p_x') \end{aligned}$$

If we multiply momentum by $2\pi/h$, use $p = \frac{h\nu}{c}$, then for photons

$$p_x \frac{2\pi}{h} = \frac{h\nu}{c} \frac{2\pi}{h} = \frac{2\pi}{\lambda} = k, \text{ magnitude of wave-propagation vector.}$$

\vec{k} is the wave-propagation vector.

Multiplying the last component of $(\vec{p}, U/c)$ by $2\pi/h$, we get

$$\frac{U}{c} \frac{2\pi}{h} = \frac{h\nu}{c} \frac{2\pi}{h} = \frac{\omega}{c} \quad \text{where } \omega \text{ is the angular frequency of the radiation.}$$

So photons have a 4-vector $[\vec{k}, \omega/c]$. ~~as well as particles~~

Transformation of this 4-vector gives us the change in frequency (Doppler shift) and direction (aberration) of the wave.