Twisted Characters of Depth-Zero Supercuspidal Representations of GL(n)

by

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Abstract

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Let F be a finite extension of $\mathbb{Q}_p$, for p an odd prime. Given an automorphism θ of $G = \mathrm{GL}_n(F)$ of finite order, any irreducible, θ-stable representation π of G may be extended to an irreducible representation π^+ of $G^+ = G \rtimes \langle\theta\rangle$. If π is supercuspidal, we obtain a Harish-Chandra-type integral formula for the character Θ_{π^+} of π^+, expressed on sufficiently regular elements of G^+. In the case that π is a depth-zero supercuspidal representation of G, we use this integral formula to compare values of Θ_{π^+} on a neighbourhood of θ in G^+ to a (finite) linear combination of characters of depth-zero supercuspidal representations of the group G_θ of θ-fixed points in G. Then, using properties of the characters in this linear combination, we compare Θ_{π^+} to a linear combination of Fourier transforms of orbital integrals on the Lie algebra $\mathrm{Lie}(G_\theta)$ of G_θ.

Dedication

To Hazel Jane Rose

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0. INTRODUCTION

Let F be a finite extension of $\mathbb{Q}_p$, for p an odd prime. Let $\mathcal{O}_F$ be the ring of integers in F, and $\mathcal{P}_F$ its prime ideal. Let G be the group of F-rational points of a connected, reductive linear algebraic group which is defined over F. Let $C_c^\infty(G)$ be the space of complex-valued functions on G which are locally constant and compactly supported. Let (π, V) be a smooth (complex) representation of G. For any $f \in C_c^\infty(G)$, define an operator $\pi(f)$ on V by $\pi(f)v = \int_G f(g)\pi(g)v\,dg$, $v \in V$, where dg is Haar measure on G. Suppose π is admissible. That is, for any compact, open subgroup $K \subset G$, the subspace $V^K \subset V$ of vectors fixed by all elements of K is finite-dimensional. Then each operator $\pi(f)$ is of trace class, and we define the *character* of π to be the distribution on G (*i.e.*, the linear functional on $C_c^\infty(G)$) given by $\Theta_\pi(f) = \operatorname{tr}\bigl(\pi(f)\bigr)$, for $f \in C_c^\infty(G)$. By a result of Harish-Chandra ([21]), Θ_π is represented by a function on G, also denoted Θ_π, which is locally constant on the dense, open subset of regular elements $G_{\mathrm{reg}} \subset G$ and locally integrable on G. That is, $\Theta_\pi(f) = \int_G f(g)\Theta_\pi(g)\,dg$ for any $f \in C_c^\infty(G)$.

Let $\mathfrak{g}$ be the Lie algebra of G, and let $\mathfrak{b}$ be an $\mathrm{Ad}G$-invariant, symmetric, bilinear form on $\mathfrak{g}$. Fix a non-trivial additive character Λ of F with conductor $\mathcal{P}_F$. The Fourier transform of an element $f \in C_c^\infty(\mathfrak{g})$ is defined by $\hat{f}(X) = \int_{\mathfrak{g}} f(Y)\Lambda\bigl(\mathfrak{b}(X,Y)\bigr)\,dY$, $X \in \mathfrak{g}$. For any $X \in \mathfrak{g}$, the homogeneous space $G/C_G(X)$ carries a unique (up to a constant) invariant measure $d\dot{x}$. The orbital integral associated to the G-orbit of X in $\mathfrak{g}$ is the distribution defined by $\mu_X(f) = \int_{G/C_G(X)} f({}^{x}X)\,d\dot{x}$, $f \in C_c^\infty(\mathfrak{g})$. Define the Fourier transform of μ_X by $\hat{\mu}_X(f) = \mu_X(\hat{f})$. Harish-Chandra also showed that the distribution $\hat{\mu}_X$ is represented by a locally integrable function on $\mathfrak{g}$ ([20]), also denoted $\hat{\mu}_X$.

In the early 1970s, Harish-Chandra ([20]) proved a local character expansion, given as follows. Let γ be a semisimple element of G. Let G_γ be the centralizer of γ in G, and let $\mathfrak{g}_\gamma$ be the Lie algebra of G_γ. Then for all regular elements $X \in \mathfrak{g}_\gamma$ which are sufficiently near 0,

$$\Theta_\pi\bigl(\gamma(\exp X)\bigr) = \textstyle\sum_{\mathcal{O} \in \mathcal{O}_\gamma(0)} c_{\mathcal{O}}(\pi)\,\hat{\mu}_{\mathcal{O}}(X). \tag{0.0.1}$$

Here, exp is the exponential map or a suitable replacement (such as a truncated exponential map), $\mathcal{O}_\gamma(0)$ is the set of nilpotent G_γ-orbits in $\mathfrak{g}_\gamma$, each $c_{\mathcal{O}}(\pi)$ is a complex number, and $\hat{\mu}_{\mathcal{O}}$ is the Fourier transform (relative to $\mathfrak{g}_\gamma$) of the orbital integral over $\mathcal{O}$. This expansion was a generalization of a result of Howe ([23]), who proved (0.0.1) in the case that $G = \mathbf{GL}_n(F)$ and $\gamma = 1$. Later, Clozel extended Harish-Chandra's result to non-connected groups in [10]. When $\gamma = 1$, the results of DeBacker in [11] give an explicit neighbourhood of 0 in $\mathfrak{g}$ (which depends on the depth of the representation π) on which (0.0.1) is valid. Here, DeBacker works in the connected case, but remarks that his results are also valid in the non-connected case. In [2], Adler and Korman generalize DeBacker's result, providing an explicit domain on which (0.0.1) is valid for more general semisimple $\gamma \in G$, and explicitly allowing the non-connected case.

It is currently not possible to obtain an explicit relation between characters and orbital integrals from (0.0.1), as it is not known in general how to compute the coefficients $c_{\mathcal{O}}(\pi)$. However, for some representations, it is possible to express the values of $\Theta_\pi \circ \exp$ near 0 as an explicit linear combination of Fourier transforms of orbital integrals involving regular G-orbits in $\mathfrak{g}$ which depend on intrinsic properties of the data used in the construction of the representation π. For example, in the 1990s, it was discovered (see

Murnaghan's work in [31, 32, 33, 34]) that the characters of many irreducible, supercuspidal representations of classical p-adic groups exhibit the following behaviour. There exists a regular elliptic G-orbit $\mathcal{O}_\pi$ in $\mathfrak{g}$ such that, sufficiently near 0, $\Theta_\pi \circ \exp$ coincides with $d(\pi)\hat{\mu}_{\mathcal{O}_\pi}$. Here, $d(\pi)$ is the formal degree of π, and the orbit $\mathcal{O}_\pi$ depends on the specific K-types which are contained in π (that is, the pairs (σ, K) where σ is an irreducible representation of a compact, open subgroup K of G such that $\pi|K$ contains σ). More recently, Adler and DeBacker ([1]) have determined explicit neighbourhoods of 0 on which such a relation holds for supercuspidal representations of general linear groups, and also for tame, very supercuspidal representations of more general reductive groups. In [26] and [27], subject to some hypotheses on G, Kim and Murnaghan obtain similar results for many irreducible, admissible representations. They express $\Theta_\pi \circ \exp$ as a linear combination of Fourier transforms of regular orbital integrals on a specific neighbourhood of 0, where the orbital integrals which appear are again determined by the K-types which occur in π. As such, expressions for Θ_π of this type can be used to distinguish properties of individual representations, in ways in which the local character expansion generally cannot be used.

In recent work ([13]), DeBacker and Reeder explicitly construct depth-zero supercuspidal "L-packets" for connected, reductive groups which are quasi-split over F and split over an unramified extension of F. Each L-packet corresponds to a tame Langlands parameter which is in "general position", and each representation in a given L-packet is induced from a Deligne-Lusztig representation associated to a depth-zero character of an unramified, elliptic maximal torus in G. By making extensive use of Deligne-Lusztig theory, they derive expressions for the characters of the depth-zero supercuspidal representations in their L-packets on various large G-domains. Each such expression is an explicit linear combination of Fourier transforms of orbital integrals. Obtaining such character formulas was an essential step in proving the stability of a certain sum of the characters of the representations in the depth-zero L-packets they construct, an important conjectured property of L-packets.

Apart from the papers of Clozel ([10]), DeBacker ([11]), and Adler and Korman ([2]), the above mentioned results deal with the connected case. Suppose that G is a non-connected, reductive p-adic group, π is an irreducible, admissible representation of G whose restriction to the identity component of G is irreducible and supercuspidal, and γ is a semisimple element of G which does not lie in the identity component of G. In this thesis, we ask whether the restriction of Θ_π to a neighbourhood of γ can be expressed as an explicit linear combination of Fourier transforms of orbital integrals, where the associated orbits lie in the Lie algebra of the centralizer of γ in G and are defined in terms of specific data related to the representation π. In particular, we consider the case of $G = \mathrm{GL}_n(F) \rtimes \langle\theta\rangle$, for θ a finite-order automorphism of $\mathrm{GL}_n(F)$. We prove such a relation on elements of the form $g \rtimes \theta$, near $1 \rtimes \theta$, under certain hypotheses on θ and the structure of $\mathrm{GL}_n(F)$ relative to θ. In general, if $G = G^0 \rtimes \langle\theta\rangle$, for G^0 the group of F-rational points of a connected, reductive F-group, and θ a finite-order automorphism of G^0, then the distribution Θ_π can be expressed as a sum of the $(\theta^i, 1)$-twisted characters of $\pi|G^0$ (see [28]), $0 \le i \le |\langle\theta\rangle| - 1$. Here, the 1 in "$(\theta^i, 1)$-twisted" represents the trivial quasi-character of G^0. In light of this, the type of relation between Θ_π and Fourier transforms of orbital integrals developed here is of interest for the theory of twisted endoscopy, which studies those representations π^0 of G^0 which satisfy $\pi^0 \circ \theta \simeq \omega \otimes \pi^0$, for some quasi-character ω of π^0. In analogue with the theory of standard endoscopy, there are expected identities between twisted characters of G^0 and stable characters of an endoscopic group for (G^0, θ, ω).

We will assume in this thesis that θ may be realized as the restriction to $\mathbf{G}(F_0)$ of an F_0-automorphism of the restriction of scalars $\mathbf{G} = \mathrm{R}_{F/F_0}\,\mathbf{GL}_n$, where F_0 is a subfield of F such that F/F_0 is cyclic. Under this assumption, we may (and do) identify $\mathbf{G}(F_0)$ with $\mathbf{GL}_n(F)$. We note here one important complication that arises in the non-connected case. The existence of a θ-stable pair $\mathbf{B} \supset \mathbf{T}$ consisting of a Borel subgroup $\mathbf{B}$ and a maximal, unramified, F_0-minisotropic F_0-torus $\mathbf{T}$ of $\mathbf{G}$ is essential to our constructions. In particular, our main result concerns the character Θ_π, where $\pi|\mathbf{GL}_n(F)$ is a depth-zero supercuspidal representation which is compactly induced from a Deligne-Lusztig representation associated to a depth-zero character of $T = \mathbf{T}(F_0)$, for such a θ-stable torus $\mathbf{T}$. Before relating Θ_π to Fourier transforms of orbital integrals, we first express it as a linear combination of characters of representations of the group of fixed points G_θ in G. These representations are compactly induced from Deligne-Lusztig representations associated to depth-zero characters of G-conjugates of the group of fixed points T_θ in T. However, this construction fails if any of the associated G-conjugates of $\mathbf{T}$ are not contained in θ-stable Borel subgroups, as then each such conjugate of T_θ is not the group of F_0-points of a maximal torus of $\mathbf{G}_\theta$ (see Corollary 1.3.3).

It should be possible to extend the arguments of this thesis to obtain similar results for twisted characters more generally. In particular, the Deligne-Lusztig theory of Digne and Michel ([14]) used in Chapter 2 is developed for general non-connected, reductive groups over finite fields. As noted in [7], the arguments of Chapter 3 should be applicable to groups other than general linear groups. As well, the proofs of the main results in §§4.4–4.6 are largely independent of the fact that we are working in a general linear group, once the hypotheses of the preceding sections and chapters are assumed. Finally, it may be possible to modify the current proofs to remove the reliance on some of the hypotheses.

Statement of results

Rather than state the general results of this thesis, we illustrate them by giving here an outline of a specific case. The details of this case appear in §5.1.

Suppose F_0 is a subfield of F of finite index d such that F/F_0 is unramified. Let θ be a generator of $\mathrm{Gal}(F/F_0)$. Letting θ act on the entries of elements of $G = \mathbf{GL}_n(F)$ defines an automorphism of G. We may realize this automorphism as the restriction to $\mathbf{G}(F_0)$ of an F_0-automorphism of the restriction of scalars $\mathbf{G} = \mathrm{R}_{F/F_0}\,\mathbf{GL}_n$, identifying $\mathbf{G}(F_0)$ with G. The subgroups $K_0 = \mathrm{GL}_n(\mathscr{O}_F)$ and $K_1 = 1 + \mathrm{M}_n(\mathscr{P}_F)$ of G are θ-stable, so θ induces an automorphism of $\mathrm{G} = K_0/K_1 \simeq \mathbf{GL}_n(k_F)$, where k_F is the residue field of F. This automorphism is given by letting θ, as an element of $\mathrm{Gal}(k_F/k_{F_0})$, act on the entries of elements of G.

Suppose π is a θ-stable, irreducible, supercuspidal representation of G. By choosing an intertwining operator $A_\pi \in \mathrm{Hom}_G(\pi, \pi\circ\theta)$ with $A_\pi^d = 1$, we may extend π to an irreducible representation π^+ of $G^+ = G \rtimes \langle\theta\rangle$ by setting $\pi^+(\theta) = A_\pi$. We first show (Corollary 3.5.5) that for elements $g \in G^+$ which commute with θ and satisfy a regularity condition, the value of the character Θ_{π^+} of π^+ at g can be expressed by the Harish-Chandra-type integral formula

$$\Theta_{\pi^+}(g) = [F : F_0]\,\frac{d(\pi^+)}{\varphi(1)} \int_{Z_\theta\backslash G} \int_{K_0} \varphi({}^{xk}g)\, dk\, d\dot{x}. \tag{0.0.2}$$

Here, dk is normalized Haar measure on K_0, Z_θ is the group of θ-fixed points in the centre Z of G, $d\dot{x}$ is invariant measure on $Z_\theta\backslash G$, $d(\pi^+)$ is the formal degree of π^+ relative to $d\dot{x}$, and φ is a matrix coefficient of π^+ satisfying $\varphi(1) \neq 0$.

Suppose $\pi = \text{c-Ind}_{ZK_0}^{G}\sigma$, for σ a θ-stable, irreducible (hence finite-dimensional) representation of ZK_0 which is trivial on K_1 and whose restriction to K_0 is cuspidal as a representation of G. Then π is an irreducible, admissible, θ-stable, depth-zero supercuspidal representation of G. If we choose an intertwining operator $A_\sigma \in \text{Hom}_{ZK_0}(\sigma, \sigma\circ\theta)$ in a compatible way, then π^+ is equivalent to $\text{c-Ind}_{(ZK_0)^+}^{G^+}\sigma^+$ (Proposition 1.5.4).

Assume that the character χ_{σ^+} of σ^+ satisfies

$$\chi_{\sigma^+}(k\theta) = \varepsilon\left(\mathsf{R}_{\mathsf{T}^+}^{\mathsf{G}^+}\lambda^+\right)(\bar{k}\theta), \qquad \left(k \in K_0\right),$$

where $\bar{k}$ is the image of k in G, $\varepsilon = \pm 1$, T is the group of k_{F_0}-points of a θ-stable, k_{F_0}-minisotropic maximal torus $\mathbf{T}$ of $\mathbf{G} = \mathrm{R}_{k_F/k_{F_0}}\mathbf{GL}_n$, λ^+ is a one-dimensional character of $\mathsf{T}^+ = \mathsf{T} \rtimes \langle\theta\rangle$ such that $\lambda^+|\mathsf{T}$ is in general position, and $\mathsf{R}_{\mathsf{T}^+}^{\mathsf{G}^+}\lambda^+$ is the Deligne-Lusztig virtual character of $\mathsf{G}^+ = \mathsf{G} \rtimes \langle\theta\rangle$ defined by Digne and Michel ([14]). If $\gamma \in K_0$ is θ-fixed and topologically unipotent, then using a formula of [14] we show (Theorem 2.2.9) that

$$\left(\mathsf{R}_{\mathsf{T}^+}^{\mathsf{G}^+}\lambda^+\right)(\bar{\gamma}\theta) = \lambda^+(\theta)\,\mathsf{Q}_{\mathsf{T}_\theta}^{\mathsf{G}_\theta}\left(\bar{\gamma}\right), \tag{0.0.3}$$

where G_θ is the group of θ-fixed points in G, $\mathsf{T}_\theta = \mathsf{T}\cap\mathsf{G}_\theta$, and $\mathsf{Q}_{\mathsf{T}_\theta}^{\mathsf{G}_\theta}$ is a Green function associated to $\mathsf{T}_\theta \subset \mathsf{G}_\theta$. In this way, we may compare the character χ_{σ^+} on such elements $\gamma\theta$ to the character of a representation σ_θ of $Z_\theta(K_0)_\theta$ which is cuspidal as a representation of G_θ. Let $\dot{\chi}_{\sigma^+}$ be the extension by zero of χ_{σ^+} to all of G^+. Then, taking $\varphi = \dot{\chi}_{\sigma^+}$ in (0.0.2), we use (0.0.3) to show that

$$\Theta_{\pi^+}(\gamma\theta) = [F : F_0]\,\varepsilon_+\,\lambda^+(\theta)\,\frac{\deg(\sigma_\theta)}{\deg(\sigma)}\,\frac{d(\pi)}{d(\pi_\theta)}\,\Theta_{\pi_\theta}(\gamma), \tag{0.0.4}$$

where $\pi_\theta = \text{c-Ind}_{Z_\theta(K_0)_\theta}^{G_\theta}\sigma_\theta$, and $\varepsilon_+ = \pm 1$ (see Theorem 4.4.2 and §4.6 for the general statement). Finally, we may use a result of [12] to obtain a θ-fixed, regular semisimple element $X_\theta \in \mathrm{M}_n(\mathscr{O}_F)$ such that $\Theta_{\pi_\theta}(\gamma) = d(\pi_\theta)\,\hat{\mu}_{X_\theta}(\gamma - 1)$, where μ_{X_θ} is the orbital integral on $\text{Lie}(G_\theta)$ associated to X_θ, and $\hat{\mu}_{X_\theta}$ is its Fourier transform. Thus, (0.0.4) becomes

$$\Theta_{\pi^+}(\gamma\theta) = [F : F_0]\,\varepsilon_+\,\lambda^+(\theta)\,\frac{\deg(\sigma_\theta)}{\deg(\sigma)}\,d(\pi)\,\hat{\mu}_{X_\theta}(\gamma - 1) \tag{0.0.5}$$

(see Theorem 4.5.2 and §4.6 for the general result).

For other types of automorphisms, we follow the same basic procedure as above. However, the situation is complicated by the fact that the integral formula analogous to (0.0.2) may not converge. In such cases, we work around this by first restricting π^+ to an appropriate subgroup H^+ of G^+. This restriction is not irreducible, and each irreducible component may make a contribution to Θ_{π^+}. Therefore, in general the result analogous to (0.0.5) will involve a (finite) linear combination of Fourier transforms of orbital integrals.

Outline

Chapter 1 is devoted to setting up notation and discussing basic concepts. By a *p-adic field*, we will always mean a non-archimedean local field of characteristic zero. In §1.2, we discuss the concrete realization of restriction of the ground field that we use. In §1.3, we discuss some properties of quasi-semisimple automorphisms of a connected reductive group $\mathbf{G}$, defined over any perfect field F. In our discussion of regular elements, after Lemma 1.3.12 we restrict to the case that $\mathbf{G} = \mathrm{R}_{F/F_0}\,\mathbf{GL}_n$ for F/F_0 a finite, abelian extension of p-adic fields. In §1.4, we state O'Meara's classification of the automorphisms of $\mathbf{GL}_n(F)$ (as given in [17]), and based on it give a preliminary restriction on the types of automorphisms of $\mathrm{R}_{F/F_0}\,\mathbf{GL}_n$ which we will consider. In §1.5, we discuss representations of a totally disconnected, locally compact group G which are stable under some finite-order automorphism θ of G. For such a representation π of G, we describe how to extend π to $G^+ = G \rtimes \langle\theta\rangle$, and state some properties of this extension. In our discussion of unitary twists of stable representations in §1.5.3, we restrict to considering only certain types of automorphisms of $G = \mathbf{GL}_n(F)$, for F a p-adic field.

Chapter 2 discusses the Deligne-Lusztig theory for non-connected, reductive algebraic groups over a finite field, as developed by Digne and Michel ([14]). In particular, for a non-connected group whose quotient modulo its identity component is cyclic and consists of semisimple elements, we extract from this theory simplified formulas for Deligne-Lusztig characters on certain types of elements in the set of rational points of connected components away from the identity.

In Chapter 3, we follow the method used by Bushnell and Henniart ([7]) to develop a Harish-Chandra-type integral formula for the character Θ_{π^+} of a representation π^+ of $G^+ = G \rtimes \langle\theta\rangle$. From this point on, we only consider the case that $G = \mathbf{GL}_n(F)$, for F a p-adic field, and the automorphism θ is of a certain form relative to the classification in §1.4. We consider π^+ to be the extension to G^+ (as defined in §1.5) of an irreducible, admissible representation π of G. In §3.1, we make some preliminary hypotheses on the automorphism θ and the structure of $\mathbf{G} = \mathrm{R}_{F/F_0}\,\mathbf{GL}_n$ and $G = \mathbf{GL}_n(F)$ relative to θ. As well, based on the form of θ that we consider, we may (and do) assume that F/F_0 is a finite, cyclic extension. We devote §3.2 and §3.3 to finding a family of operators on the space of π^+, indexed by the elements of a set of sufficiently regular elements of G^+, whose trace is equal to the value of Θ_{π^+} on those elements. In the case that θ sends scalar elements in G to either their inverse or some Galois conjugate of their inverse, it is necessary to restrict π^+ to an appropriate open, closed, normal subgroup H^+ of G^+ to ensure convergence of our integral formula. This is discussed in §3.4, and we give a decomposition for such a restriction. In §3.5, we are finally able to give the integral formula for Θ_{π^+}.

In Chapter 4, we specialize the integral formula from Chapter 3 to the case that π^+ is the extension to G^+ of an irreducible, depth-zero supercuspidal representation π of G. We then use this formula to find a relation between Θ_{π^+}, evaluated near θ, and a linear combination of Fourier transforms of orbital integrals on the Lie algebra of G_θ. Here, G_θ is the subgroup of G consisting of elements fixed by θ. In general, it is known that irreducible, depth-zero representations of connected, reductive p-adic groups are induced from representations of normalizers of maximal parahoric subgroups. In $G = \mathbf{GL}_n(F)$, there is one conjugacy class of maximal parahoric subgroup. Let K be a maximal parahoric subgroup of G, and let Z be the centre of G. Then the normalizer of K in G is ZK. Up to equivalence, an irreducible, depth-zero supercuspidal representation π of G is induced from an irreducible, smooth representation σ of ZK which is trivial on the pro-unipotent radical K' of K, and factors to an irreducible, cuspidal representation of $K/K' \simeq \mathbf{GL}_n(k_F)$. Here, k_F is the residue field of F. We assume that both K and σ are θ-stable, so that π is then also θ-stable. We discuss the details of the construction of such representations in §4.1, and add a sharpened version of one of the hypotheses from §3.1. In §4.2, we analyze the decomposition of the restriction of π^+ to the subgroup H^+ from §3.4 in more detail, given the nature of π, and use this extra information to simplify the integral formula from §3.5. We give some further hypotheses in §4.3 that will allow us to use the results of Chapter 2 to relate Θ_{π^+}, near θ, to a linear combination of characters of depth-zero, supercuspidal representations of G_θ. This relation is obtained in §4.4. In §4.5, we use the general results of DeBacker and Reeder ([13]) to express these characters on G_θ in terms of Fourier transforms of orbital integrals on its Lie algebra.

In Chapter 5, we apply the results of Chapter 4 to several specific cases. In particular, we show that our many hypotheses are lax enough as to be satisfied in a number of cases of interest. Note that the inner case is intentionally not considered specifically (though it is allowed for in our general results), as this case is effectively handled by DeBacker and Reeder in [13], without our many restrictions.

1. Preliminaries

1.1 Notation and basic facts

1.1.1 p-Adic fields. Let $\mathbb{Q}_p$ be the field of p-adic numbers, for p an odd prime, with p-adic absolute value $|\cdot|_p$. If q is a power of p, let $\mathbb{F}_q$ be the finite field with q elements. For any rational number a, let $\|a\|_p = |a|_p^{-1}$ be the *p-part of a* and $\|a\|_{p'} = a/\|a\|_p$ the *p'-part of a*. If A is a finite set, let $\|A\|_p$ (resp. $\|A\|_{p'}$) be the p-part (resp. p'-part) of the cardinality of A. Let $\bar{\mathbb{Q}}_p$ be the algebraic closure of $\mathbb{Q}_p$. By a *p-adic field*, we mean a finite extension $F \supseteq \mathbb{Q}_p$. For such a field, let $|\cdot|_F$ be the absolute value and val_F the valuation on F. Let $\mathcal{O}_F \supset \mathcal{P}_F$ be the ring of integers in F and its maximal ideal, respectively, with uniformizer ϖ_F. For $\alpha \in F^\times$, let $\mathrm{int}_F(\alpha) = \alpha/\varpi_F^{\mathrm{val}_F(\alpha)} \in \mathcal{O}_F^\times$. Let $k_F = \mathcal{O}_F/\mathcal{P}_F$ be the residue field of F, and denote its size by $q_F = |k_F|$. The residue field of $\mathbb{Q}_p$ is $\mathbb{F}_p$. For any element α (resp. subset A) of $\mathcal{O}_F$, let $\bar{\alpha}$ (resp. $\bar{A}$) be its image in k_F under the natural projection. If X is a matrix with entries in $\mathcal{O}_F$, let $\bar{X}$ be the matrix with entries in k_F obtained by applying the $\bmod \mathcal{P}_F$ map to each entry of X.

If $F \subset L$ is any finite, unramified extension, we may (and do) take a common uniformizer $\varpi = \varpi_L = \varpi_F$ for the valuations of F and L, and the inclusion $\mathcal{O}_F \hookrightarrow \mathcal{O}_L$ induces an injection $k_F \hookrightarrow k_L$. Considering k_F as a subfield of k_L, if α lies in $\mathcal{O}_F$ then its projections modulo $\mathcal{P}_F$ and $\mathcal{P}_L$ coincide. In this way, we obtain an isomorphism $\mathrm{Gal}(L/F) \simeq \mathrm{Gal}(k_L/k_F)$ by $\tau \mapsto \bar{\tau}$, where, given $\tau \in \mathrm{Gal}(L/F)$ and $\alpha \in \mathcal{O}_L$, $\bar{\tau}(\bar{\alpha}) = \overline{\tau(\alpha)}$. Therefore, $\mathrm{Gal}(L/F)$ is cyclic, and the generator which induces the Frobenius automorphism of k_L/k_F is called the Frobenius automorphism of L/F.

Lemma 1.1.1. *A subset $\mathcal{B} \subset \mathcal{O}_L$ is an $\mathcal{O}_F$-basis for $\mathcal{O}_L$ if and only if $\bar{\mathcal{B}}$ is a k_F-basis for k_L.*

Proof. The reverse direction is [9, Ch. 7, Lemma 5.4], so consider the forward direction. Let $n = [L : F]$, and suppose $\mathcal{B} = \{\xi_1, \ldots, \xi_n\}$ is an $\mathcal{O}_F$-basis for $\mathcal{O}_L$. Choose a subset $\mathcal{D} = \{\alpha_1, \ldots, \alpha_n\} \subset \mathcal{O}_L$ such that $\bar{\mathcal{D}}$ is a k_F-basis for k_L. Then by the reverse direction, $\mathcal{D}$ is an $\mathcal{O}_F$-basis for $\mathcal{O}_L$. Let A be the element of $\mathrm{End}_{\mathcal{O}_F}(\mathcal{O}_L)$ such that $A\alpha_i = \xi_i$ for each i. Clearly A is invertible, so that $A \in \mathrm{Aut}_{\mathcal{O}_F}(\mathcal{O}_L)$. If $\bar{A}$ is the image of A in $\mathrm{Aut}_{\mathcal{O}_F}(\mathcal{O}_L)/(1 + \varpi\,\mathrm{End}_{\mathcal{O}_F}(\mathcal{O}_L)) \simeq \mathrm{GL}_{k_F}(k_L)$, we have $\bar{A}\bar{\alpha}_i = \overline{A\alpha_i} = \bar{\xi}_i$ for each i, and hence $\bar{\mathcal{B}} = \{\bar{\xi_1}, \ldots, \bar{\xi_n}\}$ is a k_F-basis for k_L. $\square$

If $\mathcal{B} = \{\xi_1, \ldots, \xi_n\}$ is as in the lemma, then each ξ_i lies in $\mathcal{O}_L^\times$, since otherwise $\bar{\mathcal{B}}$ would be linearly dependent. Via our common uniformizer ϖ, we see that $\mathcal{B}$ is also an F-basis for L.

1.1.2 Linear algebra. For any positive integer n, let 0_n and 1_n denote the $n \times n$ zero and identity matrices, respectively. If $A_1, \ldots, A_m$ is a collection of square matrices, with A_i of dimension m_i, let $\mathrm{diag}(A_1, \ldots, A_m)$ be the square matrix of dimension $\sum_i m_i$ in which the A_i appear as blocks down the diagonal, with zero in all other entries.

If R is an integral domain and M is an R-module, let M^* be the linear dual of M. If M is free of rank n and $\mathcal{B}$ is a basis of M, let $[\cdot]_{\mathcal{B}} : M \to R^n$ be the coordinate transformation with respect to $\mathcal{B}$. Given another basis $\mathcal{D}$ of M, let $[\cdot]_{\mathcal{D}\mathcal{B}} : \mathrm{End}_R(M) \to \mathrm{M}_n(R)$ be the transformation which sends an R-linear endomorphism of M to the matrix which represents it relative to $\mathcal{B}$ and $\mathcal{D}$. That is,

$$[Am]_{\mathcal{D}} = [A]_{\mathcal{D}\mathcal{B}}[m]_{\mathcal{B}}, \qquad \big(A \in \mathrm{End}_R(M),\, m \in M\big).$$

If $\mathcal{D} = \mathcal{B}$, write $[A]_{\mathcal{B}}$ for $[A]_{\mathcal{B}\mathcal{B}}$. If $\mathrm{f}\colon M \times M \to R$ is a bilinear form on M, let $[\mathrm{f}]_{\mathcal{B}} = \big(\mathrm{f}(\xi_i, \xi_j)\big)$, where $\mathcal{B} = \{\xi_i\}$.

1.1.3 Groups and automorphisms. Let G be a group. Let $Z(G)$ denote its centre. For $x, y \in G$, denote left- and right-conjugation by ${}^x y = xyx^{-1}$ and $y^x = x^{-1}yx$, respectively, and let $\mathrm{Int_L}\, x$ and $\mathrm{Int_R}\, x$ denote the corresponding inner automorphisms of G. If G is an algebraic group and $\mathfrak{g}$ is its Lie algebra, denote $(\mathrm{Ad}\, g)(X)$ by ${}^g X$ for $g \in G$ and $X \in \mathfrak{g}$, and for an automorphism μ of G, let $\mathrm{d}\mu$ be its differential on $\mathfrak{g}$. If γ is an element and A a subset of G, let $A_\gamma = C_A(\gamma)$ be the centralizer of γ in A. More generally, if θ is an automorphism of G, let $G^+ = G \rtimes \langle\theta\rangle$ and let $A_\theta = G^+_\theta \cap A$ be the subset of elements in A which are fixed by θ. As a subgroup, G is normal in G^+. For any integer i let G^i denote the coset $G\theta^i$. Let A^+ be the subgroup of G^+ generated by A and θ, and let $A^i = A^+ \cap G^i$. If A is a θ-stable subgroup of G then $A^+ = A \rtimes \langle\theta\rangle$ and $A^i = A\theta^i$. Considering the automorphism θ of G as the restriction to G of the inner automorphism of G^+ associated to θ, we may write $\theta(x) = {}^\theta x$ for x in G. The centre $Z(G^+)$ of G^+ is easy to compute, and we omit the proof of the following lemma.

Lemma 1.1.2.

(1) $Z(G^+) \cap G = Z(G)_\theta$.

(2) For $i \neq 0$, *if* $Z(G^+) \cap G^i \neq \emptyset$, *then there exists* $g \in G_\theta$ *with* $\theta^i = \mathrm{Int_R}\, g$ *as an element of* $\mathrm{Aut}(G)$, *and* $Z(G^+) \cap G^i = g\theta^i Z(G)_\theta$.

Remark. The hypothesis of (2) is satisfied for some $i > 0$, for example, when θ is a non-trivial inner automorphism of G.

Let $1 - \theta\colon G \to G$ denote the map $x \mapsto x({}^\theta x^{-1})$, so that G_θ is precisely the set of elements of G which are sent to 1. The map $1 - \theta$ differentiates the cosets of G_θ in G, since for $x, y \in G$ we have $xG_\theta = yG_\theta$ (resp. $G_\theta x = G_\theta y$) if and only if $(1-\theta)(x) = (1-\theta)(y)$ (resp. $(1-\theta)(x^{-1}) = (1-\theta)(y^{-1})$). If θ has finite order d, let $\mathrm{N}_\theta\colon G \to G$ denote the map $x \mapsto x\theta(x)\cdots\theta^{d-1}(x)$. The image $\mathrm{N}_\theta(x)$ is called the *θ-norm of x*. Considering G as a subgroup of G^+, we may write $\mathrm{N}_\theta(x) = (x\theta)^d$. Note that N_θ maps every element of $(1-\theta)(G)$ to 1. If G is abelian, then both maps $1-\theta$ and N_θ are homomorphisms, and $\mathrm{N}_\theta(G) \subseteq G_\theta = \ker(1-\theta)$. If G is finite abelian and one of the maps $1-\theta\colon G \to \ker \mathrm{N}_\theta$, $\mathrm{N}_\theta\colon G \to G_\theta$ is surjective, then so is the other.

If H is a subgroup of G, let $N_G(H)$ be the normalizer of H in G, and let $W_G(H) = N_G(H)/H$. If G is an algebraic group defined over a field F and H is a closed F-subgroup of G, let $W_G(H)^F = N_{G(F)}(H)/H(F)$. Let $\widehat{H}$ be the dual group of (complex) characters of H. If G is a topological group and H is a closed subgroup, we take $\widehat{H}$ to be only the continuous characters of H. Every element $x \in G$ induces a bijection $\widehat{H} \to ({}^x H)^\wedge$ by $\lambda \mapsto {}^x\lambda$, for $\lambda \in \widehat{H}$. Here, ${}^x\lambda$ is defined by ${}^x\lambda(g) = \lambda(g^x)$, for $g \in {}^x H$. This defines an action of $N = N_G(H)$ on $\widehat{H}$ which factors to an action of $W = W_G(H)$ on $\widehat{H}$. Suppose H is stable under some automorphism θ of G. Then N is also θ-stable, with $N^+ = N_{G^+}(H)$. Moreover, $\theta|N_G(H)$ factors to an automorphism of W, with $W^+ = N^+/H$. Therefore, we may extend the action of W on $\widehat{H}$ to W^+ by setting ${}^\theta\lambda = \lambda \circ \theta^{-1}$, for $\lambda \in \widehat{H}$.

1.1.4 Algebraic geometry and algebraic groups. We will assume all varieties to be affine. If X is a variety defined over a field F, let $X(F)$ be the set of F-rational points of X. The variety X will be identified with $X(\bar{F})$, where $\bar{F}$ is the algebraic closure of F.

For most of this thesis, we will use a boldface Roman font for an algebraic group defined over a p-adic field and an italic font for the corresponding group of rational points. For an algebraic group defined over a finite field, we will use a boldface Euler font, and the corresponding group of rational points will be denoted using a normal Euler font. For example, for $\mathbf{G}$ defined over $\mathbb{Q}_p$, we write $G = \mathbf{G}(\mathbb{Q}_p)$, while for $\mathbf{G}$ defined over $\mathbb{F}_p$, we write $\mathrm{G} = \mathbf{G}(\mathbb{F}_p)$. One notable exception to this rule occurs in §1.3, where, for part of that section, $\mathbf{G}$ is taken to be an algebraic group defined over any perfect field. We will write $\mathbb{G}_\mathrm{m}$ for $\mathbf{GL}_1$ when the field of definition is understood. Elements of either $(\mathbb{G}_\mathrm{m})^n$ or its embedding in $\mathbf{GL}_n$ as the diagonal torus will often be written as n-tuples, sometimes prefixed with diag in the latter case. We will often identify $\mathbb{G}_\mathrm{m}$ with $Z(\mathbf{GL}_n)$, without comment.

Suppose $\mathbf{G}$ is a connected linear algebraic group, defined over a perfect field F. Subgroups $\mathbf{B} \supset \mathbf{T}$ consisting of a Borel subgroup $\mathbf{B}$ and a maximal torus $\mathbf{T}$ of $\mathbf{G}$ will be referred to as a *pair* in $\mathbf{G}$. A pair will be called *F-split* if the torus $\mathbf{T}$ is defined and split over F. If $\mathbf{G}$ is a torus, it contains a unique maximal F-split subtorus $\mathbf{G}_\mathrm{s} = \mathbf{G}_{\mathrm{s},F}$. The *$F$-rank of* $\mathbf{G}$ is defined to be $\operatorname{rk}\mathbf{G}_\mathrm{s}$, and is denoted $\operatorname{rk}_F \mathbf{G}$. If $\mathbf{G}$ is a quasi-split F-group, define $\operatorname{rk}_F \mathbf{G}$ to be $\operatorname{rk}_F \mathbf{T}$, for $\mathbf{B} \supset \mathbf{T}$ any pair in $\mathbf{G}$ which is defined over F. In either case, define the *F-sign of* $\mathbf{G}$ (denoted $\varepsilon_{\mathbf{G},F} = \varepsilon_\mathbf{G}$) to be $(-1)^{\operatorname{rk}_F \mathbf{G}}$. A maximal F-torus $\mathbf{T} \subset \mathbf{G}$ is called *F-minisotropic* if its F-rank is minimal over all maximal F-tori of $\mathbf{G}$. Suppose that $\mathbf{G}$ is reductive. Since $Z(\mathbf{G})$ is contained in every maximal torus of $\mathbf{G}$, a sufficient condition for a maximal F-torus $\mathbf{T} \subseteq \mathbf{G}$ to be F-minisotropic is that $\mathbf{T}_\mathrm{s} \subseteq Z(\mathbf{G})$. If $\mathbf{G}$ is also F-split, then this condition is both sufficient and necessary.

1.1.5 Totally disconnected groups and their representations. Let (π, V) be a (complex) representation of a group G. If A and W are subsets of G and V, respectively, let $A \cdot W = \operatorname{Span}\{\pi(a)w \mid a \in A,\ w \in W\}$. For $v \in V$, write $A \cdot v$ for $A \cdot \{v\}$. If H is a subgroup of G, let V^H be the subspace of H-fixed vectors of V. If (σ, W) is a representation of H and $g \in G$, let ${}^g\sigma$ (resp. σ^g) be the representation of ${}^gH$ (resp. H^g) induced by $\operatorname{Int}_\mathrm{R} g$ (resp. $\operatorname{Int}_\mathrm{L} g$). Let $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{C}$ be the obvious pairing, and let (π^*, V^*) be the representation of G defined by

$$\left\langle \pi^*(x)\lambda, v \right\rangle = \left\langle \lambda, \pi(x^{-1})v \right\rangle, \qquad \big(x \in G,\ \lambda \in V^*,\ v \in V\big).$$

Suppose that G is a totally disconnected, locally compact group. Let V_sm be the subspace of V consisting of vectors with open stabilizer in G. If $V_\mathrm{sm} = V$, then π is called a *smooth* representation of G. A one-dimensional, smooth representation of G will be referred to as a *quasi-character* of G. A smooth representation π is called *admissible* if V^K is finite-dimensional for every compact, open subgroup K of G. Let $\widetilde{V} = (V^*)_\mathrm{sm}$, and define the *contragredient* $\widetilde{\pi}$ of π to be $\pi^* | \widetilde{V}$. For any $v \in V$ and $\widetilde{v} \in \widetilde{V}$, the function

$$\varphi_{v,\widetilde{v}} : G \to \mathbb{C}, \qquad x \mapsto \langle \widetilde{v}, \pi(g)v \rangle,$$

is called a *matrix coefficient* of π. A smooth representation π is called *supercuspidal* if every matrix coefficient of π is compactly supported modulo the centre of G. Let H be a closed subgroup of G. If (σ, W) is a smooth representation of H, let $\operatorname{Ind}_H^G W$ be the space of functions $f : G \to W$ which are invariant under right-translation by the elements of some compact, open subgroup of G, and which satisfy $f(hg) = \sigma(h)f(g)$ for all $h \in H$, $g \in G$. Further, let $\text{c-Ind}_H^G W$ be the subspace of $\text{Ind}_H^G W$ consisting of those functions which have compact support modulo H. The representation $\text{Ind}_H^G \sigma \colon G \to \text{GL}(\text{Ind}_H^G W)$, where G acts on the elements of $\text{Ind}_H^G W$ by right translation, is called the representation of G *induced* from σ, and $\text{c-Ind}_H^G \sigma = \text{Ind}_H^G \sigma | \text{c-Ind}_H^G W$ is called the representation of G *compactly induced* from σ. It is well known that if H is an open, closed, compact modulo centre subgroup of G, and $\pi = \text{c-Ind}_H^G \sigma$ is irreducible, then π is admissible and supercuspidal.

1.1.6 Miscellany. If B is a subset of a set A, let $\text{ch}_B : A \to \{0,1\}$ be the characteristic function on A with respect to B.

1.2 Restriction of the ground field

In this section, we review a concrete realization of the restriction of the ground field $\text{R}_{E/F} X$ of an affine E-variety X, for E a finite extension of F. Here, we allow F to be any perfect field. Let $\bar{F}$ be the algebraic closure of F, and for a positive integer m, let $\mathbb{A}^m$ be m-dimensional affine space over $\bar{F}$. Recall that we assume all varieties to be affine. To simplify the discussion, we also assume that E/F is Galois, since we will only be concerned with that case.

1.2.1 Construction. Set $\Gamma = \text{Gal}(\bar{F}/F)$ and $\Gamma' = \text{Gal}(\bar{F}/E) \subset \Gamma$. An element $\sigma \in \Gamma$ induces a map $\sigma : \mathbb{A}^m \to \mathbb{A}^m$ by $\sigma((x_i)) = (\sigma(x_i))$. For a polynomial map $f : \mathbb{A}^m \to \mathbb{A}^\ell$, let ${}^\sigma f = \sigma f \sigma^{-1}$, so that ${}^\sigma f$ is the polynomial map obtained from f by allowing σ to act on the coefficients.

Let X be an E-variety. Setting $\Sigma = \Gamma/\Gamma' \simeq \text{Gal}(E/F)$, we can construct an E-variety

$$X^\Sigma = \textstyle\prod_{\sigma \in \Sigma} \sigma(X).$$

Given $\sigma \in \Gamma$, let $\rho_\sigma = \rho_{X,\sigma} : X^\Sigma \to \sigma(X)$ be the natural projection. Any element $\tau \in \Gamma$ permutes the elements of Σ by left-multiplication, so we also have an induced linear map $\psi_\tau = \psi_{X,\tau}$ given by

$$\psi_\tau : X^\Sigma \to \tau(X^\Sigma), \qquad \psi_\tau(x) = (x_{\tau\sigma}), \qquad \big(x = (x_\sigma) \in X^\Sigma\big).$$

Restriction of the ground field is a construction whereby, given an E-variety X, we obtain an F-variety $\text{R}X = \text{R}_{E/F} X$ and an E-isomorphism $f : \text{R}X \to X^\Sigma$ such that $\text{R}X(F)$ maps bijectively onto $X(E)$ under $\tilde{f} = \rho_{\text{id}} f$. Before we make this construction concrete, we cite the following abstract result.

Theorem 1.2.1 ([38, 11.4.16])**.** *Let X be an E-variety. There exists an F-variety $\text{R}X$ together with a surjective E-morphism $\tilde{f} \colon \text{R}X \to X$ with the following universal property. For any F-variety Y together with an E-morphism $\varphi \colon Y \to X$ there is a unique F-morphism $\Phi \colon Y \to \text{R}X$ such that $\varphi = \tilde{f}\,\Phi$. The pair $(\text{R}X, \tilde{f})$ is unique up to canonical isomorphism.*

Following [37, §3.3], we may realize $\text{R}X$ concretely as follows. Let $[E : F] = n$, and let $\{\sigma_1, \ldots, \sigma_n\}$ be a set of representatives for Σ, with σ_1 the identity automorphism, so that $\text{Gal}(E/F) = \{\sigma_i | E\}$. For any variety Y, take the product Y^Σ to have the same ordering as our set of representatives of Σ, and write $\rho_i = \rho_{Y,i}$ for ρ_{Y,σ_i}.

If $X = \mathbb{A}^1$, then X^Σ is just $\mathbb{A}^n$, and so take $\mathrm{R}X = \mathbb{A}^n$ as well. Any choice of F-basis $\mathcal{B} = \{e_i\}$ of E affords an E-morphism

$$f_{\mathcal{B}}\colon \mathbb{A}^n \to \mathbb{A}^n, \qquad f_{\mathcal{B}}(x) = \left(\sum_k x_k \sigma_i(e_k)\right), \qquad \left(x = (x_i) \in \mathbb{A}^n\right).$$

This map is linear with matrix $(\sigma_i(e_j))$. By Dedekind's theorem on the linear independence of field automorphisms, this matrix, and hence the map $f_{\mathcal{B}}$, is invertible. Notice that

$$\tilde{f}_{\mathcal{B}}(x) = (\rho_1 f_{\mathcal{B}})(x) = \sum_i x_i e_i,$$

which clearly maps $\mathrm{R}X(F)$ bijectively onto $X(E)$. Also, a simple calculation shows ${}^\sigma f_{\mathcal{B}} = \psi_\sigma f_{\mathcal{B}}$ for any $\sigma \in \Gamma$.

Now proceed inductively. If X is a product of E-varieties Y and Z for which $\mathrm{R}Y$ and $\mathrm{R}Z$ are defined, set $\mathrm{R}X = \mathrm{R}Y \times \mathrm{R}Z$. This is clearly defined over F. We have $X^\Sigma = \prod_i \left(\sigma_i(Y) \times \sigma_i(Z)\right)$, and so, choosing appropriate E-isomorphisms $f_Y\colon \mathrm{R}Y \to Y^\Sigma$ and $f_Z\colon \mathrm{R}Z \to Z^\Sigma$, we may take $f\colon \mathrm{R}X \to X^\Sigma$ by

$$f(y,z) = \Big(\big((\rho_{Y,i} f_Y)(y), (\rho_{Z,i} f_Z)(z)\big)\Big), \qquad \left(y \in \mathrm{R}Y,\ z \in \mathrm{R}Z\right).$$

Assume that $\tilde{f}_Y$ (resp. $\tilde{f}_Z$) maps $\mathrm{R}Y(F)$ (resp. $\mathrm{R}Z(F)$) bijectively onto $Y(E)$ (resp. $Z(E)$). Then since $\tilde{f} = \tilde{f}_Y \times \tilde{f}_Z$, we also have that $\tilde{f}$ maps $\mathrm{R}X(F)$ bijectively onto $X(E)$. For any $\sigma \in \Gamma$, both $\mathrm{R}Y$ and $\mathrm{R}Z$ are σ-stable since they are defined over F. By the first case, we may further assume inductively that ${}^\sigma f_Y = \psi_{Y,\sigma} f_Y$ and ${}^\sigma f_Z = \psi_{Z,\sigma} f_Z$. Since each $\rho_{Y,i}$, $\rho_{Z,i}$ is defined over F, they each commute with σ, from which we may conclude that ${}^\sigma f = \psi_{X,\sigma} f$ holds in this case as well.

Finally, we define $\mathrm{R}X$ relatively. Suppose X is a subvariety of an E-variety Y for which $\mathrm{R}Y$ is defined, and let $f_Y\colon \mathrm{R}Y \to Y^\Sigma$ be a suitable E-isomorphism. Then X^Σ is a subset of Y^Σ, and we may take $\mathrm{R}X = f_Y^{-1}(X^\Sigma)$ and $f = f_Y|\mathrm{R}X$. Let $\sigma \in \Gamma$. By the previous cases, we may assume ${}^\sigma f_Y = \psi_{Y,\sigma} f_Y$, so that

$$\sigma(\mathrm{R}X) = {}^\sigma f_Y^{-1}\left(\sigma(X^\Sigma)\right) = \left(f_Y^{-1}\psi_{Y,\sigma}^{-1}\right)\left(\sigma(X^\Sigma)\right) = f_Y^{-1}(X^\Sigma) = \mathrm{R}X.$$

Since this holds for any σ, $\mathrm{R}X$ is defined over F. Note that ${}^\sigma f = \psi_{X,\sigma} f$ also holds in this case. We may also assume that $\tilde{f}_Y$ maps $\mathrm{R}Y(F)$ bijectively onto $Y(E)$. Since $\tilde{f} = \tilde{f}_Y|\mathrm{R}X$, the map $\tilde{f}$ gives a bijection between $\mathrm{R}X(F)$ and $X(E)$.

Example. For $X = \mathbb{A}^m$, the above induction process yields $\mathrm{R}X = (\mathrm{R}\mathbb{A}^1)^m = (\mathbb{A}^n)^m$. Enumerate the coordinates of an element $x \in \mathrm{R}X$ by $x = (x_{ij} | 1 \le i \le m,\ 1 \le j \le n)$. We also have $X^\Sigma = \prod_i \sigma_i(\mathbb{A}^m) = (\mathbb{A}^m)^n$, so enumerate the coordinates of an element $x' \in X^\Sigma$ by $x' = (x'_{ij} | 1 \le i \le n,\ 1 \le j \le m)$. Any family $\mathcal{B} = \{\mathcal{B}_i\}_{i=1}^m$ of F-bases of E, with $\mathcal{B}_i = \{e_{ij}\}_{j=1}^n$, determines an E-isomorphism $f_{\mathcal{B}}\colon \mathrm{R}X \to X^\Sigma$ as follows. For $x \in \mathrm{R}X$, let $x' = f_{\mathcal{B}}(x)$, so that

$$x'_{ij} = \sum_k x_{jk}\, \sigma_i(e_{jk}). \qquad \square$$

1.2.2 Properties. If $f\colon X \to Y$ is a morphism of E-varieties, we have an induced E-morphism $f^\Sigma = \prod_i {}^{\sigma_i} f$ from X^Σ to Y^Σ. Choosing E-isomorphisms $f_X\colon \mathrm{R}X \to X^\Sigma$ and $f_Y\colon \mathrm{R}Y \to Y^\Sigma$, we can construct a morphism $\mathrm{R}f\colon \mathrm{R}X \to \mathrm{R}Y$ by setting $\mathrm{R}f = f_Y^{-1} f^\Sigma f_X$. Then, for any $\sigma \in \Gamma$,

$${}^\sigma(\mathrm{R}f) = ({}^\sigma f_Y^{-1})({}^\sigma f^\Sigma)({}^\sigma f_X) = f_Y^{-1}\,\psi_{Y,\sigma}^{-1}\,({}^\sigma f^\Sigma)\,\psi_{X,\sigma}\, f_X.$$

However, it is easy to check that $\psi_{Y,\sigma}^{-1}({}^{\sigma}f^{\Sigma})\,\psi_{X,\sigma} = f^{\Sigma}$. Therefore, ${}^{\sigma}(\mathrm{R}f) = \mathrm{R}f$ for any $\sigma \in \Gamma$, from which we conclude that $\mathrm{R}f$ is defined over F. Moreover, one can check that $\rho_{Y,i}\,f^{\Sigma} = ({}^{\sigma_i}f)\rho_{X,i}$, for $1 \le i \le n$, so that

$$\tilde{f}_Y\,\mathrm{R}f = (\rho_{Y,1}\,f_Y)(f_Y^{-1}f^{\Sigma}f_X) = f\,\rho_{X,1}\,f_X = f\,\tilde{f}_X.$$

Now suppose that X is an L-variety, where L/E is a finite Galois extension, and let $Y = \mathrm{R}_{L/E}\,X$. Let $\Gamma'' = \mathrm{Gal}(\bar{F}/L)$, $\Sigma' = \Gamma'/\Gamma''$, and $\Sigma'' = \Gamma/\Gamma''$. We have $Y \simeq X^{\Sigma'}$ and $\mathrm{R}_{L/F}\,X \simeq X^{\Sigma''}$ over L, and $\mathrm{R}_{E/F}\,Y \simeq Y^{\Sigma}$ over E. But since Σ is isomorphic to Σ''/Σ', $Y^{\Sigma} \simeq (X^{\Sigma'})^{\Sigma}$ is equal to $X^{\Sigma''}$, after reordering the factors if necessary. The universal property of restriction of the ground field allows us to conclude that $\mathrm{R}_{E/F}\,Y$ is isomorphic over F to $\mathrm{R}_{L/F}\,X$.

1.2.3 Group structure. Suppose that $X = \mathbf{G}$ is an algebraic group defined over E. Choose an E-isomorphism $f\colon \mathrm{R}\mathbf{G} \to \mathbf{G}^{\Sigma}$. Considering $(\mathbf{G}\times\mathbf{G})^{\Sigma} \simeq \mathbf{G}^{\Sigma}\times\mathbf{G}^{\Sigma}$ and $\mathrm{R}(\mathbf{G}\times\mathbf{G}) = \mathrm{R}\mathbf{G}\times\mathrm{R}\mathbf{G}$, let f^2 denote the E-isomorphism $f\times f\colon \mathrm{R}\mathbf{G}\times\mathrm{R}\mathbf{G} \to \mathbf{G}^{\Sigma}\times\mathbf{G}^{\Sigma}$. Multiplication $\mu\colon \mathbf{G}\times\mathbf{G}\to\mathbf{G}$ and inversion $\iota\colon \mathbf{G}\to\mathbf{G}$ are E-morphisms, so we have induced maps μ^{Σ}, ι^{Σ}, $\mathrm{R}\mu$, and $\mathrm{R}\iota$, the latter two defined using f and f^2. We may then give each of $\mathbf{G}^{\Sigma}$ and $\mathrm{R}\mathbf{G}$ group structure via these induced maps, respectively. Since $\rho_i\,\mu^{\Sigma} = ({}^{\sigma_i}\mu)(\rho_i\times\rho_i)$, for $1 \le i \le n$, it follows that $\tilde{f}$ is a homomorphism of algebraic groups which restricts to a group isomorphism mapping $\mathrm{R}\mathbf{G}(F)$ onto $\mathbf{G}(E)$. We will make frequent use of the following example.

Example. Consider $\mathbf{G} = \mathbb{G}_{\mathrm{m}}$ as the algebraic subset $\{(x,x^{-1}) \mid x \neq 0\}$ of $\mathbb{A}^2$. Since $\mathbf{G}$ is defined over F, $\mathbf{G}^{\Sigma} = \mathbf{G}\times\cdots\times\mathbf{G}$ (n copies). Choose an F-basis $\mathcal{B} = \{e_i\}_{i=1}^n$ of E, and let $A_{\mathcal{B}} = \big(\sigma_i(e_j)\big) \in \mathbf{GL}_n(E)$. Take $f_{\mathcal{B}}\colon (\mathbb{A}^n)^2 \to (\mathbb{A}^2)^n$ by

$$f_{\mathcal{B}}\big((x_k),(y_k)\big) = \Big(\big(\textstyle\sum_j x_j\,\sigma_k(e_j),\, \sum_j y_j\,\sigma_k(e_j)\big)\Big).$$

Following our construction, we set $\mathrm{R}\mathbf{G} = f_{\mathcal{B}}^{-1}(\mathbf{G}^{\Sigma})$, so that

$$\mathrm{R}\mathbf{G} = \Big\{\, \big((x_k),(y_k)\big) \in (\mathbb{A}^n)^2 \;\big|\; \big(\textstyle\sum_j x_j\sigma_k(e_j)\big)\big(\sum_j y_j\sigma_k(e_j)\big) = 1,\ 1 \le k \le n \Big\}.$$

Let $\{c_{ijh}\}$ be the elements of F such that $e_ie_j = \sum_h c_{ijh}\,e_k$, for $1 \le i,j \le n$. Then

$$\mathrm{R}\mathbf{G} = \Big\{\, \big((x_k),(y_k)\big) \;\big|\; \textstyle\sum_{h,i,j} x_i\,y_j\,c_{ijh}\,\sigma_k(e_h) = 1,\ 1 \le k \le n \Big\}, \tag{1.2.1}$$

with multiplication given by

$$\big((x_k),(y_k)\big)\big((x'_k),(y'_k)\big) = \Big(\big(\textstyle\sum_{i,j} x_i\,x'_j\,c_{ijk}\big), \big(\sum_{i,j} y_i\,y'_j\,c_{ijk}\big)\Big).$$

The map $f_{\mathcal{B}}|\mathrm{R}\mathbf{G}$ defines an E-isomorphism of $\mathrm{R}\mathbf{G}$ onto $\mathbf{G}^{\Sigma}$, so that $\mathrm{R}\mathbf{G}$ is an n-dimensional F-torus which splits over E. The basis $\mathcal{B}$ affords an embedding of $\mathrm{R}\mathbf{G}$ into $\mathbf{GL}_n$ as follows. For $x = \big((x_k),(y_k)\big) \in \mathrm{R}\mathbf{G}$, let $\varphi(x)$ be the $n\times n$ matrix $\big(\sum_k x_k\,c_{kji}\big)$. View the equations that determine $\mathrm{R}\mathbf{G}$ in (1.2.1) as a linear system in the indeterminates $\{y_k\}$, with coefficient matrix $A_{\mathcal{B}}\,\varphi(x)$. Since $f_{\mathcal{B}}$ is invertible, (y_k) is the unique solution to the system for a given $x \in \mathrm{R}\mathbf{G}$, and so $\varphi(x)$ is invertible. Therefore, φ maps $\mathrm{R}\mathbf{G}$ into $\mathbf{GL}_n$. For any integers $1 \le i,j,k \le n$, the identity $e_ie_je_k = e_ke_je_i$ yields relations $c_{ija}\,c_{akb} = c_{kja}\,c_{aib}$, for $1 \le a,b \le n$. Using these relations, one can check that φ is a group homomorphism. Finally, let $f = f_{\mathcal{B}}|\mathrm{R}\mathbf{G}$, as per our construction. Notice that for any $\alpha \in E^{\times}$, $\big(\varphi\,\tilde{f}^{-1}\big)(\alpha) = [\mathrm{mult}(\alpha,\cdot)]_{\mathcal{B}}$, so that $\ker(\varphi|\mathrm{R}\mathbf{G}(F))$ is trivial. The equation $\varphi(x) = 1_n$ determines a system of linear equations with coefficients in F. The existence of a unique solution over F allows us to conclude that $\ker\varphi$ is trivial and φ is injective. The image $\varphi(\mathrm{R}\mathbf{G})$ is given by $\varphi(\mathrm{R}\mathbf{G}) = \mathbf{D}^{A_{\mathcal{B}}}$, where $\mathbf{D} \simeq \mathbf{G}^{\Sigma}$ is the diagonal torus in $\mathbf{GL}_n$. □

We now record some connections between the structures of $\mathbf{G}$ and $\mathrm{R}\mathbf{G}$. The following statements are obvious.

Lemma 1.2.2.

(1) $Z(\mathbf{G}^\Sigma) = Z(\mathbf{G})^\Sigma$ *and* $Z(\mathrm{R}\mathbf{G}) = f^{-1}\big(Z(\mathbf{G})^\Sigma\big) \simeq \mathrm{R}\big(Z(\mathbf{G})\big)$.

(2) If $\mathbf{G}$ *is connected, then so is* $\mathrm{R}\mathbf{G}$. *If* $\mathbf{G}$ *is also reductive, then so is* $\mathrm{R}\mathbf{G}$.

Now assume that $\mathbf{G}$ is a connected, reductive E-group.

Lemma 1.2.3.

(1) If $\mathbf{G}$ *is* E*-quasi-split, then* $\mathrm{R}\mathbf{G}$ *is* F*-quasi-split.*

(2) If $\mathbf{T}$ *is a maximal* E*-torus of* $\mathbf{G}$*, then* $\mathrm{R}\mathbf{T}$ *is a maximal* F*-torus of* $\mathrm{R}\mathbf{G}$*,* $(\mathrm{R}\mathbf{T})_{\mathrm{s},F} = \big(\mathrm{R}(\mathbf{T}_{\mathrm{s},E})\big)_{\mathrm{s},F}$*, and* $\mathrm{rk}_F\,\mathrm{R}\mathbf{T} = \mathrm{rk}_E\,\mathbf{T}$.

(3) If $\mathbf{G}$ *is* E*-split and* $\mathbf{T}$ *is an* E*-minisotropic torus of* $\mathbf{G}$*, then* $\mathrm{R}\mathbf{T}$ *is an* F*-minisotropic torus of* $\mathrm{R}\mathbf{G}$.

Proof. For (1), it is clear that if $\mathbf{B} \supset \mathbf{T}$ is a pair in $\mathbf{G}$ which is defined over E, then $\mathrm{R}\mathbf{B} \supset \mathrm{R}\mathbf{T}$ is a pair in $\mathrm{R}\mathbf{G}$ which is defined over F. Statement (2) is discussed in [38, 16.2.6–7]. Now let $\mathbf{T} \subseteq \mathbf{G}$ be as in (3). Then $\mathbf{T}_{\mathrm{s},E} \subseteq Z(\mathbf{G})$, so by Lemma 1.2.2(1) we have $\mathrm{R}(\mathbf{T}_{\mathrm{s},E}) \subseteq Z(\mathrm{R}\mathbf{G})$. Now (2) implies that $(\mathrm{R}\mathbf{T})_{\mathrm{s},F} \subseteq Z(\mathrm{R}\mathbf{G})$, and it follows that $\mathrm{R}\mathbf{T}$ is F-minisotropic. ☐

1.2.4 Galois action. Suppose $X \subset \mathbb{A}^m$ is defined over F. Following the example of §1.2.1, but simplifying using the single F-basis $\mathcal{B} = \{e_i\}$ of E, let $f_{\mathcal{B}} \colon (\mathbb{A}^n)^m \to (\mathbb{A}^m)^n$ by $f_{\mathcal{B}}(x_{ij}) = \big(\sum_k x_{jk}\,\sigma_i(e_k)\big)$, where $n = [E : F]$. Since X is defined over F, X^Σ is just n factors of X sitting inside $(\mathbb{A}^m)^n$, and ψ_σ is an F-automorphism of X^Σ for any $\sigma \in \Gamma$. Take $\mathrm{R}X = \mathrm{R}_{E/F}\,X = f_{\mathcal{B}}^{-1}(X^\Sigma)$, and set $f_X = f_{\mathcal{B}}|\mathrm{R}X$. Let θ be an element in the centre of Γ. The action of θ on X, restricted to $X(E)$, coincides with restriction to $\mathrm{R}X(F)$ of an F-automorphism η_θ of $\mathrm{R}X$ as follows. Take $\eta_\theta = f_X^{-1}\psi_\theta f_X$. For any $\sigma \in \Gamma$,

$$^\sigma\eta_\theta = {}^\sigma(f_X^{-1})\,{}^\sigma\psi_\theta\,{}^\sigma f_X = f_X^{-1}\psi_\sigma^{-1}\psi_\theta\psi_\sigma f_X = f_X^{-1}\psi_{\sigma\theta\sigma^{-1}} f_X.$$

However, since θ is central, $\psi_{\sigma\theta\sigma^{-1}} = \psi_\theta$. Therefore, $^\sigma\eta_\theta = \eta_\theta$, and so η_θ is defined over F. Recall that $\tilde{f}_X = \rho_1 f_X$ maps $\mathrm{R}X(F)$ bijectively onto $X(E)$. In particular, for $x = (\alpha_i) \in X(E)$, we may write $x = \tilde{f}_X(x_{ij})$ for $\{x_{ij}\}$ the elements of F such that $\alpha_i = \sum_j x_{ij} e_j$. Then, using the fact that $\tilde{f}_X\,\eta_\theta = \rho_1{}^\theta f_X$, we have

$$\big(\tilde{f}_X\,\eta_\theta\big)(x_{ij}) = \Big(\sum_j x_{ij}\,\theta(e_j)\Big) = (\theta(\alpha_i)),$$

so that $\tilde{f}_X\,\eta_\theta|\mathrm{R}X(F)$ induces $\theta|X(E)$.

If X is a linear algebraic group, defined over F, then it is straightforward to check that η_θ is also a group homomorphism.

1.3 Quasi-semisimple automorphisms

In this section, we introduce the necessary background on quasi-semisimple automorphisms of a reductive algebraic group $\mathbf{G}$, and discuss two notions of regularity of elements in $\mathbf{G}^+$.

1.3.1 Definitions and basic facts. Let $\mathbf{G}$ be a connected reductive linear algebraic group defined over a perfect field F, and let $G = \mathbf{G}(F)$. We do not make any assumption on the characteristic of F at this point. Let θ be a quasi-semisimple automorphism of $\mathbf{G}$ which is defined over F. By *quasi-semisimple*, we mean that there exists a pair in $\mathbf{G}$ which is θ-stable. We cite some facts from [14].

Theorem 1.3.1 ([14, Theorem 1.8]).

(1) The connected component, $\mathbf{G}_\theta^0$, of the subgroup $\mathbf{G}_\theta$ of θ-fixed elements of $\mathbf{G}$ is reductive.

(2) Suppose $\mathbf{B} \supset \mathbf{T}$ is a θ-stable pair in G, and set $\mathbf{B}_\theta^0 = \mathbf{B} \cap \mathbf{G}_\theta^0$ and $\mathbf{T}_\theta^0 = \mathbf{T} \cap \mathbf{G}_\theta^0$. Then $\mathbf{B}_\theta^0 \supset \mathbf{T}_\theta^0$ is a pair in $\mathbf{G}_\theta^0$.

(3) For any pair $\mathbf{C} \supset \mathbf{S}$ in $\mathbf{G}_\theta^0$ there exists a θ-stable pair $\mathbf{B} \supset \mathbf{T}$ in $\mathbf{G}$ such that $\mathbf{C} = \mathbf{B} \cap \mathbf{G}_\theta^0$ and $\mathbf{S} = \mathbf{T} \cap \mathbf{G}_\theta^0$. In particular, $\mathbf{T} = C_{\mathbf{G}}(\mathbf{T}_\theta^0) = C_{\mathbf{G}}(\mathbf{T}_\theta)$.

Note that since θ is defined over F, $\mathbf{G}_\theta^0$ is as well. Let $G_\theta^0 = \mathbf{G}_\theta^0(F)$. We add the following observations, all of which are known.

Corollary 1.3.2. *Every element of $\mathbf{G}_\theta^0$ lies in a θ-stable Borel subgroup of $\mathbf{G}$. Every semisimple element of $\mathbf{G}_\theta^0$ lies in a θ-stable pair of $\mathbf{G}$.*

Proof. Let γ be an element of $\mathbf{G}_\theta^0$. Every element of $\mathbf{G}_\theta^0$ lies in a Borel subgroup of $\mathbf{G}_\theta^0$; let $\mathbf{C}$ be a Borel subgroup of $\mathbf{G}_\theta^0$ containing γ. Choose a maximal torus $\mathbf{S}$ of $\mathbf{C}$ and apply (3) from the theorem to obtain a θ-stable pair $\mathbf{B} \supset \mathbf{T}$ of $\mathbf{G}$ with $\mathbf{C}$, hence γ, contained in $\mathbf{B}$. Every semisimple element of $\mathbf{G}_\theta^0$ is contained in a maximal torus of $\mathbf{G}_\theta^0$; if γ is semisimple, we may choose $\mathbf{S}$ containing γ, so that γ lies in $\mathbf{T}$ as well. □

Corollary 1.3.3. *Let $\mathbf{T}$ be a θ-stable maximal torus of $\mathbf{G}$. Then $\mathbf{T}$ is contained in a θ-stable Borel subgroup of $\mathbf{G}$ if and only if $\mathbf{T} \cap \mathbf{G}_\theta^0$ is a maximal torus of $\mathbf{G}_\theta^0$.*

Proof. One direction is already provided by the theorem, so let us prove the other. If $\mathbf{T} \cap \mathbf{G}_\theta^0$ is a maximal torus of $\mathbf{G}_\theta^0$, choose a Borel subgroup of $\mathbf{G}_\theta^0$ containing $\mathbf{T} \cap \mathbf{G}_\theta^0$ and apply (3) of the theorem to obtain a θ-stable pair $\mathbf{B} \supset \mathbf{T}'$ of $\mathbf{G}$ such that $\mathbf{T}' \cap \mathbf{G}_\theta^0 = \mathbf{T} \cap \mathbf{G}_\theta^0$. But then $\mathbf{T}' = C_{\mathbf{G}}(\mathbf{T} \cap \mathbf{G}_\theta^0) \supseteq C_{\mathbf{G}}(\mathbf{T}) = \mathbf{T}$. Since $\mathbf{T}$ and $\mathbf{T}'$ are both maximal tori of $\mathbf{G}$, we must have equality throughout, and $\mathbf{B}$ is the required Borel subgroup of $\mathbf{G}$. □

Corollary 1.3.4. *Let $\mathbf{B} \supset \mathbf{T}$ be a θ-stable pair of $\mathbf{G}$.*

(1) $N_{\mathbf{G}}(\mathbf{T})_\theta = N_{\mathbf{G}_\theta}(\mathbf{T}_\theta) = N_{\mathbf{G}_\theta}(\mathbf{T}_\theta^0)$.

(2) $N_{\mathbf{G}_\theta^0}(\mathbf{T}_\theta^0) = N_{\mathbf{G}}(\mathbf{T}) \cap \mathbf{G}_\theta^0$.

(3) The inclusion $N_{\mathbf{G}_\theta^0}(\mathbf{T}_\theta^0) \hookrightarrow N_{\mathbf{G}}(\mathbf{T})$ induces an embedding $W_{\mathbf{G}_\theta^0}(\mathbf{T}_\theta^0) \hookrightarrow W_{\mathbf{G}}(\mathbf{T})_\theta$.

(4) If $\mathbf{T}$ is defined over F, then the inclusion $N_{G_\theta^0}(\mathbf{T}_\theta^0) \hookrightarrow N_G(\mathbf{T})$ induces an embedding $W_{\mathbf{G}_\theta^0}(\mathbf{T}_\theta^0)^F \hookrightarrow (W_{\mathbf{G}}(\mathbf{T})^F)_\theta$.

Proof. We prove $N_{\mathbf{G}}(\mathbf{T})_\theta = N_{\mathbf{G}_\theta}(\mathbf{T}_\theta)$ and $N_{\mathbf{G}}(\mathbf{T})_\theta = N_{\mathbf{G}_\theta}(\mathbf{T}_\theta^0)$ simultaneously. If g is any element of $N_{\mathbf{G}}(\mathbf{T})_\theta$, then ${}^g\mathbf{T}_\theta$ and ${}^g(\mathbf{T}_\theta^0)$ are contained in $\mathbf{T}_\theta$. Moreover, ${}^g(\mathbf{T}_\theta^0)$ is connected and contains the identity, so it must be contained in $\mathbf{T}_\theta^0$. Since $N_{\mathbf{G}}(\mathbf{T})_\theta$ is closed under inverses, the same inclusions hold for g^{-1}, and we have ${}^g\mathbf{T}_\theta = \mathbf{T}_\theta$ and ${}^g(\mathbf{T}_\theta^0) = \mathbf{T}_\theta^0$. Therefore, $N_{\mathbf{G}}(\mathbf{T})_\theta$ is contained in both $N_{\mathbf{G}_\theta}(\mathbf{T}_\theta)$ and $N_{\mathbf{G}_\theta}(\mathbf{T}_\theta^0)$.

Now suppose g_0 and g_1 lie in $N_{\mathbf{G}_\theta}(\mathbf{T}_\theta^0)$ and $N_{\mathbf{G}_\theta}(\mathbf{T}_\theta)$, respectively, and let i be either 0 or 1. Statement (3) of Theorem 1.3.1 gives $\mathbf{T} = C_{\mathbf{G}}(\mathbf{T}_\theta^0)$, so we will show that for elements $x \in \mathbf{T}$ and $t \in \mathbf{T}_\theta^0$, ${}^{g_i}x$ commutes with t. Indeed, t^{g_i} lies in $\mathbf{T}_\theta$ and thus commutes with x, hence ${}^{g_i x}t = {}^{g_i x}(t^{g_i}) = t$. Therefore, ${}^{g_i}\mathbf{T} \subseteq \mathbf{T}$. As before, since $N_{\mathbf{G}_\theta}(\mathbf{T}_\theta^0)$ and $N_{\mathbf{G}_\theta}(\mathbf{T}_\theta)$ are closed under inverses, we must have ${}^{g_i}\mathbf{T} = \mathbf{T}$. Thus, both $N_{\mathbf{G}_\theta}(\mathbf{T}_\theta^0)$ and $N_{\mathbf{G}_\theta}(\mathbf{T}_\theta)$ are contained in $N_{\mathbf{G}}(\mathbf{T})_\theta$, and we have shown (1). Statement (2) now follows from (1).

Let $\alpha\colon N_{\mathbf{G}_\theta^0}(\mathbf{T}_\theta^0) \to W_{\mathbf{G}}(\mathbf{T})$ be the composition of inclusion $N_{\mathbf{G}_\theta^0}(\mathbf{T}_\theta^0) \hookrightarrow N_{\mathbf{G}}(\mathbf{T})$ (afforded by (2)) and the natural projection $N_{\mathbf{G}}(\mathbf{T}) \to W_{\mathbf{G}}(\mathbf{T})$. Since θ acts on $W_{\mathbf{G}}(\mathbf{T})$ by acting on coset representatives, α has image in $W_{\mathbf{G}}(\mathbf{T})_\theta$. The kernel of α is $\mathbf{T} \cap \mathbf{G}_\theta^0 = \mathbf{T}_\theta^0$, so α factors to an embedding $W_{\mathbf{G}_\theta^0}(\mathbf{T}_\theta^0) \hookrightarrow W_{\mathbf{G}}(\mathbf{T})_\theta$, giving (3). Similarly, let $\alpha^F : N_{G_\theta^0}(\mathbf{T}_\theta^0) \to W_{\mathbf{G}}(\mathbf{T})^F$ be the composition of inclusion $N_{G_\theta^0}(\mathbf{T}_\theta^0) \hookrightarrow N_G(\mathbf{T})$ and the natural projection $N_G(\mathbf{T}) \to W_{\mathbf{G}}(\mathbf{T})^F$. Then α^F has kernel $\mathbf{T}(F) \cap G_\theta^0 = \mathbf{T}_\theta^0(F)$ and image in $(W_G(\mathbf{T})^F)_\theta$, where θ factors to an automorphism of $W_G(\mathbf{T})^F$ in the natural way. This proves (4). ☐

We add some results over F.

Proposition 1.3.5. *If* $\mathbf{G}$ *contains an* F*-split,* θ*-stable pair* $\mathbf{B} \supset \mathbf{T}$*, then* $\mathbf{G}_\theta^0$ *is* F*-split.*

Proof. This is clear from Theorem 1.3.1(2) and the fact that any subtorus of an F-split torus is defined and splits over F. ☐

Lemma 1.3.6. *Suppose* $\mathbf{B} \supset \mathbf{T}$ *is a* θ*-stable pair in* $\mathbf{G}$*, with* $\mathbf{T}$ *defined over* F*, such that* $\mathbf{T}_{\mathrm{s}} \subseteq Z(\mathbf{G})$*. Then* $\mathbf{T}_\theta^0$ *is an* F*-minisotropic* F*-torus of* $\mathbf{G}_\theta^0$*.*

Proof. Note that the condition on $\mathbf{T}$ implies that it is F-minisotropic in $\mathbf{G}$. Since $\mathbf{T}$ lies in a θ-stable Borel subgroup of $\mathbf{G}$, we have that $\mathbf{T}_\theta^0$ is a maximal F-torus of $\mathbf{G}_\theta^0$. It suffices to show that $(\mathbf{T}_\theta^0)_{\mathrm{s}} \subseteq Z(\mathbf{G}_\theta^0)$. Now, $(\mathbf{T}_\theta^0)_{\mathrm{s}}$ is an F-split subtorus of $\mathbf{T}$, so we have $(\mathbf{T}_\theta^0)_{\mathrm{s}} \subseteq (\mathbf{T}_{\mathrm{s}})_\theta \subseteq Z(\mathbf{G})_\theta \subseteq Z(\mathbf{G}_\theta)$. But then since $(\mathbf{T}_\theta^0)_{\mathrm{s}}$ is connected, we have $(\mathbf{T}_\theta^0)_{\mathrm{s}} \subseteq Z(\mathbf{G}_\theta)^0 \subseteq Z(\mathbf{G}_\theta^0)$. ☐

The following is the situation we will later find ourselves in.

Corollary 1.3.7. *Suppose* $\mathbf{G} = \mathrm{R}_{E/F}\,\mathbf{H}$*, where* E *is a finite, Galois extension of* F*, and* $\mathbf{H}$ *is a connected, reductive,* E*-split* E*-group. Let* $\mathbf{S}$ *be an* E*-minisotropic torus of* $\mathbf{H}$ *such that* $\mathbf{T} = \mathrm{R}_{E/F}\,\mathbf{S}$ *is* θ*-stable and is contained in a* θ*-stable Borel subgroup of* $\mathbf{G}$*. Then* $\mathbf{T}_\theta^0$ *is* F*-minisotropic in* $\mathbf{G}_\theta^0$*.*

Proof. By Lemma 1.2.3(3), $\mathbf{T}$ is F-minisotropic in $\mathbf{G}$. From the proof of this lemma, we have $\mathbf{T}_{\mathrm{s},F} \subseteq Z(\mathbf{G})$. We may now apply Lemma 1.3.6. ☐

Suppose $\mathbf{B} \supset \mathbf{T}$ is a pair in $\mathbf{G}$, with $\mathbf{T}$ θ-stable. The following will help us keep track of those G-conjugates of this pair which are θ-stable. Define the set-valued functions $\boldsymbol{\mathfrak{X}}_{\mathbf{T}}$ and $\mathfrak{X}_{\mathbf{T}}$ on $N_{\mathbf{G}}(\mathbf{T}) \times N_{\mathbf{G}}(\mathbf{T})$ by

$$\boldsymbol{\mathfrak{X}}_{\mathbf{T}}(n_1, n_2) = \left\{ x \in \mathbf{G} \mid x^{-1} n_2 \, ({}^{n_1\theta}x) \in \mathbf{T} \right\},$$

$$\mathfrak{X}_{\mathbf{T}}(n_1, n_2) = \boldsymbol{\mathfrak{X}}_{\mathbf{T}}(n_1, n_2) \cap G,$$

for $n_1, n_2 \in N_{\mathbf{G}}(\mathbf{T})$. Notice that $\mathfrak{X}_{\mathbf{T}}(n_1,n_2)$ is a union of right-cosets of $\mathbf{G}_{n_2n_1\theta}$. Let $\widetilde{\mathfrak{X}}_{\mathbf{T}}(n_1,n_2)$ be the image of $\mathfrak{X}_{\mathbf{T}}(n_1,n_2)$ in $\mathbf{G}_{n_2n_1\theta}\backslash\mathbf{G}$ under the natural projection. If $\mathbf{G}_{n_2n_1\theta}$ is defined over F, let $\widetilde{\mathcal{X}}_{\mathbf{T}}(n_1,n_2)$ be the image of $\mathcal{X}_{\mathbf{T}}(n_1,n_2)$ in $G_{n_2n_1\theta}\backslash G$, where $G_{n_2n_1\theta} = \mathbf{G}_{n_2n_1\theta}(F)$.

Now, ${}^{\theta}\mathbf{B} \supset \mathbf{T}$ is also a pair in $\mathbf{G}$, and so there exists $n_o \in N_{\mathbf{G}}(\mathbf{T})$, with unique image in $W_{\mathbf{G}}(\mathbf{T})$, such that ${}^{n_o\theta}\mathbf{B} = \mathbf{B}$.

Lemma 1.3.8. *Let $\mathbf{B} \supset \mathbf{T}$ and n_o be as above. Then for $n \in N_{\mathbf{G}}(\mathbf{T})$ and $x \in \mathbf{G}$, we have $x \in \mathfrak{X}_{\mathbf{T}}(n_o,n)$ if and only if the pair ${}^{x}\mathbf{B} \supset {}^{x}\mathbf{T}$ is stable under the automorphism $\mathrm{Int}_{\mathrm{L}}(nn_o)\circ\theta$ of $\mathbf{G}$. For $x \in \mathfrak{X}_{\mathbf{T}}(n_o,n)$, $({}^{x}\mathbf{T})_{nn_o\theta} = {}^{x}\mathbf{T}_{n_o\theta}$.*

Proof. The forward direction is straightforward computation. For the reverse implication, suppose $x \in \mathbf{G}$ such that the pair ${}^{x}\mathbf{B} \supset {}^{x}\mathbf{T}$ is stable under $\mathrm{Int}_{\mathrm{L}}(nn_o)\circ\theta$. Since the pair $\mathbf{B} \supset \mathbf{T}$ is stable under $\mathrm{Int}_{\mathrm{L}}(n_o)\circ\theta$, it follows that $x^{-1}n({}^{n_o\theta}x) \in N_{\mathbf{G}}(\mathbf{B}) \cap N_{\mathbf{G}}(\mathbf{T})$. However, $N_{\mathbf{G}}(\mathbf{B}) = \mathbf{B}$ and $\mathbf{B} \cap N_{\mathbf{G}}(\mathbf{T}) = \mathbf{T}$, and thus $x \in \mathfrak{X}_{\mathbf{T}}(n_o,n)$. The second assertion is easily verified. □

Corollary 1.3.9. *There exists a θ-stable G-conjugate of the pair $\mathbf{B} \supset \mathbf{T}$ if and only if $\mathcal{X}_{\mathbf{T}}(n_o,n_o^{-1}) \neq \emptyset$.*

Now assume θ is of finite order, say d_θ. The group $\mathbf{G}^+$ is also defined over F, and $G^+ = \mathbf{G}^+(F)$. For non-trivial θ, $\mathbf{G}^+$ is non-connected with components $\{\mathbf{G}^i\}_{i=0}^{d_\theta-1}$. Moreover, $\mathbf{G}^+$ is linear, since any embedding ρ of $\mathbf{G}$ into $\mathbf{GL}_n$ induces an embedding ρ^+ of $\mathbf{G}^+$ into $\mathbf{GL}_{nd_\theta}$ as follows. For $g \in \mathbf{G}$ and $0 \le m < d_\theta$, the $(i,j)^{\text{th}}$ $n\times n$ block $\rho^+(g\theta^m)_{i,j}$ of $\rho^+(g\theta^m)$ is given by

$$\rho^+(g\theta^m)_{i,j} = \begin{cases} \rho(\theta^{i-1}(g)), & j = (m+i-1 \mod n)+1, \\ 0, & \text{otherwise.} \end{cases}$$

Note that if ρ is an F-embedding, then so is ρ^+. The image of θ under this embedding is the element

$$\begin{pmatrix} 0_n & 1_n & & \\ & \ddots & \ddots & \\ & & \ddots & 1_n \\ 1_n & & & 0_n \end{pmatrix} \in \mathbf{GL}_{nd_\theta}.$$

If $\gcd(d_\theta, \operatorname{char} F) = 1$, then this image of θ is semisimple, so that θ is a semisimple automorphism of $\mathbf{G}$.

1.3.2 Regular elements. In this section, we discuss two notions of "regular elements" of $\mathbf{G}^+$. First, we examine how the familiar notion of regularity in $\mathbf{G}$ behaves relative to θ. For the first two results, F is any perfect field, $\mathbf{G}$ is any connected, reductive F-group, and θ is any quasi-semisimple F-automorphism of $\mathbf{G}$. After Corollary 1.3.11, we assume that θ has finite order. Following the example concerning Lemma 1.3.12, we change the setup, and take θ to be a finite-order F_0-automorphism of $\mathbf{G} = \mathrm{R}_{F/F_0}\mathbf{GL}_n$, for F/F_0 a finite, abelian extension of p-adic fields. Of particular importance is Lemma 1.3.12, which states that if $g \in G$ is θ-fixed and $g\theta^i$ satisfies a regularity condition in G^+, then g is regular in G.

It is most convenient for us to define $\gamma \in \mathbf{G}$ to be *regular in* $\mathbf{G}$ if $\mathbf{G}^0_\gamma = C_{\mathbf{G}}(\gamma)^0$ is a maximal torus of $\mathbf{G}$. Let $\mathbf{G}_{\text{reg}}$ denote the set of regular elements in $\mathbf{G}$. For a subset $\mathbf{A} \subseteq \mathbf{G}$, let $\mathbf{A}_{\text{reg}} = \mathbf{A} \cap \mathbf{G}_{\text{reg}}$.

Lemma 1.3.10. *Any element of* $\mathbf{G}_\theta^0$ *which is regular in* $\mathbf{G}$ *is regular in* $\mathbf{G}_\theta^0$.

Proof. Suppose γ is θ-fixed and regular in $\mathbf{G}$. Since γ is semisimple, by Corollary 1.3.2 there exists a θ-stable pair $\mathbf{B} \supset \mathbf{T}$ of $\mathbf{G}$ containing γ. Since γ is regular, we must have $\mathbf{T} = \mathbf{G}_\gamma^0$. Then, $\mathbf{S} = \mathbf{G}_\gamma^0 \cap \mathbf{G}_\theta^0$ is a maximal torus in $\mathbf{G}_\theta^0$ by (2) of Theorem 1.3.1. Since $\gamma \in \mathbf{S}$ and $\mathbf{S}$ is connected, we have $\mathbf{S} \subseteq (\mathbf{G}_\theta^0)_\gamma^0$. On the other hand, since $(\mathbf{G}_\theta^0)_\gamma^0$ is connected we have $(\mathbf{G}_\theta^0)_\gamma^0 \subseteq \mathbf{T}$, and so $(\mathbf{G}_\theta^0)_\gamma^0 \subseteq \mathbf{T} \cap \mathbf{G}_\theta^0 = \mathbf{S}$. Therefore, we have $(\mathbf{G}_\theta^0)_\gamma^0 = \mathbf{S}$, and so γ is regular in $\mathbf{G}_\theta^0$. □

For a given θ-stable maximal torus $\mathbf{T}$, we obtain a condition for $\mathbf{T}_\theta^0 = \mathbf{T} \cap \mathbf{G}_\theta^0$ to be a maximal torus in $\mathbf{G}_\theta^0$ (see also Corollary 1.3.3), based on the existence of θ-fixed regular elements in $\mathbf{T}$. This condition is necessary and sufficient under the assumption that $\mathbf{G}_{\mathrm{reg}} \cap \mathbf{G}_\theta^0$ is non-empty.

Corollary 1.3.11. *Let* $\mathbf{T}$ *be a* θ*-stable maximal torus of* $\mathbf{G}$.

(1) If $\mathbf{G}_{\mathrm{reg}} \cap \mathbf{T}_\theta^0$ *is non-empty, then* $\mathbf{T}$ *is contained in a* θ*-stable Borel subgroup of* $\mathbf{G}$.

(2) Assume $\mathbf{G}_{\mathrm{reg}} \cap \mathbf{G}_\theta^0$ *is non-empty. If* $\mathbf{T}$ *is contained in a* θ*-stable Borel subgroup of* $\mathbf{G}$*, then* $\mathbf{G}_{\mathrm{reg}} \cap \mathbf{T}_\theta^0$ *is non-empty.*

Proof. Suppose $\gamma \in \mathbf{G}_{\mathrm{reg}} \cap \mathbf{T}_\theta^0$. Again, we apply Corollary 1.3.2 to obtain a θ-stable pair $\mathbf{B} \supset \mathbf{S}$ of $\mathbf{G}$ containing γ. Since γ is regular in $\mathbf{G}$, we must have $\mathbf{S} = \mathbf{T}$, and so $\mathbf{B}$ is the Borel subgroup of $\mathbf{G}$ required to prove (1). For (2), take $\gamma \in \mathbf{G}_{\mathrm{reg}} \cap \mathbf{G}_\theta^0$. Then by Lemma 1.3.10, γ is regular in $\mathbf{G}_\theta^0$ and $\mathbf{S} = (\mathbf{G}_\theta^0)_\gamma^0$ is a maximal torus in $\mathbf{G}_\theta^0$. Since $\mathbf{T}$ is contained in a θ-stable Borel subgroup of $\mathbf{G}$, $\mathbf{T}_\theta^0$ is also a maximal torus of $\mathbf{G}_\theta^0$, and we may choose $g \in \mathbf{G}_\theta^0$ such that $\mathbf{T}_\theta^0 = {}^g\mathbf{S} = (\mathbf{G}_\theta^0)_{{}^g\gamma}^0$. Therefore, ${}^g\gamma$ is the required element of $\mathbf{G}_{\mathrm{reg}} \cap \mathbf{T}_\theta^0$. □

Now we assume that θ is of finite order d_θ. As usual, write $G = \mathbf{G}(F)$. Let $x \in \mathbf{G}^i$, and let ℓ be the order of θ^i. Then ℓ is the smallest positive integer such that $x^\ell \in \mathbf{G}$. Call x $\mathbf{G}$*-regular* (or sometimes G*-regular* if $x \in G^i$) if x^ℓ is regular in $\mathbf{G}$. For any subset A of $\mathbf{G}^+$ (resp. G^+), let $A_{\mathbf{G}\text{-reg}}$ (resp. $A_{G\text{-reg}}$) denote the set of elements of A which are $\mathbf{G}$-regular (resp. G-regular). Obviously, $\mathbf{G}_{\mathbf{G}\text{-reg}}$ is just $\mathbf{G}_{\mathrm{reg}}$, the set of regular elements in $\mathbf{G}$. If $x \in \mathbf{G}^i$ for some $1 \le i < d_\theta$, then ℓ is the least common multiple of i and d_θ. Writing $x = x_0\theta^i$ for some $x_0 \in \mathbf{G}$, we have that x is $\mathbf{G}$-regular precisely when $\mathrm{N}_{\theta^i}(x_0)$ is regular in $\mathbf{G}$.

Lemma 1.3.12. *Let* $g \in \mathbf{G}_\theta$*. There exists an integer* i *such that* $g\theta^i \in \mathbf{G}^i_{\mathbf{G}\text{-reg}}$ *if and only if* $g \in \mathbf{G}_{\mathrm{reg}}$.

Proof. If $g \in \mathbf{G}_{\mathrm{reg}}$, take $i = 0$. For the other direction, choose an integer $0 \le i < d_\theta$ such that $g\theta^i \in \mathbf{G}^i_{\mathbf{G}\text{-reg}}$, and let ℓ be the order of θ^i. Then g^ℓ, and hence g, is regular in $\mathbf{G}$. □

Example. It is not true for $g \in (\mathbf{G}_{\mathrm{reg}})_\theta$ that $g\theta^i \in \mathbf{G}^i_{\mathbf{G}\text{-reg}}$ for all i. For example, let F_0 be a p-adic field in which -1 is not a square. Choose $\imath \in \bar{F}_0$ such that $\imath^2 = -1$, and let $F = F_0(\imath)$. For $x \in \mathbf{GL}_4$, define $\theta(x) = J^{-1}\,{}^{\mathrm{t}}x^{-1}J$, where

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Then $g = \mathrm{diag}(\imath, 2, -\imath, 1/2)$ is regular and θ-fixed, but $(g\theta)^2 = g^2 = \mathrm{diag}(-1, 4, -1, 1/4)$ is not regular. □

For the remainder, we take θ to be a finite-order F_0-automorphism of $\mathbf{G} = \mathrm{R}_{F/F_0}\mathbf{GL}_n$, where F/F_0 is a finite, abelian extension of p-adic fields (see §1.4). Here, we allow the possibility that $F_0 = F$. Identify the groups $\mathbf{G}(F_0)$ and $G = \mathbf{GL}_n(F)$ (see §1.2.3). Via this identification, $\theta|\mathbf{G}(F_0)$ induces an (abstract group) automorphism of G, which we also denote θ. We may then also identify $\mathbf{G}^+(F_0)$ with G^+. Using the characterization that an element in $\mathbf{GL}_n$ is regular if and only if its eigenvalues are distinct, it is easy to see that an element of $\mathbf{G}(F_0)$ is regular in $\mathbf{G}$ if and only if the corresponding element in $\mathbf{GL}_n(F)$ is regular.

Lemma 1.3.13. *For $0 \le i < d_\theta$, $(G^i)_{G\text{-reg}}$ is dense in G^i.*

Proof. Since θ is defined over F_0, $\mathbf{G}$-regularity of elements in $\mathbf{G}^i(F_0) = \mathbf{G}^i \cap \mathbf{G}^+(F_0)$ can be characterized by the non-vanishing of a certain polynomial function with coefficients in F_0. Therefore, $\big(\mathbf{G}^i(F_0)\big)_{G\text{-reg}}$ is dense in $\mathbf{G}^i(F_0)$ in the F_0-topology. Since the map $\mathbf{G}(F_0) \to G$ corresponding to the identification $\mathbf{G}(F_0) \simeq G$ is continuous, $(G^i)_{G\text{-reg}}$ is dense in G^i in the F-topology. □

Let $\mathbf{g} = \mathrm{Lie}(\mathbf{G}) = \mathrm{R}_{F/F_0}\mathbf{M}_n$, and identify $\mathbf{g}(F_0)$ with $\mathfrak{g} = \mathbf{M}_n(F)$. As an element of G^+, θ acts on $\mathfrak{g}$ by

$$(\mathrm{Ad}\,\theta)(X) = {}^{\theta}X = \mathrm{d}\theta(X), \qquad \big(X \in \mathfrak{g}\big).$$

Following [7, Appendix A], we make the following definition. Let $x \in G^+$ be called *quasi-regular* if it satisfies either of the following equivalent conditions:

(i) $\mathfrak{k}_x = \ker(\mathrm{Ad}\,x - 1)$ contains no non-zero nilpotent element of $\mathfrak{g}$;

(ii) if U is the unipotent radical of a parabolic subgroup of G and $\mathfrak{u} \subset \mathfrak{g}$ is the Lie algebra of U, then $\mathfrak{k}_x \cap \mathfrak{u} = \{0\}$.

Let G^+_{qr} denote the set of quasi-regular elements of G^+. For $A \subseteq G^+$, let $A_{\mathrm{qr}} = A \cap G^+_{\mathrm{qr}}$. Since $\mathrm{d}\theta$ preserves nilpotency, so does $\mathrm{Ad}\,y$ for any $y \in G^+$. From this, we may conclude that for any $x, y \in G^+$, ${}^{y}x$ lies in G^+_{qr} if and only if x does.

Example. Let $n = 2$ and $\theta(x) = J^{-1}({}^{\mathrm{t}}x^{-1})J$, for $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Let $g = \left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$ and consider $x = g\theta$ in G^1. Then $\mathfrak{k}_x$ is the linear subspace of $\mathfrak{g}$ spanned by g. Since $g^2 = 1$, x is quasi-regular. □

Lemma 1.3.14. *For $0 \le i < d_\theta$, $(G^i)_{G\text{-reg}} \subseteq (G^i)_{\mathrm{qr}}$.*

Proof. Let $x \in (G^i)_{G\text{-reg}}$ and let ℓ be the order of θ^i. Since $G_{\mathrm{reg}} \subseteq G_{\mathrm{qr}}$, $\ker(\mathrm{Ad}\,x^\ell - 1)$ contains no non-zero nilpotent elements of $\mathfrak{g}$. But then we must have $x \in (G^i)_{\mathrm{qr}}$, since $\ker(\mathrm{Ad}\,x - 1) \subseteq \ker(\mathrm{Ad}\,x^\ell - 1)$. □

Proposition 1.3.15. *For $0 \le i < d_\theta$, the set $(G^i)_{\mathrm{qr}}$ is open and dense in G^i in the F-topology.*

Proof. Combining Lemmas 1.3.13 and 1.3.14, we have that $(G^i)_{\mathrm{qr}}$ is dense in G^i. To show that $(G^i)_{\mathrm{qr}}$ is open, we copy the proof of [7, (A.3)]. For a matrix $X = (x_{jk}) \in \mathfrak{g}$, define $\|X\| = \max_{j,k} |x_{jk}|_F$. Let $\mathscr{S} = \{X \in \mathfrak{g} \mid \|X\| = 1\}$. Then $\mathscr{S}$ is compact in $\mathfrak{g}$, and the subset $\mathscr{S}_{\mathrm{nilp}}$ of nilpotent elements of $\mathscr{S}$ is closed and hence compact.

Let $\{g_m\}_{m\ge 1} \subset G^i \smallsetminus (G^i)_{\mathrm{qr}}$ be a convergent sequence with limit $g \in G^i$. For each $m \ge 1$, g_m not quasi-regular implies that there exists a non-zero, nilpotent $X_m \in \ker(\mathrm{Ad}\,g - 1)$. Scaling if necessary, we may assume each X_m is in $\mathscr{S}_{\mathrm{nilp}}$. Since $\mathscr{S}_{\mathrm{nilp}}$ is compact, there exists a convergent subsequence $\{X_{m_k}\}_{k\ge 1}$ and an element $X \in \mathscr{S}_{\mathrm{nilp}}$ such that $X_m \to X$ as $k \to \infty$. But then also ${}^{g_{m_k}}X_{m_k} \to X$, and so ${}^{g}X = X$. This implies $g \notin (G^i)_{\mathrm{qr}}$, and so $G^i \smallsetminus (G^i)_{\mathrm{qr}}$ is closed. Therefore, $(G^i)_{\mathrm{qr}}$ is open in G^i. □

Corollary 1.3.16. *G^+_{qr} is open and dense in G^+.*

1.4 Automorphisms of $\mathbf{GL}_n(F)$

In this section, we examine certain automorphisms of $G = \mathbf{GL}_n(F)$. Identify $Z(G)$ with $F^\times$.

1.4.1 Arbitrary fields. For any field F, O'Meara has classified the automorphisms of G. From [17, Theorem 21], we have the following.

Theorem 1.4.1 (O'Meara)**.** *Suppose $n \geq 3$. For any $\theta \in \mathrm{Aut}(G)$, there exist a group homomorphism $\chi : G \to F^\times$, a field automorphism $\tau \in \mathrm{Aut}(F)$, and an element $J \in G$ such that*

$$\theta(g) = \chi(g)\, J^{-1}\, \tau\big(\theta_0(g)\big)\, J, \qquad \big(g \in G\big), \tag{1.4.1}$$

where θ_0 is either the identity map or the map $g \mapsto {}^{\mathrm{t}}g^{-1}$ on G, and τ acts on an element of G by acting on its entries.

In this thesis, we will ignore the automorphisms of G that are not of the form of (1.4.1) in the case that $n = 2$. In particular, we restrict our attention to automorphisms of finite order that are of the given form, regardless of the value of n.

Remark. An automorphism of G may have several expressions of the form of (1.4.1). Obviously, we may replace J by any element of $F^\times J$. For another example, consider $n = 2$ and $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Then the automorphism $g \mapsto J^{-1}\, {}^{\mathrm{t}}g^{-1}\, J$ may also be written $g \mapsto (\det g)^{-1} g$.

We will also only consider certain automorphisms which are induced by restricting an F_0-automorphism of $\mathbf{G} = \mathrm{R}_{F/F_0} \mathbf{GL}_n$ to $\mathbf{G}(F_0) \simeq G$, where F/F_0 is some finite, Galois extension. Fix such a subfield $F_0 \subseteq F$, allowing the possibility that $F_0 = F$, and for convenience write $\mathrm{R} = \mathrm{R}_{F/F_0}$. Let $d = [F : F_0]$ and, as in §1.2.1, set $\Sigma = \mathrm{Gal}(\bar{F}/F_0)/\mathrm{Gal}(\bar{F}/F) \simeq \mathrm{Gal}(F/F_0)$.

Proposition 1.4.2.

(1) Let $\chi : \mathbf{G} \to Z(\mathbf{G}) \simeq \mathrm{R}\mathbb{G}_{\mathrm{m}}$ be any F_0-homomorphism. Then the map $G \to F^\times$ induced by $\chi | \mathbf{G}(F_0)$ is of the form

$$g \mapsto \textstyle\prod_{\sigma \in \Sigma} \sigma(\det g)^{m_\sigma}, \qquad \big(g \in G\big), \tag{1.4.2}$$

for some collection of integers $\{m_\sigma\}_{\sigma \in \Sigma}$.

(2) Assume that F/F_0 is an abelian extension. Let $\theta_0 : \mathbf{GL}_n \to \mathbf{GL}_n$ be either the identity map or transpose-inverse. Suppose $J \in \mathbf{GL}_n(F)$, $\tau \in \mathrm{Gal}(F/F_0)$, and $\chi : G \to F^\times$ is as in (1.4.2) *for some $\{m_\sigma\} \subset \mathbb{Z}$. Then the automorphism of G given by* (1.4.1) *coincides with that induced by the restriction to $\mathbf{G}(F_0)$ of some F_0-automorphism of $\mathbf{G}$.*

Proof. Recall that $\mathbb{G}_{\mathrm{m}} = \mathbf{GL}_1$. Since $\mathbf{GL}_m$ is defined over F_0 for any m, we have that $\mathbf{GL}_m^\Sigma$ is just a direct product of d copies of $\mathbf{GL}_m$, for either $m = 1$ or $m = n$. Write $f_m = f_{\mathbf{GL}_m} : \mathrm{R}\mathbf{GL}_m \to \mathbf{GL}_m^\Sigma$. Identify $\mathbb{G}_{\mathrm{m}}$ with $Z(\mathbf{GL}_n)$. We may assume that we have constructed $\mathrm{R}\mathbb{G}_{\mathrm{m}}$ and $\mathbf{G} = \mathrm{R}\mathbf{GL}_n$ such that $f_1^{-1} = f_n^{-1} | \mathbb{G}_{\mathrm{m}}^\Sigma$. Now suppose $\chi : \mathbf{G} \to \mathrm{R}\mathbb{G}_{\mathrm{m}}$ is an F_0-homomorphism. Then, composing the restriction of $f_1 \circ \chi \circ f_m^{-1}$ to the i^{th}

factor of $\mathbf{GL}_n^\Sigma$ with the projection onto the j^{th} factor of $\mathbb{G}_{\mathrm{m}}^\Sigma$, we obtain an F-homomorphism $\mathbf{GL}_n \to \mathbb{G}_{\mathrm{m}}$. Since any such map must be of the form $x \mapsto (\det x)^{k_{ij}}$ for some $k_{ij} \in \mathbb{Z}$, (1) follows. Now consider (2). For $\sigma \in \mathrm{Gal}(F/F_0)$ and $m \geq 1$, let $\eta_{\sigma,m}$ be the F_0-automorphism of $\mathrm{R}\mathbf{GL}_m$ as constructed in §1.2.4. Then the required F_0-automorphism of $\mathbf{G}$ is the product of the morphisms in the collection $\{\eta_{\sigma,1} \circ \mathrm{R}(\det^{m_\sigma})\}_{\sigma\in\Sigma}$ (whose images lie in the centre of $\mathbf{G}$), and the morphism $\mathrm{R}(\mathrm{Int}_{\mathrm{R}}\, J) \circ \eta_{\tau,n} \circ \mathrm{R}\theta_0$. $\square$

Accordingly, we will only consider automorphisms of G which have the form considered in Proposition 1.4.2(2). Let θ be such an element of $\mathrm{Aut}(G)$. Lift τ to an element of $\mathrm{Gal}(\bar{F}/F_0)$, and consider θ as an automorphism of the abstract group $\mathbf{GL}_n = \mathbf{GL}_n(\bar{F})$. We will also use θ to denote the F_0-automorphism of $\mathrm{R}\mathbf{GL}_n$ from the proof of Proposition 1.4.2(2). The following is easily verified.

Lemma 1.4.3. *Suppose* $\mathbf{H}$ *is a closed* F*-subgroup of* $\mathbf{GL}_n$ *which is* θ*-stable. Then* $\mathrm{R}\mathbf{H}$ *is an* θ*-stable* F_0*-subgroup of* $\mathbf{G}$.

1.4.2 p-Adic fields. Now consider the case that F is a p-adic field, and let θ be of the form of (1.4.1), with χ of the form of (1.4.2). For this section, we need only assume that F/F_0 is a finite, Galois extension; we do not assume that $\mathrm{Gal}(F/F_0)$ is abelian. The first properties to note are that θ stabilizes $\mathcal{O}_F^\times$, and it stabilizes $K_0 = \mathrm{GL}_n(\mathcal{O}_F)$ if and only if $J \in N_G(K_0) = F^\times K_0$. Define the *valuation of* θ, denoted $\mathrm{val}\,\theta$, by $\mathrm{val}\,\theta = \mathrm{val}_F\big(\theta(\varpi_F)\big)$. For the given form of θ, it is easy to see that

$$\mathrm{val}\,\theta = \mathrm{val}\,\theta_0 + n\textstyle\sum_{\sigma\in\Sigma} m_\sigma,$$

and $\mathrm{val}_F\big(\det\theta(x)\big) = (\mathrm{val}\,\theta)\big(\mathrm{val}_F(\det x)\big)$ for any $x \in G$. If θ has finite order, then we must have $\mathrm{val}\,\theta = \pm 1$, and in the case $\mathrm{val}\,\theta = -1$, θ must have even order. The following is immediate from the above formula for $\mathrm{val}\,\theta$, showing that if θ has finite order, then the possible values for the m_σ are not completely arbitrary.

Lemma 1.4.4. *Write* $S = \sum_{\sigma\in\Sigma} m_\sigma$.

(1) If $\mathrm{val}\,\theta = \mathrm{val}\,\theta_0$, *then* $S = 0$.

(2) If $\mathrm{val}\,\theta = -\mathrm{val}\,\theta_0$, *then either* $n = 1$ *and* $S = 2\,\mathrm{val}\,\theta$, *or* $n = 2$ *and* $S = \mathrm{val}\,\theta$.

Matters of convergence of certain integrals will be affected by the value of $\mathrm{val}\,\theta$, due to the following (see §3.4).

Lemma 1.4.5. *Suppose* $\mathrm{val}\,\theta = \pm 1$. *The quotient* $F^\times/(F^\times)_\theta$ *is compact if and only if* $\mathrm{val}\,\theta = 1$.

Proof. If $\mathrm{val}\,\theta = -1$, then we must have $(F^\times)_\theta \subseteq \mathcal{O}_F^\times$, and $F^\times/\mathcal{O}_F^\times$ is not compact. So suppose $\mathrm{val}\,\theta = 1$. Then Lemma 1.4.4 implies that in either case of $\mathrm{val}\,\theta_0 = \pm 1$, we have $F_0^\times \subseteq (F^\times)_\theta$. Therefore, it suffices to show that $F^\times/F_0^\times$ is compact. However, this is immediate from the property $\big|\varpi_{F_0}\big|_F = |\varpi_F|_F^e$, where e is the ramification index of F/F_0, since this implies $F^\times \subseteq \omega F_0^\times$ for ω the compact set $\cup_{i=0}^e \varpi_F^i \mathcal{O}_F^\times$. $\square$

1.5 Stable representations

In this section, we explore the relationship between a representation π of G and a fixed finite-order automorphism $\theta \in \mathrm{Aut}(G)$, in particular in the case where π is induced from a θ-stable subgroup of G. In §1.5.1 and §1.5.2, we take G to be any totally disconnected, locally compact group. In §1.5.3, we take $G = \mathbf{GL}_n(F)$, for F a p-adic field, and assume θ is of the form of (1.4.1), with χ of the form of (1.4.2).

1.5.1 Definitions and basic facts. To begin, suppose G is any abstract group, $\theta \in \mathrm{Aut}(G)$ is of finite order d_θ, and (π, V) is a (complex) representation of G. If $\pi \cong \pi \circ \theta$, then π is called *θ-stable*. In this case, if there exists an intertwining operator $A_\pi \in \mathrm{Hom}_G(\pi, \pi \circ \theta)$ with the property $A_\pi^{d_\theta} = 1$, then we may extend π to a representation (π^+, V) of G^+ by setting

$$\pi^+(g\theta^i) = \pi(g)A_\pi^i, \qquad \big(g \in G,\ 0 \le i < d_\theta\big).$$

Note that such an operator A_π will always exist for any pair (G, π) to which Schur's Lemma applies.

Now assume that G is a totally disconnected, locally compact group. Let (π, V) be an admissible, irreducible, θ-stable representation of G, and fix a choice of A_π with $A_\pi^{d_\theta} = 1$. Then π^+ is also admissible and irreducible. Note that since G is open in G^+, the subspaces of elements of V^* which are smooth with respect to π^* and $(\pi^+)^*$, respectively, coincide.

Proposition 1.5.1. *There exist the following relationships between the properties of π and π^+.*

(1) The extension π^+ is unitary if and only if π is unitary.

(2) The extension π^+ is supercuspidal if and only if π is supercuspidal and $Z(G)/Z(G)_\theta$ is compact.

Proof. First suppose π^+ is unitary. Then any G^+-invariant, positive definite, Hermitian inner product on V is also G-invariant. On the other hand, if π is unitary, let $\langle\cdot,\cdot\rangle$ be a G-invariant, positive definite, Hermitian inner product on V. For $v, w \in V$, set $\langle v, w\rangle^+ = \sum_{i=0}^{d_\theta - 1}\langle A_\pi^i v, A_\pi^i w\rangle$. It is straightforward to verify that this defines a G^+-invariant, positive definite, Hermitian inner product on V. This proves (1).

Now suppose π^+ is supercuspidal. Choose elements $v \in V$, $\tilde{v} \in \widetilde{V}$, and let $\varphi = \varphi_{v,\tilde{v}}$ be the corresponding matrix coefficient of π. Then the pair $v, \tilde{v}$ also determine a matrix coefficient of π^+ which is an extension of φ to G^+. Denote this extension by φ^+. Since π^+ is supercuspidal, $\mathrm{supp}\,\varphi^+$ is compact modulo $Z(G^+)$. Let ω be a compact subset of G^+ such that $\mathrm{supp}\,\varphi^+ \subseteq \omega Z(G^+)$. Then $\mathrm{supp}\,\varphi \subseteq \big(\omega Z(G^+)\big) \cap G$. Recall that $Z(G)_\theta = Z(G^+) \cap G$ (Lemma 1.1.2(1)). Set $g_0 = 1$, let $\Lambda = \{0 \le i < d_\theta \mid Z(G^+) \cap G^i \ne \emptyset\}$, and for each $i \in \Lambda$ with $i \ne 0$, let $g_i \in G_\theta$ as in Lemma 1.1.2(2). Note that $g_i\theta^i \in Z(G^+)$ for each $i \in \Lambda$. Let $\Delta = \bigcup_{i\in\Lambda} g_i\theta^i\big(\omega \cap G^{d_\theta - i}\big) \subseteq G$. We may assume that ω was chosen with $\omega \cap G \ne \emptyset$, so that $\Delta \ne \emptyset$. Then

$$\mathrm{supp}\,\varphi \subseteq \big(\omega Z(G^+)\big) \cap G = \bigcup_{i=0}^{d_\theta - 1}\big(\omega \cap G^{d_\theta - i}\big)\big(Z(G^+) \cap G^i\big) = \Delta Z(G)_\theta \subseteq \Delta Z(G).$$

Since Δ is compact, it follows that π is supercuspidal. To see that $Z(G)/Z(G)_\theta$ must be compact, notice that $Z(G)$ acts on V by (non-zero) scalars, since π is irreducible. Therefore, if v and $\tilde{v}$ are chosen such that $\langle\tilde{v}, v\rangle \ne 0$, then, as above, $Z(G) \subseteq \mathrm{supp}\,\varphi \subseteq \Delta Z(G)_\theta$.

Finally, assume $Z(G)/Z(G)_\theta$ is compact and π is supercuspidal. Again, choose elements $v \in V$ and $\tilde{v} \in \widetilde{V}$, and let φ^+ be the corresponding matrix coefficient of π^+. For $0 \le i < d_\theta$, define

$$\varphi^i : G \to \mathbb{C}, \qquad g \mapsto \varphi^+(g\theta^i).$$

Then each φ^i is the matrix coefficient of π corresponding to the elements $A_\pi^i v \in V$ and $\tilde{v} \in \widetilde{V}$, and therefore has compact support modulo $Z(G)$. For each i, we may choose a compact subset $\omega_i \subseteq G$ such that

$$\mathrm{supp}(\varphi^+|G^i) = (\mathrm{supp}\,\varphi^i)\theta^i \subseteq \omega_i Z(G)\theta^i.$$

Since $Z(G)/Z(G)_\theta$ is compact, we may also choose a compact subset $\Delta \subset G$ such that $Z(G) \subseteq \Delta Z(G)_\theta$. Therefore, $\operatorname{supp}\varphi^+ \subseteq \omega Z(G^+)$, where ω is the compact set $\omega = \bigcup_{i=0}^{d-1} \omega_i \Delta\theta^i$. It follows that π^+ is supercuspidal, completing the proof of (2). □

1.5.2 Induced θ-stable representations. Let (σ, W) be a smooth, irreducible representation of an open, closed, θ-stable subgroup $H \subset G$, and consider the representation (π, V) of G with $\pi = \text{c-Ind}_H^G \sigma$. Since H is open, we may embed W in V as follows. For $w \in W$, let f_w be the element of V defined by

$$f_w(x) = \begin{cases} \sigma(x)w, & x \in H, \\ 0, & x \notin H. \end{cases}$$

Proposition 1.5.2. *There exist an injection and surjection, respectively,*

$$\Phi\colon \operatorname{Hom}_H(\sigma, \sigma\circ\theta) \hookrightarrow \operatorname{Hom}_G(\pi, \pi\circ\theta),$$
$$\Psi\colon \operatorname{Hom}_G(\pi, \pi\circ\theta) \twoheadrightarrow \operatorname{Hom}_H(\sigma, \sigma\circ\theta),$$

such that Φ preserves invertibility, and $\Psi\circ\Phi = \mathrm{id}$.

Proof. First, define

$$\big(\Phi(A)f\big)(x) = Af(x^\theta), \qquad \big(A \in \operatorname{Hom}_H(\sigma,\sigma\circ\theta),\ f \in V,\ x \in G\big).$$

This map is clearly linear. Take elements $A \in \operatorname{Hom}_H(\sigma, \sigma\circ\theta)$, $f \in V$, and $x \in G$. The function $\Phi(A)f$ has compact support modulo H, since $\operatorname{supp}\Phi(A)f \subseteq {}^\theta(\operatorname{supp} f)$. For $h \in H$, we have

$$(\Phi(A)f)(hx) = Af((hx)^\theta) = \sigma(h)Af(x^\theta) = \sigma(h)\big(\Phi(A)f\big)(x),$$

so $\Phi(A)f$ transforms properly under left-translation by elements of H. If f is right K-invariant for some compact, open subgroup $K \subset G$, then $\Phi(A)f$ is right ${}^\theta K$-invariant. We have now shown that the image of $\Phi(A)$ is contained in V. For any $g \in G$, we have

$$\big(\Phi(A)\pi(g)f\big)(x) = Af\big((x\,{}^\theta g)^\theta\big) = \big(\pi({}^\theta g)\Phi(A)f\big)(x),$$

hence $\Phi(A)$ is an element of $\operatorname{Hom}_G(\pi, \pi\circ\theta)$. Now suppose $\Phi(A) = 0$. Then $Af(x^\theta) = 0$ for all $x \in G$ and $f \in V$. Taking $x = 1$ and $f = f_w$ for any $w \in W$, we see that $Aw = 0$ for all $w \in W$, and therefore A is the zero endomorphism of W. This shows that Φ is injective. Finally, suppose A is invertible. Then $\Phi(A)$ is also invertible, with inverse

$$\big(\Phi(A)^{-1}f\big)(x) = A^{-1}f({}^\theta x), \qquad \big(f \in V,\ x \in G\big),$$

where similar arguments to those above show that $\Phi(A)^{-1} \in \operatorname{Hom}_G(\pi\circ\theta, \pi)$.

Now define

$$\Psi(B)w = (Bf_w)(1), \qquad \big(B \in \operatorname{Hom}_G(\pi, \pi\circ\theta),\ w \in W\big).$$

Again, this map is clearly linear. Take elements $B \in \operatorname{Hom}_G(\pi, \pi\circ\theta)$, $w \in W$, and $h \in H$. Then,

$$\Psi(B)\,\sigma(h)w = (Bf_{\sigma(h)w})(1) = \big(\pi({}^\theta h)Bf_w\big)(1) = (Bf_w)({}^\theta h) = \sigma({}^\theta h)\,\Psi(A)w,$$

so that $\Psi(B) \in \mathrm{Hom}_H(\sigma, \sigma \circ \theta)$.

For any $A \in \mathrm{Hom}_H(\sigma, \sigma \circ \theta)$ and $w \in W$,

$$(\Psi \circ \Phi)(A)w = \big(\Phi(A)f_w\big)(1) = Af_w(1) = Aw,$$

so that $(\Psi \circ \Phi)(A) = A$. Note that this also shows that Ψ is surjective. $\square$

Corollary 1.5.3. *If σ is θ-stable, then so is π.*

Assume that σ is θ-stable. Fix an intertwining operator $A_\sigma \in \mathrm{Hom}_H(\sigma, \sigma \circ \theta)$ and let $A_\pi = \Phi(A_\sigma) \in \mathrm{Hom}_G(\pi, \pi \circ \theta)$, where Φ is as in Proposition 1.5.2. Note that if A_σ is normalized so that $A_\sigma^{d_\theta} = 1$, then also $A_\pi^{d_\theta} = 1$. Using these operators we may define representations π^+ and σ^+ of G^+ and H^+, respectively, as in §1.5.1.

Proposition 1.5.4. *The extension π^+ is equivalent to* $\text{c-Ind}_{H^+}^{G^+} \sigma^+$.

Proof. Let U be the space of $\tau = \text{c-Ind}_{H^+}^{G^+} \sigma^+$. Define $A_\tau : V \to U$ by

$$(A_\tau f)(x\theta^i) = (A_\pi^i f)(x) = A_\sigma^i f(x^{\theta^i}), \qquad \big(f \in V,\ x \in G,\ i \in \mathbb{Z}\big).$$

Take elements $f \in V$, $x \in G$, $h \in H$, and $i, j \in \mathbb{Z}$. The function $A_\tau f$ has compact support modulo H^+, since $\operatorname{supp} A_\tau f = \bigcup_{\ell=0}^{d_\theta - 1} \theta^\ell \cdot \operatorname{supp} f$. We have

$$(A_\tau f)(h\theta^i \cdot x\theta^j) = A_\sigma^{i+j} f(h^{\theta^{i+j}}\, x^{\theta^j}) = \sigma(h) A_\sigma^{i+j} f(x^{\theta^j}) = \sigma^+(h\theta^i)(A_\tau f)(x\theta^j),$$

so $A_\tau f$ transforms properly under left-translation by elements of H^+. If f is right K-invariant for some compact, open subgroup K of G, then so is $A_\tau f$, and K is a compact, open subgroup of G^+. Thus the image of A_τ is indeed contained in V_τ. For any $g \in G$, we have

$$(A_\tau \pi^+(g\theta^i) f)(x\theta^j) = \big(A_\pi^{i+j} f\big)(x \cdot {}^{\theta^j} g) = (A_\tau f)(x\theta^j g\theta^i) = (\tau(g\theta^i) A_\tau f)(x\theta^j).$$

Therefore, $A_\tau \in \mathrm{Hom}_G(\pi^+, \tau)$, and it remains to show that A_τ is invertible. Suppose $A_\tau f \equiv 0$. Then $f = A_\tau f | G \equiv 0$, and so A_τ is injective. Now let $f' \in U$ and $f = f' | G$. Let $\omega \subset G^+$ be a compact subset such that $\operatorname{supp} f' \subseteq \omega H^+$, and let $\omega' = \bigcup_{\ell=0}^{d_\theta - 1} (\omega \cap G^\ell)\theta^{-\ell}$. Then ω' is a compact subset of G, and

$$\operatorname{supp} f = \operatorname{supp} f' \cap G \subseteq \omega H^+ \cap G = \omega' H.$$

Also, since f' transforms properly under left-translation by elements of H, so does f. And if f' is right K-invariant for some compact, open subgroup K of G^+, then f is right $(K \cap G)$-invariant. Therefore, $f \in V$, and furthermore,

$$(A_\tau f)(x\theta^i) = A_\sigma^i f\big(x^{\theta^i}\big) = \sigma^+(\theta^i) f'\big(x^{\theta^i}\big) = f'(x\theta^i),$$

since $\theta \in H^+$. Thus, A_τ is surjective. $\square$

1.5.3 Unitary twists. We now take $G = \mathbf{GL}_n(F)$, for F a p-adic field, and assume θ is of the form of (1.4.1), with χ of the form of (1.4.2). In this section, we show that for any irreducible, supercuspidal, θ-stable representation π of G, there exists a twist of π which is unitary and θ-stable.

Lemma 1.5.5. *If π is an irreducible, smooth, θ-stable representation of G, then there exists a θ-fixed quasi-character ν of G such that $\pi \otimes \nu$ has unitary central character.*

Proof. Let ω_π be the central character of π. Since π is irreducible and θ-stable, ω_π must be θ-fixed. If $\operatorname{val}\theta = -1$, then $(1-\theta)(\varpi_F) \in \varpi_F^2 \mathscr{O}_F^\times$. Then, from the relation $\omega_\pi\big(\theta(\varpi_F)\big) = \omega_\pi(\varpi_F)$, we see that $|\omega_\pi(\varpi_F)|_\infty^2 = 1$. Therefore, we may take $\nu \equiv 1$. Now suppose $\operatorname{val}\theta = 1$. Then, as in §1.4.2, we have $\operatorname{val}_F\big(\det\theta(x)\big) = \operatorname{val}_F(\det x)$ for any $x \in G$. Let $s \in \mathbb{C}$ such that $\omega_\pi(z) = |z|_F^s\, \omega_\pi\big(\operatorname{int}_F(z)\big)$ for any $z \in Z$. Then if we set $\nu(x) = |\det x|_F^{-s/n}$, the twist $\pi \otimes \nu$ has unitary central character, and clearly ν is θ-fixed in the present case. □

Corollary 1.5.6. *If π is an irreducible, supercuspidal, θ-stable representation of G, then there exists a quasi-character ν of G such that $\pi \otimes \nu$ is irreducible, supercuspidal, θ-stable, and unitary.*

Proof. Let ν be as in the lemma. Then $\pi \otimes \nu$ is unitary, and since ν is θ-fixed, we have

$$\operatorname{Hom}_G\big(\pi \otimes \nu, (\pi \otimes \nu)\circ\theta\big) = \operatorname{Hom}_G(\pi, \pi \circ \theta).$$ □

2. A CHARACTER FORMULA OVER THE RESIDUE FIELD

The θ-stable representations of $G = \mathbf{GL}_n(F)$, for F a p-adic field, which we will consider will be constructed from θ-stable, cuspidal representations of $\mathsf{G} = \mathbf{GL}_n(k_F)$, where $\theta \in \mathrm{Aut}(\mathsf{G})$ is induced from the restriction of $\theta \in \mathrm{Aut}(G)$ to an appropriate θ-stable, maximal parahoric subgroup of G. These cuspidal representations of G will in turn be constructed using Deligne-Lusztig induction from characters of θ-stable tori of G. Extending these θ-stable representations to G^+ and G^+, respectively, the associated characters of G^+ will figure into our character formula at the level of G^+. In [14], Digne and Michel develop a Deligne-Lusztig theory for non-connected, reductive algebraic groups over a finite field. In this section, we use their results to obtain a character formula over k_F which allows us to reduce to Green functions on the set of unitary elements in the group of fixed points G_θ.

2.1 Definitions and basic facts

Let $\mathbf{G}$ be a connected, reductive algebraic group defined over a finite field $k = \mathbb{F}_q$, for q a power of p, and let θ be a quasi-semisimple automorphism of $\mathbf{G}$ which is defined over k and of finite order d_θ. Assume that $\gcd(d_\theta, p) = 1$. We make the following definitions, as in [14, Definition 1.2]. A *Borel subgroup* of $\mathbf{G}^+$ is one of the form $\mathbf{B}' = N_{\mathbf{G}^+}(\mathbf{B})$ for some Borel subgroup $\mathbf{B}$ of $\mathbf{G}$. A *torus* of $\mathbf{G}^+$ is a subgroup of the form $\mathbf{T}' = N_{\mathbf{G}^+}(\mathbf{T}, \mathbf{B})$ for some pair $\mathbf{B} \supset \mathbf{T}$ in $\mathbf{G}$.

Fix a pair $\mathbf{B} \supset \mathbf{T}$ in $\mathbf{G}$, with $\mathbf{T}$ defined over k and θ-stable. Then ${}^\theta\mathbf{B} \supset \mathbf{T}$ is also a pair in $\mathbf{G}$, and so there exists a unique $w_o \in W_{\mathbf{G}}(\mathbf{T})$ such that ${}^{w_o\theta}\mathbf{B} = \mathbf{B}$ ([38, 6.4.12]). Fix a choice of representative $n_o \in \mathbf{G}$ of w_o, and set $\vartheta = n_o\theta$. The automorphism $\mathrm{Int}_{\mathrm{L}}\,\vartheta$ of $\mathbf{G}^+$ restricts to a quasi-semisimple automorphism of $\mathbf{G}$, as it stabilizes the pair $\mathbf{B} \supset \mathbf{T}$. This automorphism is defined over k if $n_o \in \mathbf{G}(k)$. Let $\mathbf{T}'$ be the torus of $\mathbf{G}^+$ corresponding to the pair $\mathbf{B} \supset \mathbf{T}$. Notice that $\mathbf{T}'$ is not necessarily defined over k; however, if $\mathbf{B}$ is defined over k then clearly $\mathbf{T}'$ is as well.

Proposition 2.1.1.

(1) The element $w_o\theta$ of $W_{\mathbf{G}}(\mathbf{T})^+$ has order d_θ.

(2) $[\mathbf{T}' : \mathbf{T}] = d_\theta$, with $\mathbf{T}' \cap \mathbf{G}^i = \mathbf{T}\vartheta^i$ for $0 \le i < d_\theta$.

(3) If $w_o \in W_{\mathbf{G}}(\mathbf{T})_\theta$, then θ normalizes $\mathbf{T}'$.

(4) If $w_o = 1$ (that is, if $\mathbf{B}$ is θ-stable), then $\mathbf{T}' = \mathbf{T}^+$ and $\mathbf{T}'$ is defined over k.

Proof. Let $y = (w_o\theta)^{d_\theta} = \mathrm{N}_\theta(w_o) \in W_{\mathbf{G}}(\mathbf{T})$. Since $w_o\theta$ stabilizes $\mathbf{B}$, so does y. Therefore, y must be trivial, proving (1). Fix an integer $0 \le i < d_\theta$. Since ϑ normalizes $\mathbf{B}$, we have that for any $x \in \mathbf{G}^i$, $x \in N_{\mathbf{G}^+}(\mathbf{B})$ if and only if $x\vartheta^{-i} \in N_{\mathbf{G}}(\mathbf{B}) = \mathbf{B}$. Thus, $N_{\mathbf{G}^+}(\mathbf{B}) \cap \mathbf{G}^i = \mathbf{B}\vartheta^i$. But ϑ also normalizes $\mathbf{T}$, so that for any $b \in \mathbf{B}$, $b\vartheta^i \in N_{\mathbf{G}^+}(\mathbf{T})$ if and only if $b \in N_{\mathbf{B}}(\mathbf{T})$. Since $\mathbf{G}$ is reductive, $N_{\mathbf{B}}(\mathbf{T}) = \mathbf{T}$, and (2) follows. If $w_o \in W_{\mathbf{G}}(\mathbf{T})_\theta$, then there exists $t \in \mathbf{T}$ such that ${}^\theta\vartheta = t\vartheta$. Since $\mathbf{T}$ is θ-stable, we have $(t\vartheta)^i\vartheta^{-i} \in \mathbf{T}$, and so

$$ {}^\theta(\mathbf{T}' \cap \mathbf{G}^i) = \mathbf{T}\,{}^\theta(\vartheta^i) = \mathbf{T}(t\vartheta)^i\,\vartheta^{-i}\,\vartheta^i = \mathbf{T}\vartheta^i = \mathbf{T}' \cap \mathbf{G}^i. $$

This proves (3). Finally, if B is θ-stable, then n_o lies in T, hence $\vartheta^i\theta^{-i}$ does as well. Therefore,

$$\mathsf{T}' \cap \mathbf{G}^i = \mathsf{T}\vartheta^i\theta^{-i}\theta^i = \mathsf{T}\theta^i = \mathsf{T}^+ \cap \mathbf{G}^i,$$

and we have (4). □

2.2 Deligne-Lusztig induction

2.2.1 On the identity component. Let $\mathrm{G} = \mathbf{G}(k)$. Assume that $\mathrm{T} = \mathsf{T}(k)$ is *non-degenerate* in the sense that $\mathsf{T} = C_{\mathbf{G}}(\mathsf{T})^0$ (see [8, §3.6]). In this case, $W_{\mathbf{G}}(\mathsf{T})^k$ is isomorphic to $W = W_{\mathrm{G}}(\mathrm{T})$ ([8, 3.6.5]). For $\lambda \in \mathrm{Irr}(\mathrm{T})$, let $\mathsf{R}_{\mathsf{T}}^{\mathsf{G}}\lambda$ be the corresponding Deligne-Lusztig (virtual) character of G. Let $\mathsf{Q}_{\mathsf{T}}^{\mathsf{G}}$ be the Green function associated to T, defined on the unipotent set in G by $\mathsf{Q}_{\mathsf{T}}^{\mathsf{G}}(u) = (\mathsf{R}_{\mathsf{T}}^{\mathsf{G}}1)(u)$, for unipotent $u \in \mathrm{G}$.

Proposition 2.2.1. *For any $\lambda \in \mathrm{Irr}(\mathrm{T})$, we have* $\mathsf{R}_{\mathsf{T}}^{\mathsf{G}}({}^\theta\lambda) = \left(\mathsf{R}_{\mathsf{T}}^{\mathsf{G}}\lambda\right)\circ\theta^{-1}$.

Proof. Let $g \in \mathrm{G}$ have Jordan decomposition $g = su$ so that g^θ has Jordan decomposition $g^\theta = rv$, where $r = s^\theta$, $v = u^\theta$. Then [8, 7.2.8] gives

$$(\mathsf{R}_{\mathsf{T}}^{\mathsf{G}}\lambda)(g^\theta) = |\mathrm{G}_r^0|^{-1}\sum_{\substack{x\in\mathrm{G}\\ r\in{}^x\mathrm{T}}} \mathsf{Q}_{{}^x\mathsf{T}}^{\mathsf{G}_r^0}(v)\,\lambda(r^x).$$

Since T is θ-stable, we may make the change of variables $y = \theta(x)$ in the sum to obtain

$$\begin{aligned}(\mathsf{R}_{\mathsf{T}}^{\mathsf{G}}\lambda)(g^\theta) &= \left|(\mathrm{G}_s^0)^\theta\right|^{-1}\sum_{\substack{y\in\mathrm{G}\\ s\in{}^y\mathrm{T}}} \mathsf{Q}_{({}^y\mathsf{T})^\theta}^{(\mathsf{G}_s^0)^\theta}(v)\,\lambda(s^{y\theta})\\ &= |\mathrm{G}_s^0|^{-1}\sum_{\substack{y\in\mathrm{G}\\ s\in{}^y\mathrm{T}}} \mathsf{Q}_{{}^y\mathsf{T}}^{\mathsf{G}_s^0}(u)\,{}^\theta\lambda(s^y)\\ &= \left(\mathsf{R}_{\mathsf{T}}^{\mathsf{G}}({}^\theta\lambda)\right)(g).\end{aligned}$$ □

Recall that $\lambda \in \mathrm{Irr}(\mathrm{T})$ is said to be in *general position* if its stabilizer in W is trivial. In this case, one of $\pm\mathsf{R}_{\mathsf{T}}^{\mathsf{G}}\lambda$ is an irreducible character of G ([8, 7.3.5]). If $\mathsf{R}_{\mathsf{T}}^{\mathsf{G}}\lambda$ is θ-stable, then from Proposition 2.2.1 we have $\mathsf{R}_{\mathsf{T}}^{\mathsf{G}}\lambda = \mathsf{R}_{\mathsf{T}}^{\mathsf{G}}({}^\theta\lambda)$. Therefore, there exists a unique element $w_\lambda \in W = W_{\mathrm{G}}(\mathrm{T})$ such that ${}^{w_\lambda\theta}\lambda = \lambda$ ([8, 7.3.4]).

Proposition 2.2.2. *Let $\lambda \in \mathrm{Irr}(\mathrm{T})$ be in general position. If $\mathsf{R}_{\mathsf{T}}^{\mathsf{G}}\lambda$ is θ-stable, then the element $w_\lambda\theta$ of W^+ has order dividing d_θ.*

Proof. Since $w_\lambda\theta$ stabilizes λ, so does $(w_\lambda\theta)^{d_\theta}$. However, $(w_\lambda\theta)^{d_\theta}$ lies in W, and therefore λ in general position implies $(w_\lambda\theta)^{d_\theta} = 1$. □

Let T' be the finite group $\mathsf{T}' \cap \mathrm{G}^+$.

Lemma 2.2.3. *Suppose n_o can be chosen to be an element of* G, *and $\lambda \in \mathrm{Irr}(\mathrm{T})$ is the restriction to* T *of some irreducible character of* T'. *If λ is in general position, then λ is ϑ-stable and $\mathsf{R}_{\mathsf{T}}^{\mathsf{G}}\lambda$ is θ-stable.*

Proof. If n_o is k-rational then, as in Proposition 2.1.1(2), T' is the union of the cosets $\mathrm{T}\vartheta^i$, $0 \le i \le d_\theta - 1$. Therefore, for any element $t \in \mathrm{T}$, t and t^ϑ are T'-conjugate, and so λ must be ϑ-stable. Thus, ${}^{w\theta}\lambda = \lambda$, for w the image of n_o in W. Since λ is in general position, w is the unique element of W with this property. Therefore, we have $w_\lambda = w$, and $\mathsf{R}_{\mathsf{T}}^{\mathsf{G}}\lambda$ is θ-stable. □

2.2.2 Extension to G^+**.** Let $\lambda' \in \operatorname{Irr}(\mathsf{T}')$. Using [14], we obtain a (virtual) character $\mathsf{R}_{\mathsf{T}'}^{\mathsf{G}^+}\lambda'$ of G^+ (see [14, Definition 2.2 (i)]), which satisfies a reduction formula similar to that for Deligne-Lusztig characters in the connected case.

Theorem 2.2.4 (Digne-Michel)**.** *Let $g \in \mathsf{G}^+$ with Jordan decomposition $g = su$. If $u \in (\mathsf{G}^+)^0_s$, then*

$$(\mathsf{R}_{\mathsf{T}'}^{\mathsf{G}^+}\lambda')(g) = \left|\mathsf{T}'\right|^{-1}\left|(\mathsf{G}^+)^0_s\right|^{-1}\sum_{\substack{x\in\mathsf{G}^+\\ s\in{}^x\mathsf{T}'}}\left|({}^x\mathsf{T}')^0_s\right|\mathsf{Q}_{({}^x\mathsf{T}')^0_s}^{(\mathsf{G}^+)^0_s}(u)\,{}^x\lambda'(s). \tag{2.2.1}$$

Otherwise, $(\mathsf{R}_{\mathsf{T}'}^{\mathsf{G}^+}\lambda')(g) = 0$.

Remark. There is a minor typographical error in the formula given in [14, Proposition 2.6 (i)]. The correct formula should appear similarly as in the connected case (see [15, Proposition 12.2 (i)]).

Proof. This is [14, Proposition 2.6 (i)], combined with the following observations. For any $x \in \mathsf{G}^+$ such that $s \in {}^x\mathsf{T}'$, let $\mathbf{Y}_{s,x}$ be the Deligne-Lusztig variety corresponding to $\mathsf{R}_{({}^x\mathsf{T}')^0_s}^{(\mathsf{G}^+)^0_s}$. Then, as in *loc. cit.*, for any unipotent elements $v \in (\mathsf{G}^+)_s$ and $w \in ({}^x\mathsf{T}')_s$, define

$$\mathsf{Q}_{({}^x\mathsf{T}')^0_s}^{(\mathsf{G}^+)^0_s}(v,w) = \begin{cases}\operatorname{tr}\big((v,w)\,\big|H_c^*(\mathbf{Y}_{s,x})\big), & vw \in (\mathsf{G}^+)^0_s,\\ 0, & \text{otherwise.}\end{cases}$$

Since $\mathbf{G}^+/\mathbf{G} \simeq \langle\theta\rangle$ consists of semisimple elements, the unipotent elements of G^+ all lie in G ([14, Remark 2.7]). Hence, for any $x \in \mathsf{G}^+$, the only unipotent element in $({}^x\mathsf{T}')_s$ is the identity, and so the inner sum in the formula given in [14, Proposition 2.6 (i)] is trivial. Now, if $u \notin (\mathsf{G}^+)^0_s$, then for each $x \in \mathsf{G}^+$ such that $s \in {}^x\mathsf{T}'$, we have $\mathsf{Q}_{({}^x\mathsf{T}')^0_s}^{(\mathsf{G}^+)^0_s}(u,1) = 0$, and so $(\mathsf{R}_{\mathsf{T}'}^{\mathsf{G}^+}\lambda')(g) = 0$. On the other hand, if $u \in (\mathsf{G}^+)^0_s$, then

$$\mathsf{Q}_{({}^x\mathsf{T}')^0_s}^{(\mathsf{G}^+)^0_s}(u,1) = \left(\mathsf{R}_{({}^x\mathsf{T}')^0_s}^{(\mathsf{G}^+)^0_s}1\right)(u) = \mathsf{Q}_{({}^x\mathsf{T}')^0_s}^{(\mathsf{G}^+)^0_s}(u). \qquad \square$$

We now consider the above formula on specific elements of G^+ which will be of interest later. Recall that n_o is any representative of the unique element $w_o \in W_{\mathbf{G}}(\mathbf{T})$ such that ${}^{w_o\theta}\mathbf{B} = \mathbf{B}$, for $\mathbf{B}$ our fixed Borel subgroup of $\mathbf{G}$. In all of the cases we will consider, we will be able to choose $n_o = 1$. Therefore, to simplify questions of rationality and Jordan decomposition in the current general discussion, it is reasonable to assume for the remainder of Chapter 2 that we at least may choose n_o to lie in G. Under this assumption, $\mathbf{T}'$ is defined over k, with $\mathsf{T}' = \mathbf{T}'(k) = \bigsqcup_i \mathsf{T}\vartheta^i$. We are particularly interested in the values of $\mathsf{R}_{\mathsf{T}'}^{\mathsf{G}^+}\lambda'$ on elements of G^1 with semisimple part $n\vartheta$ for some $n \in N_{\mathsf{G}}(\mathsf{T})$. For any $x \in \mathsf{G}^+$, let $(\mathsf{G}^0_x)_{\mathrm{unip}} = \mathsf{G}^0_x \cap \mathsf{G}_{\mathrm{unip}}$, where $\mathsf{G}_{\mathrm{unip}}$ is the set of unipotent elements in G.

Corollary 2.2.5. *Let $n \in N_{\mathsf{G}}(\mathsf{T})$ such that $n\theta \in \mathsf{G}^+$ is semisimple. For $u \in (\mathsf{G}^0_{n\theta})_{\mathrm{unip}}$,*

$$(\mathsf{R}_{\mathsf{T}'}^{\mathsf{G}^+}\lambda')(un\theta) = \left|\mathsf{T}/\mathsf{T}^0_\vartheta\right|^{-1}\sum_{x\in\widetilde{\mathfrak{X}}_{\mathsf{T}}(n_o,nn_o^{-1})}{}^x\lambda'(n\theta)\sum_{y\in\mathsf{G}^0_{n\theta}\backslash\mathsf{G}_{n\theta}}\mathsf{Q}_{{}^x\mathsf{T}^0_\vartheta}^{\mathsf{G}^0_{n\vartheta}}\left(u^y\right). \tag{2.2.2}$$

Remark. See §1.3.1 for the definition of $\widetilde{\mathfrak{X}}_{\mathsf{T}}(n_o, nn_o^{-1})$.

Proof. Setting $m = nn_o^{-1}$, we prove the equivalent formula

$$(\mathsf{R}_{\mathsf{T}'}^{\mathsf{G}^+}\lambda')(um\vartheta) = \left|\mathsf{T}/\mathsf{T}^0_\vartheta\right|^{-1}\sum_{\substack{x\in\mathsf{G}_{m\vartheta}\backslash\mathsf{G}\\ x^{-1}m\,{}^\vartheta x\in\mathsf{T}}}{}^x\lambda'(m\vartheta)\sum_{y\in\mathsf{G}^0_{m\vartheta}\backslash\mathsf{G}_{m\vartheta}}\mathsf{Q}_{{}^x\mathsf{T}^0_\vartheta}^{\mathsf{G}^0_{m\vartheta}}\left(u^y\right), \tag{2.2.3}$$

which follows from (2.2.1) and the following. First, clearly $(\mathrm{G}^+)^0_{m\vartheta} = \mathrm{G}^0_{m\vartheta}$, and since conjugation in G^+ preserves connected components, also $({}^x\mathrm{T}')^0_{m\vartheta} = ({}^x\mathrm{T})^0_{m\vartheta}$ for any $x \in \mathrm{G}^+$. Choose $\{\vartheta^j\}_{j=0}^{d_\theta-1}$ as a set of representatives for $\mathrm{G}\backslash\mathrm{G}^+$, and convert the sum in (2.2.1) to a double sum as

$$(\mathrm{R}_{\mathrm{T}'}^{\mathrm{G}^+}\lambda')(um\vartheta) = \left|\mathrm{T}'\right|^{-1}\left|\mathrm{G}^0_{m\vartheta}\right|^{-1} \sum_{\substack{x\in\mathrm{G}\\ m\vartheta\in{}^x(\mathrm{T}\vartheta)}} \left|({}^x\mathrm{T})^0_{m\vartheta}\right| \mathrm{Q}^{\mathrm{G}^0_{m\vartheta}}_{({}^x\mathrm{T})^0_{m\vartheta}}(u) \sum_{j=0}^{d_\theta-1} {}^{x\vartheta^j}\lambda'(m\vartheta),$$

using the fact that ϑ normalizes T. For $x \in \mathrm{G}$, we have $m\vartheta \in {}^x(\mathrm{T}\vartheta)$ if and only if $x^{-1}m\,{}^\vartheta x \in \mathrm{T}$. Suppose x satisfies this condition. Then ${}^x\mathbf{B} \supset {}^x\mathrm{T}$ is an $m\vartheta$-stable pair in $\mathbf{G}$. By Theorem 1.3.1, $({}^x\mathrm{T})^0_{m\vartheta}$ is indeed a maximal torus of $\mathrm{G}^0_{m\vartheta}$, and the Green function $\mathrm{Q}^{\mathrm{G}^0_{m\vartheta}}_{({}^x\mathrm{T})^0_{m\vartheta}}$ is defined. It is easy to check that $({}^x\mathrm{T})_{m\vartheta} = {}^x\mathrm{T}_\vartheta$, so that $({}^x\mathrm{T})^0_{m\vartheta} = {}^x\mathrm{T}^0_\vartheta$. The cardinality of this last set is equal to $\left|\mathrm{T}^0_\vartheta\right|$. Moreover, λ' is a class function on T', so ${}^{x\vartheta^j}\lambda'(m\vartheta) = {}^x\lambda'(m\vartheta)$ for each j. Noting that $\left[\mathrm{T}' \colon \mathrm{T}\right] = d_\theta$, we now have

$$(\mathrm{R}_{\mathrm{T}'}^{\mathrm{G}^+}\lambda')(um\vartheta) = \left|\mathrm{T}/\mathrm{T}^0_\vartheta\right|^{-1}\left|\mathrm{G}^0_{m\vartheta}\right|^{-1} \sum_{\substack{x\in\mathrm{G}\\ x^{-1}m\,{}^\vartheta x\in\mathrm{T}}} \mathrm{Q}^{\mathrm{G}^0_{m\vartheta}}_{{}^x\mathrm{T}^0_\vartheta}(u)\, {}^x\lambda'(m\vartheta),$$

Finally, the map $x \mapsto (m\vartheta)^x$ is constant on right-cosets of $\mathrm{G}_{m\vartheta}$, so

$$\begin{aligned}(\mathrm{R}_{\mathrm{T}'}^{\mathrm{G}^+}\lambda')(um\vartheta) &= \left|\mathrm{T}/\mathrm{T}^0_\vartheta\right|^{-1}\left|\mathrm{G}^0_{m\vartheta}\right|^{-1} \sum_{\substack{x\in\mathrm{G}_{m\vartheta}\backslash\mathrm{G}\\ x^{-1}m\,{}^\vartheta x\in\mathrm{T}}} {}^x\lambda'(m\vartheta) \sum_{y\in\mathrm{G}_{m\vartheta}} \mathrm{Q}^{\mathrm{G}^0_{m\vartheta}}_{{}^{yx}\mathrm{T}^0_\vartheta}(u)\\ &= \left|\mathrm{T}/\mathrm{T}^0_\vartheta\right|^{-1}\left|\mathrm{G}^0_{m\vartheta}\right|^{-1} \sum_{\substack{x\in\mathrm{G}_{m\vartheta}\backslash\mathrm{G}\\ x^{-1}m\,{}^\vartheta x\in\mathrm{T}}} {}^x\lambda'(m\vartheta) \sum_{y\in\mathrm{G}^0_{m\vartheta}\backslash\mathrm{G}_{m\vartheta}} \sum_{y_0\in\mathrm{G}^0_{m\vartheta}} \mathrm{Q}^{\mathrm{G}^0_{m\vartheta}}_{{}^{y_0yx}\mathrm{T}^0_\vartheta}(u)\\ &= \left|\mathrm{T}/\mathrm{T}^0_\vartheta\right|^{-1} \sum_{\substack{x\in\mathrm{G}_{m\vartheta}\backslash\mathrm{G}\\ x^{-1}m\,{}^\vartheta x\in\mathrm{T}}} {}^x\lambda'(m\vartheta) \sum_{y\in\mathrm{G}^0_{m\vartheta}\backslash\mathrm{G}_{m\vartheta}} \mathrm{Q}^{\mathrm{G}^0_{m\vartheta}}_{{}^{yx}\mathrm{T}^0_\vartheta}(u),\end{aligned}$$

where the last manipulation follows from the fact that Green functions associated to conjugate tori are equal. Formula (2.2.3) is now obtained by noting that for any $y \in \mathrm{G}_{m\vartheta}$, the associated inner automorphism of $\mathbf{G}_{m\vartheta}$ normalizes $\mathbf{G}^0_{m\vartheta}$, so that

$$\mathrm{Q}^{\mathrm{G}^0_{m\vartheta}}_{{}^{yx}\mathrm{T}^0_\vartheta}(u) = \mathrm{Q}^{{}^y\mathrm{G}^0_{m\vartheta}}_{{}^{yx}\mathrm{T}^0_\vartheta}\left({}^y(u^y)\right) = \mathrm{Q}^{\mathrm{G}^0_{m\vartheta}}_{{}^{x}\mathrm{T}^0_\vartheta}\left(u^y\right). \qquad \square$$

Corollary 2.2.6. *Let n be as in Corollary 2.2.5. If there does not exist an $\left(\mathrm{Int}_{\mathrm{L}}(n)\circ\theta\right)$-stable G-conjugate of the pair $\mathbf{B} \supset \mathrm{T}$, then $(\mathrm{R}_{\mathrm{T}'}^{\mathrm{G}^+}\lambda')(un\theta) = 0$ for all unipotent elements $u \in \mathrm{G}_{n\theta}$.*

Proof. If $u \in \mathrm{G}_{n\theta} \smallsetminus \mathrm{G}^0_{n\theta}$, then from Theorem 2.2.4 we have $(\mathrm{R}_{\mathrm{T}'}^{\mathrm{G}^+}\lambda')(un\theta) = 0$, regardless of the existence or non-existence of such a G-conjugate of $\mathbf{B} \supset \mathrm{T}$. On the other hand, if $u \in \mathrm{G}^0_{n\theta}$, then by Lemma 1.3.9, the outer sum of (2.2.2) is void under the given hypothesis. $\square$

Remark. If one is interested in the values of $\mathrm{R}_{\mathrm{T}'}^{\mathrm{G}^+}\lambda'$ on elements $u\theta$, for u a unipotent element of G_θ, then the only interesting situation is when some G-conjugate of $\mathbf{B} \supset \mathrm{T}$ is θ-stable. If ${}^g\mathbf{B} \supset {}^g\mathrm{T}$ is such a pair, for some $g \in \mathrm{G}$, then $(1-\theta)(g^{-1}) \equiv n_0 \mod \mathrm{T}$. Using this, it is not hard to show that $(\mathrm{R}_{{}^g(\mathrm{T}')}^{\mathrm{G}^+}{}^g(\lambda'))(u\theta) = (\mathrm{R}_{\mathrm{T}'}^{\mathrm{G}^+}\lambda')(u\theta)$. Therefore, one may as well assume that $\mathbf{B} \supset \mathrm{T}$ is θ-stable, and take $n_o = 1$.

2.2.3 A specific situation. As before, fix $\lambda' \in \mathrm{Irr}(\mathsf{T}')$. We are interested, in particular, in the situation that $\mathrm{Res}^{\mathsf{G}^+}_{\mathsf{G}}\, \varepsilon \mathsf{R}^{\mathsf{G}^+}_{\mathsf{T}'}\lambda'$ is the character of an irreducible, θ-stable Deligne-Lusztig representation σ of G attached to T, for some sign $\varepsilon = \pm 1$, and $\varepsilon \mathsf{R}^{\mathsf{G}^+}_{\mathsf{T}'}\lambda'$ is the character of σ^+ for some choice of normalized intertwining operator $A_\sigma \in \mathrm{Hom}(\sigma, \sigma\circ\theta)$. This will restrict the choice of λ'.

Lemma 2.2.7. $\mathrm{Res}^{\mathsf{G}^+}_{\mathsf{G}}\,\mathsf{R}^{\mathsf{G}^+}_{\mathsf{T}'} = \mathsf{R}^{\mathsf{G}}_{\mathsf{T}}\,\mathrm{Res}^{\mathsf{T}'}_{\mathsf{T}}$.

Proof. This is [14, Corollary 2.4 (i)], where in the present case $\mathsf{T}'\mathsf{G} = \mathsf{G}^+$. □

Corollary 2.2.8. *Let $\chi' = \varepsilon_{\mathsf{G}}\,\varepsilon_{\mathsf{T}}\,\mathsf{R}^{\mathsf{G}^+}_{\mathsf{T}'}\lambda'$ and $\chi = \mathrm{Res}^{\mathsf{G}^+}_{\mathsf{G}}\chi'$. Then both χ and χ' are irreducible if and only if the character $\lambda = \mathrm{Res}^{\mathsf{T}'}_{\mathsf{T}}\lambda'$ of T is irreducible and in general position.*

Remark. See [14, Corollary 2.5] for the sign $\varepsilon_{\mathsf{G}}\,\varepsilon_{\mathsf{T}}$.

Proof. The reverse implication is immediate from Lemma 2.2.7, so suppose χ and χ' are both irreducible. Express λ as a (non-negative) integral combination $\sum_{\psi\in\mathrm{Irr}(\mathsf{T})} c_\psi \psi$. Using the lemma,

$$\begin{aligned}1 &= \langle \mathrm{Res}^{\mathsf{G}^+}_{\mathsf{G}}\chi, \mathrm{Res}^{\mathsf{G}^+}_{\mathsf{G}}\chi\rangle_{\mathsf{G}}\\ &= \langle \mathsf{R}^{\mathsf{G}}_{\mathsf{T}}\lambda, \mathsf{R}^{\mathsf{G}}_{\mathsf{T}}\lambda\rangle_{\mathsf{G}}\\ &= \sum_{\phi,\psi\in\mathrm{Irr}(\mathsf{T})} c_\phi c_\psi \langle \mathsf{R}^{\mathsf{G}}_{\mathsf{T}}\phi, \mathsf{R}^{\mathsf{G}}_{\mathsf{T}}\psi\rangle_{\mathsf{G}}.\end{aligned}$$

Since $\langle \mathsf{R}^{\mathsf{G}}_{\mathsf{T}}\phi, \mathsf{R}^{\mathsf{G}}_{\mathsf{T}}\psi\rangle_{\mathsf{G}} = \left|\{w \in W_{\mathsf{G}}(\mathsf{T}) \mid {}^w\phi = \psi\}\right|$ ([8, Theorem 7.3.4]), the above sum must reduce to a single term $c_\psi^2\langle \mathsf{R}^{\mathsf{G}}_{\mathsf{T}}\psi, \mathsf{R}^{\mathsf{G}}_{\mathsf{T}}\psi\rangle_{\mathsf{G}}$, for some $\psi \in \mathrm{Irr}(\mathsf{T})$. Conclude that $\lambda = \psi$ is in general position. □

In light of the corollary, we wish to analyze (2.2.2) under the assumption that our fixed character $\lambda' \in \mathrm{Irr}(\mathsf{T}')$ is the extension of some $\lambda \in \mathrm{Irr}(\mathsf{T})$ in general position. In particular, λ' is then 1-dimensional, hence multiplicative. Also, combining this assumption with Lemma 2.2.3 implies that λ is ϑ-stable and $\mathsf{R}^{\mathsf{G}}_{\mathsf{T}}\lambda$ is θ-stable. So it is possible to simplify (2.2.2), in the case that $n = n_o$, under a mild condition on $(1-\vartheta)|\mathsf{T}$. By Proposition 2.1.1(1), the automorphism $t \mapsto {}^\vartheta t$ of T has finite order d_θ, so we may consider the ϑ-norm homomorphism $\mathrm{N}_\vartheta : \mathsf{T} \to \mathsf{T}$.

Theorem 2.2.9. *Suppose $\mathrm{Res}^{\mathsf{T}'}_{\mathsf{T}}\lambda'$ is irreducible and in general position, and let $u \in (\mathsf{G}^0_\vartheta)_{\mathrm{unip}}$. If $(1-\vartheta)\colon \mathsf{T} \to \ker \mathrm{N}_\vartheta$ is surjective, then the natural map*

$$\mathsf{T}_\vartheta\backslash\mathsf{T} \to \widetilde{\mathfrak{X}}_{\mathsf{T}}(n_o, 1) = \mathsf{G}_\vartheta\backslash\mathfrak{X}_{\mathsf{T}}(n_o, 1)$$

is a bijection, and

$$(\mathsf{R}^{\mathsf{G}^+}_{\mathsf{T}'}\lambda')(u\vartheta) = \lambda'(\vartheta)\left|\mathsf{T}_\vartheta/\mathsf{T}^0_\vartheta\right|^{-1} \sum_{y\in\mathsf{G}^0_\vartheta\backslash\mathsf{G}_\vartheta} \mathsf{Q}^{\mathsf{G}^0_\vartheta}_{{}^y\mathsf{T}^0_\vartheta}(u). \tag{2.2.4}$$

If, in addition, $\mathbf{G}_\vartheta$ is connected, then

$$(\mathsf{R}^{\mathsf{G}^+}_{\mathsf{T}'}\lambda')(u\vartheta) = \lambda'(\vartheta)\,\mathsf{Q}^{\mathsf{G}_\vartheta}_{\mathsf{T}_\vartheta}(u). \tag{2.2.5}$$

Proof. Since T is ϑ-stable, it is contained in $\mathfrak{X}_{\mathsf{T}}(n_o,1)$. Given $t \in \mathsf{T}$, the cosets $\mathsf{T}_\vartheta t \subseteq \mathsf{T}$ and $\mathsf{G}_\vartheta t \subseteq \mathsf{G}$ are both uniquely determined by the element $t^{-1}({}^\vartheta t)$. Therefore, the map $\mathsf{T}_\vartheta\backslash\mathsf{T} \to \widetilde{\mathfrak{X}}_{\mathsf{T}}(n_o,1)$ induced by inclusion $\mathsf{T} \hookrightarrow \mathfrak{X}_{\mathsf{T}}(n_o,1)$ is both well-defined and injective. For $x \in \mathsf{G}$, the coset $\mathsf{G}_\vartheta x$ is uniquely determined by the element $x' = x^{-1}({}^\vartheta x)$. If $x \in \mathfrak{X}_{\mathsf{T}}(n_o,1)$, then $x' \in \mathsf{T}$, and in fact $x' \in \ker \mathrm{N}_\vartheta$. If we assume $(1-\vartheta)(\mathsf{T}) = \ker \mathrm{N}_\vartheta$, then there exists $t \in \mathsf{T}$ such that $x' = t^{-1}({}^\vartheta t)$, so that $\mathsf{G}_\vartheta x = \mathsf{G}_\vartheta t$. Thus $\mathsf{G}_\vartheta x$ is the image of $\mathsf{T}_\vartheta t$ under the given map, showing surjectivity.

Using this bijection in formula (2.2.2), in the case $n = n_o$, gives

$$(\mathsf{R}_{\mathsf{T}'}^{\mathsf{G}^+}\lambda')(u\vartheta) = \lambda'(\vartheta)\left|\mathsf{T}/\mathsf{T}_\vartheta^0\right|^{-1} \sum_{t\in\mathsf{T}_\vartheta\backslash\mathsf{T}} \lambda(t^{-1}\,{}^\vartheta t) \sum_{y\in\mathsf{G}_\vartheta^0\backslash\mathsf{G}_\vartheta} \mathsf{Q}_{{}^{yt}\mathsf{T}_\vartheta^0}^{\mathsf{G}_\vartheta^0}(u).$$

But obviously ${}^{yt}\mathsf{T}_\vartheta^0 = {}^{y}\mathsf{T}_\vartheta^0$ for any $t \in \mathsf{T}$. Also, since λ is ϑ-stable, it is trivial on $(1-\vartheta)(\mathsf{T})$. Formula (2.2.4) now follows, and (2.2.5) is immediate if we also assume that $\mathbf{G}_\vartheta$ is connected. □

3. A CHARACTER FORMULA OVER THE p-ADIC FIELD

In an appendix of [7], Bushnell and Henniart provide a detailed survey of the development of an explicit character formula for supercuspidal representations of $\mathbf{GL}_n(F) \rtimes \langle\theta\rangle$, in the case where θ is a generator of the Galois group of a finite, cyclic field extension F/F_0. In this section, we follow this development very closely to obtain a character formula in the case of a more general automorphism θ. The main difference here is the need to work around an issue with convergence to be able to handle certain classical cases (see §3.4), as the representations of $\mathbf{GL}_n(F) \rtimes \langle\theta\rangle$ we consider will not always be supercuspidal (see Proposition 1.5.1(2)).

3.1 Preliminary hypotheses

Let F be a p-adic field, and let $G = \mathbf{GL}_n(F)$. Let $F_0 \supset \mathbb{Q}_p$ be a (not necessarily proper) subfield of F such that F/F_0 is a finite, abelian extension. Fix $\theta \in \mathrm{Aut}(G)$ of finite order d_θ. It is necessary to make several assumptions on θ. We first restrict to the form of θ to which all of the cases we consider conform.

Hypothesis H1. *The automorphism $\theta \in \mathrm{Aut}(G)$ is of the form of* (1.4.1)*, with χ trivial. We allow the possibilities that $J = 1$ and/or $F_0 = F$.*

Remark. There should be little difficulty in extending the results of this thesis to the cases where θ is of the form of (1.4.1), but with χ non-trivial and of the form of (1.4.2). Two possible issues might be determining when such an automorphism has finite order, and finding a suitable replacement for $\mathfrak{b}$, the $\mathrm{Ad}\,G^+$-invariant, non-degenerate, symmetric, bilinear form on $\mathfrak{g} = \mathrm{Lie}(G)$ to be introduced in §3.2.

Since χ is trivial, we may as well take F_0 to be the fixed field of τ, so that $\mathrm{Gal}(F/F_0)$ is cyclic and generated by τ. The two possibilities of θ_0 are now distinguished by $\mathrm{val}\,\theta$. For all $g \in G$, we have

$$\det\theta(g) = \tau(\det g)^{\mathrm{val}\,\theta},$$
$$\det\big((1-\theta)(g)\big) = (\det g)\,\tau(\det g)^{-\mathrm{val}\,\theta}.$$

Write $Z = Z(G)$. As usual, we identify Z with $F^\times$. Based on the assumption that θ has finite order, we have the following. First, $[F:F_0]$ must divide d_θ. For $\mathrm{val}\,\theta = 1$, we must have $\mathrm{N}_\tau(J^{-1})^{d_\theta/[F:F_0]} \in Z$. For $\mathrm{val}\,\theta = -1$, d_θ must be even, and $\mathrm{N}_{\tau\circ\theta_0}(J^{-1})^{d_\theta/\mathrm{lcm}(2,[F:F_0])} \in Z$.

Fix a k_{F_0}-basis $\bar{\mathcal{B}}_{F/F_0}$ for k_F, and let $\widehat{\mathcal{B}}_{F/F_0}$ be any lift of $\bar{\mathcal{B}}_{F/F_0}$ into $\mathscr{O}_F$. Then $\mathcal{B}_{F/F_0} = \bigcup_{i=0}^{e-1} \varpi_F^i \widehat{\mathcal{B}}_{F/F_0}$ is an $\mathscr{O}_{F_0}$-basis for $\mathscr{O}_F$, hence an F_0-basis for F, where e is the ramification index of F/F_0. Also, as in §1.4.1, lift τ to an element of $\mathrm{Gal}(\bar{F}/F_0)$, and take $\{\mathrm{id}, \tau, \dots, \tau^{[F:F_0]-1}\}$ as a set of representatives of $\Sigma = \mathrm{Gal}(\bar{F}/F_0)/\mathrm{Gal}(\bar{F}/F)$. Let $\mathbf{G} = \mathrm{R}_{F/F_0}\,\mathbf{GL}_n$, constructed using Σ and the single basis $\mathcal{B}_{F/F_0}$ (see §1.2). Then $\mathbf{G}$ is a connected, reductive algebraic group defined over F_0 such that $\mathbf{G}(F_0) \simeq G$. We identify G with $\mathbf{G}(F_0)$ via the isomorphism afforded by $\mathcal{B}_{F/F_0}$. By Lemma 1.4.2(2), θ is the restriction to G of some F_0-automorphism of $\mathbf{G}$, which we will also denote by θ. From the proof of *loc. cit.*, we may take this $\theta \in \mathrm{Aut}_{F_0}(\mathbf{G})$ to have finite order d_θ. Recall that θ is then semisimple. Let $\mathbf{g} = \mathrm{R}_{F/F_0}\,\mathbf{M}_n$ be the Lie algebra of $\mathbf{G}$, and identify $\mathbf{g}(F_0)$ with $\mathfrak{g} = \mathbf{M}_n(F)$. Let $\mathrm{d}\theta\colon \mathbf{g} \to \mathbf{g}$ be the differential of θ. The action of $d\theta$ on $\mathfrak{g}$ is easy to compute in either case. For $X \in \mathfrak{g}$, we have

$$\mathrm{d}\theta(X) = \begin{cases} J^{-1}\,\tau(X)\,J, & \operatorname{val}\theta = 1, \\ -J^{-1}\,\tau({}^{\mathrm{t}}X)\,J, & \operatorname{val}\theta = -1. \end{cases}$$

Note that $\mathrm{d}\theta$ is an F_0-linear operator on $\mathfrak{g}$, but not an F-linear operator if τ is non-trivial.

We continue with a restriction on the order d_θ of θ.

Hypothesis H2. *The residual characteristic p does not divide d_θ.*

We make the following hypothesis to significantly reduce notational complexity, and because it is satisfied in all the cases we will consider.

Hypothesis H3. *The subgroup $\mathbf{G}_\theta$ of θ-fixed points of $\mathbf{G}$ is connected.*

Remark. One obvious example where $\mathbf{G}_\theta$ fails to be connected is when $\mathbf{G}_\theta$ is an orthogonal group.

Let $K_0 = \mathrm{GL}_n(\mathscr{O}_F)$. In the cases we will consider, we will choose θ so that K_0 is θ-stable. For now, it is enough to assume the following.

Hypothesis H4. *There exists a G-conjugate L_0 of K_0 such that L_0^+ is compact.*

Remark. This holds if there exists a θ-stable G-conjugate of K_0.

We fix now an irreducible, admissible, θ-stable representation (π, V_π) of G. By Schur's Lemma, there exists an intertwining operator $A_\pi \in \mathrm{Hom}_G(\pi, \pi\circ\theta)$ with $A_\pi^{d_\theta} = 1$. As in §1.5.1, use A_π to construct the extended representation (π^+, V_π) of G^+. The remainder of Chapter 3 will be concerned with the analysis of Θ_{π^+}.

3.2 Harish-Chandra's submersion principle

In this section, we verify the analogues of two results of [22] in the present setting. Let b be the $\mathrm{Ad}\,G$-invariant, non-degenerate, symmetric, F_0-bilinear form on $\mathfrak{g}$ given by

$$\mathrm{b}(X,Y) = \mathrm{Tr}_{F/F_0}\,\mathrm{tr}(XY), \qquad \big(X, Y \in \mathfrak{g}\big). \tag{3.2.1}$$

It is easy to check that b is $\mathrm{d}\theta$-invariant in both cases of $\operatorname{val}\theta = \pm 1$, under Hypothesis H1, and so is $\mathrm{Ad}\,G^+$-invariant. Let P be a parabolic subgroup of G.

Theorem 3.2.1. *For g in G^+_{qr}, the map*

$$\phi_g : G^+ \times P \to G^+, \qquad (x,p) \mapsto ({}^x g)p$$

is submersive.

Proof. Here we follow the proof of [7, Theorem (A.4)]. Observe that

$$\phi_g(xy, pq) = \phi_{{}^y g}(x,p)q, \qquad \big(x, y \in G^+,\ p, q \in P\big),$$

so it suffices to verify that ϕ_g is submersive at $(x,p) = (1,1)$.

The map ϕ_g is the composition of the following three maps,

$$G^+ \times P \to G^+ \times P, \quad (x,p) \mapsto (xgx^{-1}g^{-1}, p), \qquad G^+ \times P \to G^+ \times P, \quad (x,p) \mapsto (xg, p), \qquad G^+ \times P \to G^+, \quad (x,p) \mapsto xp,$$

with respective differentials

$$\mathcal{T}_1 G^+ \oplus \mathcal{T}_1 P \to \mathcal{T}_1 G^+ \oplus \mathcal{T}_1 P, \quad (X,Y) \mapsto ((1-\mathrm{Ad}\,g)X, Y), \qquad \mathcal{T}_1 G^+ \oplus \mathcal{T}_1 P \to \mathcal{T}_g G^+ \oplus \mathcal{T}_1 P, \quad (X,Y) \mapsto (Xg, Y), \qquad \mathcal{T}_g G^+ \oplus \mathcal{T}_1 P \to \mathcal{T}_g G^+, \quad (X,Y) \mapsto X + gY.$$

Thus,

$$\left(\mathrm{d}_{(1,1)}\phi_g\right)(X,Y) = Xg - gX + gY = g(\mathrm{Ad}\,g^{-1} \cdot X - X + Y).$$

So $\mathrm{d}_{(1,1)}\phi_g$ is surjective if and only if $(\mathrm{Ad}\,g^{-1} - 1)\mathfrak{g} + \mathfrak{p} = \mathfrak{g}$, where $\mathfrak{p} = \mathrm{Lie}(P)$. The orthogonal complement of $\mathfrak{p}$ with respect to $\mathfrak{b}$ is $\mathfrak{u} = \mathrm{Lie}(U)$, where U is the unipotent radical of P. On the other hand, a simple calculation using the $\mathrm{Ad}\,G^+$-invariance of $\mathfrak{b}$ gives that the orthogonal complement of $(\mathrm{Ad}\,g^{-1} - 1)\mathfrak{g}$ is $\ker(\mathrm{Ad}\,g - 1)$. But since $g \in G^+_{\mathrm{qr}}$, we have $\ker(\mathrm{Ad}\,g - 1) \cap \mathfrak{u} = \{0\}$. Taking orthogonal complements of both sides now yields the result. □

Choose a Haar measure dx on G^+ and a left Haar measure $d_l x$ on P.

Theorem 3.2.2 (Harish-Chandra [21, Theorem 11])**.** *Fix g in G^+_{qr}. There exists an embedding*

$$C_c^\infty(G^+ \times P) \hookrightarrow C_c^\infty(G^+), \qquad \alpha \mapsto f_{\alpha,g},$$

such that

$$\int_{G^+ \times P} \alpha(x,p)\,\Phi(\phi_g(x,p))\,dx\,d_l p = \int_{G^+} f_{\alpha,g}(x)\,\Phi(x)\,dx, \qquad (\Phi \in C_c^\infty(G^+)).$$

Lemma 3.2.3. *For fixed $\alpha \in C_c^\infty(G^+ \times P)$, the mapping*

$$G^+_{\mathrm{qr}} \to C_c^\infty(G^+), \qquad g \mapsto f_{\alpha,g},$$

is locally constant.

Proof. We use the method of proof of [22, Lemma 1]. It follows from Theorem 3.2.1 that the map

$$G^+_{\mathrm{qr}} \times G^+ \times P \to G^+_{\mathrm{qr}} \times G^+, \qquad (g,x,p) \mapsto (g, \phi_g(x,p)),$$

is submersive. Thus we have a mapping

$$C_c^\infty\left(G^+_{\mathrm{qr}} \times G^+ \times P\right) \to C_c^\infty\left(G^+_{\mathrm{qr}} \times G^+\right), \qquad \beta \mapsto \psi_\beta,$$

such that

$$\int_{G^+_{\mathrm{qr}}} \int_{G^+} \int_P \beta(g,x,p)\,\Phi(g,\phi_g(x,p))\,d_l p\,dx\,dg = \int_{G^+_{\mathrm{qr}}} \int_{G^+} \psi_\beta(g,x)\,\Phi(g,x)\,dx\,dg,$$

for any $\Phi \in C_c^\infty\left(G^+_{\mathrm{qr}} \times G^+\right)$. Now, $C_c^\infty\left(G^+_{\mathrm{qr}} \times G^+\right) = C_c^\infty\left(G^+_{\mathrm{qr}}\right) \otimes C_c^\infty(G^+)$, so we may write $\Phi = \lambda \otimes \rho$ for some $\lambda \in C_c^\infty\left(G^+_{\mathrm{qr}}\right)$, $\rho \in C_c^\infty(G^+)$. Thus,

$$\int_{G^+_{\mathrm{qr}}} \lambda(g) \int_{G^+} \int_P \beta(g,x,p)\,\rho(\phi_g(x,p))\,d_l p\,dx\,dg = \int_{G^+_{\mathrm{qr}}} \lambda(g) \int_{G^+} \psi_\beta(g,x)\,\rho(x)\,dx\,dg.$$

Since this is true for any λ, conclude that

$$\int_{G^+}\int_P \beta(g,x,p)\,\rho(\phi_g(x,p))\,d_l p\,dx = \int_{G^+}\psi_\beta(g,x)\,\rho(x)\,dx, \qquad \left(g\in G^+_{\mathrm{qr}},\ \rho\in C_c^\infty\left(G^+\right)\right).$$

Fix $g_0\in G^+_{\mathrm{qr}}$. We also have $C_c^\infty\left(G^+_{\mathrm{qr}}\times G^+\times P\right) = C_c^\infty\left(G^+_{\mathrm{qr}}\right)\otimes C_c^\infty\left(G^+\times P\right)$, so take β to be $\mu\otimes\alpha$ for some $\mu\in C_c^\infty\left(G^+_{\mathrm{qr}}\right)$ with $\mu(g_0)=1$. Since μ is locally constant, there exists a neighbourhood ω_o of g_0 in G^+_{qr} such that $\mu\equiv 1$ on ω_o. Hence $\beta(g,x,p)=\alpha(x,p)$ for $g\in\omega_o$, $x\in G^+$, and $p\in P$, and therefore,

$$\begin{aligned}\int_{G^+}\psi_\beta(g,x)\,\rho(x)\,dx &= \int_{G^+}\int_P \beta(g,x,p)\,\rho(\phi_g(x,p))\,d_l p\,dx\\ &= \int_{G^+}\int_P \alpha(x,p)\,\rho(\phi_g(x,p))\,d_l p\,dx = \int_{G^+} f_{\alpha,g}(x)\,\rho(x)\,dx.\end{aligned}$$

Since this is true for all $\rho\in C_c^\infty\left(G^+\right)$ and $g\in\omega_o$, we conclude that $f_{\alpha,g}(x)=\psi_\beta(g,x)$ for $g\in\omega_o$, $x\in G^+$. The result now follows. □

3.3 A representative for Θ_{π^+}

Following [7], for each $g\in G^+_{\mathrm{qr}}$ we construct an operator on V_π whose trace is equal to $\Theta_{\pi^+}(g)$.

Let $\mathrm{End}_0(V_\pi)\subset\mathrm{End}_{\mathbb{C}}(V_\pi)$ be the space of linear maps $T\colon V_\pi\to V_\pi$ such that the maps $G^+\to\mathrm{End}_{\mathbb{C}}(V_\pi)$ given by $g\mapsto\pi^+(g)T$, $g\mapsto T\pi^+(g)$ are both locally constant. For each integer $m\geq 1$, let $K_m = 1+\mathrm{M}_n(\mathscr{P}_F^m)$. For any G-conjugate L_0 of K_0 as in H4 and open subgroup $K\subset L_0^+$, define

$$\Upsilon_K^{L_0}\colon G^+_{\mathrm{qr}}\to\mathrm{End}_{\mathbb{C}}(V_\pi), \qquad g\mapsto\int_K \pi^+({}^k g)\,dk,$$

where dk is normalized Haar measure on K. For the remainder of this section, fix a choice of L_0 and $K\subset L_0^+$ as above, and to simplify notation write $\Upsilon=\Upsilon_K^{L_0}$. Note that L_0^+ is open. We have a Cartan decomposition of both G and G^+ associated to L_0 as follows. Let D be the maximal torus of diagonal matrices in G, and set

$$A=\left\{\,a_e=\mathrm{diag}(\varpi_F^{e_1},\varpi_F^{e_2},\dots,\varpi_F^{e_n})\;\middle|\; e=(e_1,\dots,e_n)\in\mathbb{Z}^n,\quad e_1\leq e_2\leq\cdots\leq e_n\right\}\subset D.$$

Let g_0 be an element of G such that $L_0={}^{g_0}K_0$. Then, from the usual Cartan decomposition $G=K_0AK_0$, we immediately get $G=L_0A'L_0$ and $G^+=L_0A'L_0^+$, where $A'={}^{g_0}A$. Let $B\supset D$ be the standard Borel subgroup of G consisting of invertible, upper-triangular matrices, and take $P={}^{g_0}B$.

Theorem 3.3.1. *The map Υ is locally constant with image in* $\mathrm{End}_0(V_\pi)$.

Proof. For the most part, we follow the proof of [7, Theorem (A.8)] quite closely. The group K contains an open subgroup K', necessarily of finite index, which is normal in L_0^+. We obtain normalized Haar measure on K' by taking dk restricted to K and renormalizing. Then,

$$\Upsilon(g)=\left[K:K'\right]^{-1}\sum_{k\in K/K'}\pi^+(k)\,\Upsilon_{K'}^{L_0}(g)\,\pi^+(k)^{-1}, \qquad \left(g\in G^+_{\mathrm{qr}}\right).$$

Therefore, without loss of generality, we may assume that K itself is normal in L_0^+. Let $C_1,C_2,\dots,C_r$ be the distinct cosets of K in L_0^+, with $K=C_1$. For $1\leq j\leq r$, define $\Upsilon_j\colon G^+_{\mathrm{qr}}\to\mathbb{C}$ by

$$\Upsilon_j(g)=\int_{C_j}\pi^+({}^{k^+}g)\,dk^+, \qquad \left(g\in G^+_{\mathrm{qr}}\right),$$

where dk^+ is the extension of dk to L_0^+. Then for $g \in G_{\mathrm{qr}}^+$ and $l \in L_0^+$,

$$\pi^+(l)\,\Upsilon_j(g) = \int_{C_j} \pi^+(lk^+g(k^+)^{-1})\,dk^+ = \int_{lC_j} \pi^+(k'g(k')^{-1}l)\,dk',$$

using the change of variables $k' = lk^+$. Choosing $j = j(l)$ such that $lC_j = K$, we have

$$\pi^+(l)\,\Upsilon_j(g) = \Upsilon(g)\,\pi^+(l).$$

Since π^+ is irreducible, $V_\pi = G^+ \cdot v_0$ for some $v_0 \in V_\pi$. Fix an open, normal subgroup L of L_0^+ such that $v_0 \in V_\pi^L$. Then V_π is generated as a G^+-module by V_π^L, which is finite-dimensional by admissibility of π^+.

We may choose an integer $m \geq 1$ such that L contains the open subgroup ${}^{g_0}K_m$. Let $B_m = B \cap K_m$ and $P_m = {}^{g_0}B_m$. Then P_m is a compact, open subgroup of P such that $(P_m)^a \subset {}^{g_0}K_m \subset L$ for any $a \in A'$. For $1 \leq j \leq r$, let $\alpha_j \in C_c^\infty(G^+ \times P)$ be the characteristic function of $C_j \times P_m$. By Theorem 3.2.2, for any $g \in G_{\mathrm{qr}}^+$ there exists a unique function $f_{\alpha_j,g} \in C_c^\infty(G^+)$ such that

$$\int_{C_j \times P_m} \Phi(kgk^{-1}p)\,dk\,d_lp = \int_{G^+} f_{\alpha_j,g}(x)\,\Phi(x)\,dx, \qquad \big(\Phi \in C_c^\infty(G^+)\big),$$

for $1 \leq j \leq r$. Since π^+ is smooth, we may apply this formula to the coefficients of π^+ (see the Corollary of [21, Part V, §2]). Therefore, for $g \in G_{\mathrm{qr}}^+$, $v \in V_\pi$, and $\tilde{v} \in \widetilde{V}_\pi$, we have

$$\begin{aligned}\Big\langle\, \tilde{v}, \pi^+(f_{\alpha_j,g})v \,\Big\rangle &= \int_{G^+} f_{\alpha_j,g}(x) \big\langle\, \tilde{v}, \pi^+(x)v \,\big\rangle\,dx \\ &= \int_{C_j \times P_m} \big\langle\, \tilde{v}, \pi^+(kgk^{-1}p)v \,\big\rangle\,dk\,d_lp = \Big\langle\, \tilde{v}, \int_{P_m} \Upsilon_j(g)\,\pi(p)\,d_lp\,v \,\Big\rangle.\end{aligned}$$

Since this is true for all v and $\tilde{v}$, conclude that

$$\pi^+(f_{\alpha_j,g}) = \int_{P_m} \Upsilon_j(g)\,\pi(p)\,d_lp, \qquad \big(g \in G_{\mathrm{qr}}^+\big).$$

Fix $x_0 \in G_{\mathrm{qr}}^+$. By Lemma 3.2.3, there exist a neighbourhood ω of x_0 in G_{qr}^+ and an open, normal subgroup $\widetilde{K}$ of L_0^+, contained in K, such that $f_{\alpha_j,g}$ is $\widetilde{K}$-bi-invariant for each $1 \leq j \leq r$ and $g \in \omega$. Consider elements $l \in L_0$, $a \in A'$, and $v \in V_\pi^L$. From above, we have

$$\Upsilon(g)\,\pi(la)v = \pi(l)\,\Upsilon_j(g)\,\pi(a)v, \qquad \big(g \in \omega,\ j = j(l)\big).$$

We also have

$$\pi^+(f_{\alpha_j,g})\pi(a)\,v = \int_{P_m} \Upsilon_j(g)\pi(p)\pi(a)\,v\,d_lp = \Upsilon_j(g)\pi(a)\int_{P_m} \pi(p^a)\,v\,d_lp = c_j(a)\Upsilon_j(g)\pi(a)v,$$

for some $c_j(a) > 0$, since $p^a \in L$ for all $p \in P_m$. Thus,

$$\Upsilon(g)\pi(la)\,v = \pi(l)\Upsilon_j(g)\pi(a)\,v = c_j(a)^{-1}\pi(l)\pi^+(f_{\alpha_j,g})\pi(a)\,v.$$

Let $e_{\widetilde{K}} = \mathrm{meas}(\widetilde{K})^{-1}\,\mathrm{ch}_{\widetilde{K}}$. Then,

$$\begin{aligned}\pi^+(e_{\widetilde{K}})\Upsilon(g)\pi(la)\,v &= c_j(a)^{-1}\pi^+(e_{\widetilde{K}})\pi(l)\pi^+(f_{\alpha_j,g})\pi(a)\,v \\ &= c_j(a)^{-1}\pi(l)\pi^+(e_{\widetilde{K}} * f_{\alpha_j,g})\pi(a)\,v,\end{aligned}$$

where $\pi^+(e_{\widetilde{K}})$ and $\pi(l)$ commute since $\widetilde{K}$ is normal in L_0^+. Furthermore, since $f_{\alpha_j,g}$ is $\widetilde{K}$-bi-invariant for $g \in \omega$, we have $e_{\widetilde{K}} * f_{\alpha_j,g} = f_{\alpha_j,g}$, and so

$$\pi^+(e_{\widetilde{K}})\Upsilon(g)\pi(la)v = c_j(a)^{-1}\pi(l)\pi^+(f_{\alpha_j,g})\pi(a)v = \Upsilon(g)\pi(la)v.$$

Therefore, $\pi^+(e_{\widetilde{K}})\Upsilon(g)$ acts as $\Upsilon(g)$ on $L_0A' \cdot V_\pi^L$. Recall that $V_\pi = G^+ \cdot V_\pi^L = (L_0A'L_0^+)\cdot V_\pi^L$. However, since L is normal in L_0^+, V_π^L is L_0^+-invariant, and so $V_\pi = (L_0A')\cdot V_\pi^L$. Hence $\pi^+(e_{\widetilde{K}})\Upsilon(g) = \Upsilon(g)$ on all of V_π, for any $g \in \omega$.

From the definition of $\Upsilon(g)$, it is clear that since $\widetilde{K} \subset K$, $\pi^+(k)$ commutes with $\Upsilon(g)$ for any $k \in \widetilde{K}$. It follows that $\pi^+(e_{\widetilde{K}})$ commutes with $\Upsilon_K(g)$ as well. Thus,

$$\Upsilon(g) = \pi^+(e_{\widetilde{K}})\Upsilon(g) = \Upsilon(g)\,\pi^+(e_{\widetilde{K}}),$$

and so since $\widetilde{K}$ is normal in K, one may calculate that $\Upsilon(kgk') = \Upsilon(g)$ for $k,k' \in \widetilde{K}$ and $g \in \omega$. Hence $g \mapsto \Upsilon(g)$ is a locally constant mapping of G^+_{qr} into $\mathrm{End}_{\mathbb{C}}(V_\pi)$.

To see that the maps

$$x \mapsto \pi^+(x)\Upsilon_K(g), \qquad x \mapsto \Upsilon_K(g)\pi^+(x), \qquad \left(x \in G^+\right).$$

are locally constant, consider that for $k \in \widetilde{K}$, we have

$$\pi^+(xk)\Upsilon(g) = \pi^+(xk)\pi^+(e_{\widetilde{K}})\Upsilon(g) = \pi^+(x)\pi^+(e_{\widetilde{K}})\Upsilon(g) = \pi^+(x)\Upsilon(g),$$

and similarly $\Upsilon(g)\pi^+(kx) = \Upsilon(g)\pi^+(x)$ using $\Upsilon(g) = \Upsilon(g)\,\pi^+(e_{\widetilde{K}})$. Thus, we have shown $\Upsilon(g) \in \mathrm{End}_0(V_\pi)$. □

Notice that for $g \in G^+_{\mathrm{qr}}$, the existence of a compact, open subgroup $\widetilde{K} \subset G^+$ such that

$$\Upsilon(g) = \pi^+\left(e_{\widetilde{K}}\right)\Upsilon(g)$$

implies that the image of $\Upsilon(g)$ lies in the space $V_\pi^{\widetilde{K}}$, which is finite-dimensional by admissibility of π^+. Therefore, each $\Upsilon(g)$ has finite rank.

Corollary 3.3.2. *The character of π^+ is represented on G^+_{qr} by*

$$\Theta_{\pi^+} : G^+_{\mathrm{qr}} \to \mathbb{C}, \qquad g \mapsto \mathrm{tr}\big(\Upsilon(g)\big).$$

In other words, for $f \in C_c^\infty\left(G^+_{\mathrm{qr}}\right)$, we have $\Theta_{\pi^+}(f) = \mathrm{tr}(\pi^+(f)) = \int_{G^+} f(g)\Theta_{\pi^+}(g)\,dg$.

Proof. For $f \in C_c^\infty\left(G^+\right)$, set $f^0(x) = \int_K f(x^k)\,dk$, $x \in G^+$. Then $\operatorname{supp} f^0 \subset K(\operatorname{supp} f)K$, so f^0 has compact support. Also, if f is left-invariant under some open, normal subgroup of K, then f^0 has the same property. Thus, $f^0 \in C_c^\infty\left(G^+\right)$. Moreover, $\mathrm{tr}(\pi^+(f^0)) = \mathrm{tr}(\pi^+(f))$.

Now, suppose $f \in C_c^\infty\left(G^+_{\mathrm{qr}}\right)$, and let $x \in G^+$ such that $f^0(x) \neq 0$. Then there exists $k \in K$ such that $f(x^k) \neq 0$, so $x^k \in G^+_{\mathrm{qr}}$. Therefore, $x \in G^+_{\mathrm{qr}}$, and so $f^0 \in C_c^\infty\left(G^+_{\mathrm{qr}}\right)$. Since $x \mapsto \Upsilon(x)$ is locally constant, the map

$$G^+_{\mathrm{qr}} \to \mathbb{C}, \qquad x \mapsto f(x)\,\mathrm{tr}\big(\Upsilon(x)\big),$$

lies in $C_c^\infty\left(G^+_{\mathrm{qr}}\right)$. Therefore,

$$\begin{aligned}\int_{G^+} f(x)\operatorname{tr}\big(\Upsilon(x)\big)\,dx &= \operatorname{tr}\left(\int_{G^+} f(x)\Upsilon(x)\,dx\right) = \operatorname{tr}\left(\int_{G^+}\int_K f(x)\pi^+({}^k x)\,dk\,dx\right)\\ &= \operatorname{tr}\left(\int_{G^+\times K} f(x^k)\pi^+(x)\,dk\,dx\right) = \operatorname{tr}\left(\int_{G^+} f^0(x)\pi^+(x)\,dx\right)\\ &= \operatorname{tr}(\pi^+(f^0)) = \operatorname{tr}(\pi^+(f)). \qquad \square\end{aligned}$$

Corollary 3.3.3. *For $\Upsilon = \Upsilon_K^{L_0}$, the map*

$$G^+_{\mathrm{qr}} \to \mathbb{C}, \qquad\qquad g \mapsto \operatorname{tr}\big(\Upsilon(g)\big)$$

is independent of the choice of conjugate L_0 of K_0 and open subgroup K of L_0^+.

3.4 Restriction to an appropriate subgroup to ensure convergence

We now assume that π is also supercuspidal. We have the familiar Harish-Chandra character formula ([21])

$$\Theta_\pi(g) = \frac{d(\pi)}{\varphi(1)}\int_{Z'\backslash G}\int_K \varphi\big({}^{xk}g\big)\,dk\,d\dot{x}, \qquad\qquad \big(g \in G_{\mathrm{reg}}\big),$$

for Z' any closed subgroup of $Z(G)$ such that $Z(G)/Z'$ is compact, K any compact, open subgroup of G with normalized Haar measure dk, and φ any sum of matrix coefficients of π with $\varphi(1) \neq 0$. However, this does not generalize to G^+, as in some cases the centre $Z(G^+)$ is too small and the integral $\int_{Z'\backslash G^+}\int_K \varphi\big({}^{xk}g\big)\,dk\,d\dot{x}$ (for appropriate $Z' \subset Z(G^+)$, K, dk, and sum of matrix coefficients φ of π^+) may not converge for certain quasi-regular $g \in G^+$. Proposition 1.5.1(2) and Lemma 1.4.5 suggest that we should not have such a problem when $\operatorname{val}\theta = 1$. However, we are also interested in cases with $\operatorname{val}\theta = -1$, even though π^+ will not be supercuspidal. To demonstrate that what follows is necessary, a detailed example of this failure to converge for a specific case with $\operatorname{val}\theta = -1$ is provided in the appendix. We will work around this problem by restricting π to an appropriate θ-stable subgroup $H \subset G$ such that $Z(H)/Z(H)_\theta$ is compact and $\pi_o = \pi|H$ has finite length. For $g \in H^+_{\mathrm{qr}}$, we will then arrive at an integral formula for $\Theta_{\pi^+}(g)$ by summing the trace of the operator $\Upsilon_K^{L_0}(g)$ restricted to each irreducible component of $\pi_o^+ = \pi^+|H^+$, for appropriate L_0 and K as in the previous section.

Set

$$H = \begin{cases} G, & \operatorname{val}\theta = 1,\\ \left\{\, g \in G \mid \det g \in \mathcal{O}_F^\times \,\right\}, & \operatorname{val}\theta = -1,\end{cases}$$

and let $Z_0 = Z(H) \cap G_\theta^0 \subseteq Z_\theta$. Then H is an open, closed, normal, θ-stable subgroup of G such that

(i) π_o is supercuspidal,

(ii) $Z(H)/Z_0$ is compact,

(iii) $G_\theta \subseteq H_\theta$,

(iv) G/ZH is finite and cyclic, and

(v) H contains every G-conjugate of K_0.

It will be necessary to determine how π_o^+ decomposes. Clifford theory tells us that π_o should decompose with multiplicity one.

Lemma 3.4.1 ([3, Lemma 2.1])**.** *If A is a normal subgroup of a group B such that $B/Z(B)A$ is finite and cyclic, then any irreducible representation of B decomposes with multiplicity one when restricted to A.*

Corollary 3.4.2.

(1) As an H-space, V_π decomposes into a direct sum $V_\pi = U_1 \oplus \cdots \oplus U_M$ of irreducible, inequivalent subspaces. The length M of the decomposition is equal to the number of one-dimensional (continuous) characters of G in the collection

$$X_{ZH}(\pi) = \left\{ \nu \in \widehat{G} \;\middle|\; \nu|ZH \equiv 1 \text{ and } \pi \otimes \nu \cong \pi \right\}.$$

(2) As an H^+-space, V_π decomposes with multiplicity one.

Proof. Statement (1) is a direct consequence of Lemma 3.4.1 and [16, Lemma 2.1]. The subgroup ZH^+ is normal in G^+, and by Lemma 1.1.2 and the fact that $G_\theta \subseteq H_\theta$, we have $Z(G^+)ZH^+ = ZH^+$. Since π^+ is irreducible and $G^+/Z(G^+)ZH^+ = G^+/ZH^+ \simeq G/ZH$, (2) also follows from Lemma 3.4.1. □

We may use the above decomposition of π_o to obtain the decomposition of π_o^+. For $1 \le i \le M$, write $U_i^+ = \sum_{j=0}^{d_\theta - 1} A_\pi^j U_i$.

Lemma 3.4.3. *Let W be a non-zero subspace of V_π. Then W is H^+-irreducible if and only if $W = U_i^+$ for some integer $1 \le i \le M$.*

Proof. First, suppose $W \subseteq V_\pi$ is a non-zero H^+-irreducible subspace. Since the quotient $H^+/Z(H^+)H$ has order at most 2, we may apply Lemma 3.4.1 to see that W decomposes with multiplicity one as an H-space. Let $W' \subseteq W$ be a non-zero H-irreducible subspace of W. Since π_o decomposes with multiplicity one, $W' = U_i$ for some i. Then W also contains, hence is equal to, the non-trivial H^+-invariant subspace U_i^+.

Now fix $1 \le i \le M$. Let $V_\pi = W_1 \oplus \cdots \oplus W_{M'}$ be the decomposition of π_o^+ afforded by Corollary 3.4.2(2). Using Lemma 3.4.1 to decompose each summand W_j with multiplicity one into H-irreducible spaces, we see that we must have $U_i \subseteq W_j$ for some $1 \le j \le M'$. As before, H^+-irreducibility of W_j implies $U_i^+ = W_j$. □

The lemma shows that to obtain the decomposition of π_o^+, we need only group together those summands in the decomposition of π_o that lie in the same A_π-orbit.

Proposition 3.4.4. *After reordering the U_i, if necessary, there exists an integer $1 \le M' \le M$ such that V_π decomposes as an H^+-space into a direct sum $V_\pi = U_1^+ \oplus \cdots \oplus U_{M'}^+$ of irreducible, inequivalent subspaces.*

Finally, we obtain a little more information about the summands in the above decomposition of π_o^+, by examining how the intertwining operator A_π interacts with decomposition of π_o.

Lemma 3.4.5.

(1) The operator A_π permutes the summands in the decomposition in Corollary 3.4.2(1).

(2) *For each integer $1 \le i \le M$, let m_i be the smallest positive integer such that $A_\pi^{m_i} U_i = U_i$. Then m_i divides d_θ, and U_i^+ is equal to the direct sum $\bigoplus_{j=0}^{m_i-1} A_\pi^j U_i$.*

Proof. Fix an integer $1 \le i \le M$. Since H is θ-stable, $A_\pi U_i$ is H-irreducible. Statement (1) is then immediate from the fact that π_o decomposes with multiplicity one. Now consider (2). Since $A_\pi^{d_\theta} U_i = U_i$, such a smallest positive integer m_i exists and must satisfy $m_i \le d_\theta$. Then we also have $A_\pi^{d_\theta - \ell m_i} U_i = U_i$, for any $\ell \in \mathbb{Z}$. If m_i does not divide d_θ, then there exists $\ell \in \mathbb{Z}$ with $0 < d_\theta - \ell m_i < m_i$, which contradicts the minimality of m_i. The equality $U_i^+ = \sum_{j=0}^{m_i-1} A_\pi^j U_i$ is obvious from the definition of m_i. By (1), the only way that this sum could fail to be direct is if there is a repeated summand. That is, there exist $0 \le j < k \le m_i - 1$ such that $A_\pi^k U_i = A_\pi^j U_i$. But this again contradicts the minimality of m_i. □

The preceding lemma gives us a convenient way to order the constituents in the decomposition of Proposition 3.4.4. For the rest of Chapter 3, assume that $m_1 \le m_2 \le \cdots \le m_{M'}$.

3.5 An integral formula

As in §3.3, choose a G-conjugate L_0 of K_0 such that L_0^+ is compact, and an open subgroup $K \subseteq L_0^+$. In this section, we develop an integral formula for the character of π^+ on elements of H_{qr}^+ by examining the trace of the operator $\Upsilon = \Upsilon_K^{L_0}$ on each constituent in the decomposition of Proposition 3.4.4. First, we show that in most cases, there is no contribution from constituents whose H-decomposition has length greater than one.

Lemma 3.5.1. *Fix integers $1 \le \ell \le M$ and $1 \le i < d_\theta$ such that m_ℓ is greater than 1 and does not divide i. Let $h \in H$ such that $h^+ = h\theta^i \in H_{\mathrm{qr}}^i$. Then $\mathrm{tr}(\Upsilon(h^+)|U_\ell^+) = 0$.*

Proof. The condition $m_\ell > 1$ says that $U_\ell^+ \ne U_\ell$. By the proof of Theorem 3.3.1, there exists an open, normal subgroup $\widetilde{K} \subset K \subset L_0^+$ such that

$$\Upsilon(h^+) = \Upsilon(h^+)\,\pi^+\left(e_{\widetilde{K}}\right) = \pi^+\left(e_{\widetilde{K}}\right)\,\Upsilon(h^+).$$

Let $L = \widetilde{K} \cap L_0$. Then L is a compact, open subgroup of H^+ contained in L_0, with $\dim(U_\ell^+)^L < \infty$, and

$$\Upsilon(h^+)U_\ell^+ \subseteq (U_\ell^+)^{\widetilde{K}} \subseteq (U_\ell^+)^L.$$

By the H-invariance of the summands in the definition of U_ℓ^+, we have

$$(U_\ell^+)^L = \bigoplus_{j=0}^{m_\ell-1} \left(A_\pi^j U_\ell\right)^L.$$

Now,

$$\Upsilon(h^+) = \int_K \pi^+\left({}^k h^+\right) dk = \int_K \pi^+\left(\theta^i\,(kh)^{\theta^i}\,k^{-1}\right) dk = A_\pi^i \int_K \pi\left((kh)^{\theta^i}\,k^{-1}\right) dk,$$

where $(kh)^{\theta^i}\,k^{-1} \in H$ for any $k \in K$. Therefore, for each j, $\Upsilon(h^+)\left(A_\pi^j U_\ell\right) \subset A_\pi^{i+j} U_\ell$. The result now follows from the fact that since $m_\ell \nmid i$, we have $A_\pi^{i+j} U_\ell \ne A_\pi^j U_\ell$, and so the projection of $\Upsilon(h^+)(A_\pi^j U_\ell)^L$ onto $(A_\pi^j U_\ell)^L$ is trivial. □

Corollary 3.5.2. *If $h \in H$ such that $h\theta \in H_{\mathrm{qr}}^i$, then $\mathrm{tr}(\Upsilon(h\theta)|U_\ell^+) = 0$ for any $1 \le \ell \le M$ with $m_\ell > 1$.*

Fix $1 \le j \le M$, and let $\pi_j^+ = \pi_o^+ | U_j^+$. Recall that we have assumed that π is supercuspidal. The following compactness lemma will ensure that the pieces of our integral formula will all converge.

Lemma 3.5.3. *Assume that π is unitary. Then π^+ is also unitary, so let $\langle \cdot, \cdot \rangle$ be a G^+-invariant Hermitian inner product on V_π. Let φ be a matrix coefficient of π_j^+ of the form*

$$\varphi(x) = \langle \pi_j^+(x) u, v \rangle, \qquad (x \in H^+),$$

for some $u, v \in U_j^+$. Suppose C is a compact subset of H_{qr}^+. Then, there exists a subset $\Delta \subset H^+$ which is compact modulo Z_0 and satisfies the following. For $g \in C$ and $x \in H^+$,

$$\int_K \varphi({}^{xk}g)\, dk = 0,$$

unless $x \in \Delta$.

Proof. Consider C as a compact subset of G_{qr}^+. From the proof of Theorem 3.3.1, for any $g \in C$ there exist a neighbourhood $\omega(g) \subset G_{\mathrm{qr}}^+$ of g and a compact, open subgroup $\widetilde{K}(g)$ of G^+ such that $\Upsilon(x) = \pi(e_{\widetilde{K}(g)})\Upsilon(x)$, for all $x \in \omega(g)$. Since C is compact, there exists a finite collection $\{g_i\} \subset C$ such that $C \subseteq \bigcup_i \omega(g_i)$. Set $\widetilde{K} = \left(\bigcap_i \widetilde{K}(g_i)\right) \cap H^+$. Then $\widetilde{K}$ is a compact, open subgroup of H^+. For $x \in C \subset H^+$, U_j^+ is $\Upsilon(x)$-invariant. Let $x \in C$. Following through the details of the proof of Theorem 3.3.1, and noting that $\pi^+\left(e_{\widetilde{K}}\right)|U_j^+ = \pi_j^+\left(e_{\widetilde{K}}\right)$, we see that $\Upsilon(x)|U_j^+ = \pi_j^+\left(e_{\widetilde{K}}\right)\Upsilon(x)|U_j^+$. Therefore, the image of $\Upsilon(x)|U_j^+$ lies in $(U_j^+)^{\widetilde{K}}$, which has finite dimension by admissibility of π^+. Let $\{v_1, \ldots, v_r\}$ be an orthonormal basis of $(U_j^+)^{\widetilde{K}}$. Then,

$$\Upsilon(x) w = \sum\nolimits_{\ell,m} \langle \Upsilon(x) v_\ell, v_m \rangle \langle w, v_\ell \rangle v_m, \qquad (w \in U_j^+).$$

Thus, for $g \in C$,

$$\begin{aligned}
\int_K \varphi({}^{xk}g)\, dk = \int_K \left\langle \pi_j^+({}^k g)\pi_j^+(x^{-1})u, \pi_j^+(x^{-1})v \right\rangle dk &= \left\langle \Upsilon(g)\pi_j^+(x^{-1})u, \pi_j^+(x^{-1})v \right\rangle \\
&= \sum\nolimits_{\ell,m} \langle \Upsilon(g) v_\ell, v_m \rangle \langle \pi_j^+(x^{-1})u, v_\ell \rangle \langle v_m, \pi_j^+(x^{-1})v \rangle \\
&= \sum\nolimits_{\ell,m} \langle \Upsilon(g) v_\ell, v_m \rangle \overline{\langle \pi_j^+(x) v_\ell, u \rangle} \langle \pi_j^+(x) v_m, v \rangle.
\end{aligned}$$

Now, for $1 \le m \le r$, the function

$$\psi_m : H^+ \to \mathbb{C}, \qquad x \mapsto \langle \pi_j^+(x) v_m, v \rangle,$$

restricted to the component $H\theta^i$, can be considered a matrix coefficient of π_o. Since π_o is supercuspidal, there exists a collection of compact sets $\{\omega_{m,i}\}_{i=0}^{d_\theta - 1}$, with each $\omega_{m,i} \subset H$, such that $\operatorname{supp}\psi_m \subseteq \bigcup_i \omega_{m,i} Z(H)\theta^i$. But $Z(H)/Z_0$ is compact, so there exists a compact subset $\Delta_o \subset H$ such that $Z(H) \subset \Delta_o Z_0$, and

$$\operatorname{supp}\psi_m \subseteq \bigcup\nolimits_i \omega_{m,i}(\Delta_o Z_0)\theta^i = \left(\bigcup\nolimits_i \omega_{m,i}\Delta_o\theta^i\right) Z_0.$$

Therefore, $\int_K \varphi({}^{xk}g)\, dk = 0$ unless

$$x \in \bigcup\nolimits_m \operatorname{supp}\psi_m \subseteq \left(\bigcup\nolimits_{m,i} \omega_{m,i}\Delta_o\theta^i\right) Z_0,$$

so take Δ to be this last set. $\square$

We may now state a Harish-Chandra-type integral formula on U_j^+.

Theorem 3.5.4. *Let φ be a matrix coefficient of π_j^+, and let $d(\pi_j^+)$ be the formal degree of π_j^+ relative to the Haar measure $d\dot{x}$ on $Z_0\backslash H^+$. Then for $g \in H^+_{\mathrm{qr}}$,*

$$\varphi(1)\operatorname{tr}\big(\Upsilon(g)|U_j^+\big) = d(\pi_j^+)\int_{Z_0\backslash H^+}\int_K \varphi({}^{xk}g)\,dk\,d\dot{x}. \tag{3.5.1}$$

Proof. Note that Z_0 has finite index in $Z(H^+)$. First, assume that π is unitary. Let $\langle\cdot,\cdot\rangle$ be as in Lemma 3.5.3. In this case, it suffices to consider φ of the form $\varphi(x) = \langle \pi_j^+(x)u, v\rangle$, for some $u, v \in U_j^+$. As in the proof of the Lemma 3.5.3, let $\{v_1,\dots,v_r\}$ be an orthonormal basis of $(U_j^+)^{\widetilde{K}}$. Using Schur orthogonality,

$$\begin{aligned} d(\pi_j^+)&\int_{Z_0\backslash H^+}\int_K \varphi({}^{xk}g)\,dk\,d\dot{x} \\ &= \textstyle\sum_{\ell,m}\langle \Upsilon(g)v_\ell, v_m\rangle\, d(\pi_j^+)\displaystyle\int_{Z_0\backslash H^+}\overline{\langle \pi_j^+(x)v_\ell, u\rangle}\langle \pi_j^+(x)v_m, v\rangle\, d\dot{x} \\ &= \textstyle\sum_{\ell,m}\langle \Upsilon(g)v_\ell, v_m\rangle\langle u, v\rangle\langle v_m, v_\ell\rangle = \varphi(1)\sum_m \langle \Upsilon(g)v_m, v_m\rangle = \varphi(1)\operatorname{tr}\big(\Upsilon(g)|U_j^+\big). \end{aligned}$$

Note that convergence of the integral is guaranteed by Lemma 3.5.3, since the inner integral has compact support mod Z_0.

Now we drop the assumption that π is unitary. By Corollary 1.5.6, there exists a θ-fixed quasi-character ν of G such that the twist $\pi' = \pi\otimes\nu$ is θ-stable and unitary. Using the same intertwining operator A_π to define an extension $(\pi')^+$ to G^+, we have $(\pi')^+ = \pi^+\otimes\nu^+$, where ν^+ is the extension to G^+ defined by $\nu^+(\theta) = 1$. The restriction $(\pi')^+_o = (\pi')^+|H^+$ has the same decomposition as π^+_o. Let $(\pi')^+_j$ be the restriction of $(\pi')^+_o$ to U_j^+. Then $(\pi')^+_j = \pi^+_j\otimes\nu^+_o$, where $\nu^+_o = \nu^+|H^+$, and so $d\big((\pi')^+_j\big) = d(\pi^+_j)$ by definition.

Let $u \in U_j^+$ and $\tilde{u} \in \widetilde{U}_j^+$ such that $\varphi(h) = \langle \tilde{u}, \pi^+(h)u\rangle$, for any $h \in H^+$. Since $\ker \nu^+_o$ is open, the linear functional $\tilde{u}$ is also smooth with respect to $(\pi')^+_j$. Therefore, the function $\varphi'(h) = \nu^+(h)\varphi(h)$, for $h \in H^+$, is a matrix coefficient of $(\pi')^+_j$. Moreover, $\varphi'(1) = \varphi(1)$, and $\varphi'({}^{xk}g) = \nu^+(g)\varphi({}^{xk}g)$, for any $x, g \in H^+$ and $k \in K$.

Finally, define $\Upsilon' = (\Upsilon')^{L_0}_K$ with respect to $(\pi')^+$, as in §3.3. Then $\Upsilon'(g) = \nu^+(g)\Upsilon(g)$, for any $g \in G^+_{\mathrm{qr}}$. Applying (3.5.1) to $(\pi')^+_j$, Υ', and φ', we have

$$\begin{aligned} \varphi(1)\operatorname{tr}\big(\Upsilon(g)|U_j^+\big) &= \nu^+(g)^{-1}\varphi'(1)\operatorname{tr}\big(\Upsilon'(g)|U_j^+\big) \\ &= \nu^+(g)^{-1}d\big((\pi')^+_j\big)\int_{Z_0\backslash H^+}\int_K \varphi'({}^{xk}g)\,dk\,d\dot{x} = d(\pi_j^+)\int_{Z_0\backslash H^+}\int_K \varphi({}^{xk}g)\,dk\,d\dot{x}, \end{aligned}$$

for any $g \in H^+_{\mathrm{qr}}$. □

Assuming we have ordered the U_j as in Proposition 3.4.4, the theorem allows us to obtain an integral formula for

$$\Theta_{\pi^+}(g) = \operatorname{tr}\big(\Upsilon(g)\big) = \textstyle\sum_{j=1}^{M'}\operatorname{tr}\big(\Upsilon(g)|U_j^+\big), \qquad \big(g \in H^+_{\mathrm{qr}}\big),$$

by applying (3.5.1) to each term in the sum.

Corollary 3.5.5. *Fix an integer $1 \le i < d_\theta$, and set*

$$J = J(i) = \big\{\, 1 \le j \le M' \;\big|\; m_j \text{ divides } i \,\big\}.$$

For each index $j \in J$, take

L_j, *a conjugate of* K_0 *such that* L_j^+ *is compact;*
M_j, *an open subgroup of* L_j^+*;*
$d_j m$, *normalized Haar measure on* M_j*; and*
φ_j, *a matrix coefficient of* π_j^+ *with* $\varphi_j(1) \neq 0$.

Then, for $g \in H^i_{\mathrm{qr}}$,

$$\Theta_{\pi^+}(g) = \sum_{j \in J} \frac{d(\pi_j^+)}{\varphi_j(1)} \int_{Z_0 \backslash H^+} \int_{M_j} \varphi_j({}^{xm}g)\, d_j m\, d\dot{x}. \tag{3.5.2}$$

If each subgroup M_j *is chosen so that* $\theta \in N_{H^+}(M_j)$*, then for* $g \in (H^i_\theta)_{\mathrm{qr}}$,

$$\Theta_{\pi^+}(g) = |\langle\theta\rangle| \sum_{j \in J} \frac{d(\pi_j^+)}{\varphi_j(1)} \int_{Z_0 \backslash H} \int_{M_j} \varphi_j({}^{xm}g)\, d_j m\, d\dot{x}. \tag{3.5.3}$$

The formulas in the corollary do not follow immediately from (3.5.1), as they compare $\Theta_{\pi^+}(g)$ to a sum involving terms $\mathrm{tr}\big(\Upsilon^{L_j}_{M_j}(g)|U_j^+\big)$, where the compact, open subgroup $M_j \subseteq L_j^+$ may vary from term to term. To justify this, we first need the following lemma.

Lemma 3.5.6. *For* $1 \le j \le M$*, the function*

$$H^+_{\mathrm{qr}} \to \mathbb{C}, \qquad g \mapsto \mathrm{tr}\big(\Upsilon^{L_0}_K(g)|U_j^+\big)$$

is independent of the choice of conjugate L_0 *of* K_0 *and open subgroup* $K \subseteq L_0^+$.

Proof. Fix $g \in H^+_{\mathrm{qr}}$. Let L_0' be another G-conjugate of K_0 such that $(L_0')^+$ is compact, and let K' be an open subgroup of $(L_0')^+$. We wish to show that $\mathrm{tr}\big(\Upsilon^{L_0}_K(g)|U_j^+\big) = \mathrm{tr}\big(\Upsilon^{L_0'}_{K'}(g)|U_j^+\big)$. Consider the following cases.

Case 1. $K' = K \subseteq L_0^+ \cap (L_0')^+$.

Clearly, the definition of $\Upsilon^{L_0}_K$ does not depend on the choice of L_0, and its properties depend only on the existence of *some* such G-conjugate of K_0 such that $K \subseteq L_0^+$. Therefore, $\mathrm{tr}\big(\Upsilon^{L_0}_K(g)|U_j\big) = \mathrm{tr}\big(\Upsilon^{L_0'}_K(g)|U_j\big)$.

Case 2. $L_0' = L_0$*,* K' *a normal subgroup of* K.

As in the proof of Theorem 3.3.1, we have

$$\Upsilon^{L_0}_K(g) = [K : K']^{-1} \sum_{k \in K/K'} \pi^+(k)\, \Upsilon^{L_0}_{K'}(g)\, \pi^+(k)^{-1}.$$

Now, U_j^+ is K-invariant, since $K \subseteq L_0^+ \subset H^+$. Therefore,

$$\begin{aligned}\mathrm{tr}\big(\Upsilon^{L_0}_K(g)|U_j^+\big) &= [K : K']^{-1} \sum_{k \in K/K'} \mathrm{tr}\Big(\pi^+(k)\big(\Upsilon^{L_0}_{K'}(g)|U_j^+\big)\pi^+(k)^{-1}\Big) \\ &= [K : K']^{-1} \sum_{k \in K/K'} \mathrm{tr}\big(\Upsilon^{L_0}_{K'}(g)|U_j^+\big) = \mathrm{tr}\big(\Upsilon^{L_0}_{K'}(g)|U_j^+\big).\end{aligned}$$

Case 3. $L_0' = L_0$*,* K' *arbitrary.*

There exists an open subgroup $K'' \subseteq K \cap K'$ which is normal in L_0^+, hence normal in both K and K'. By the previous case, $\mathrm{tr}\big(\Upsilon^{L_0}_K(g)|U_j^+\big) = \mathrm{tr}\big(\Upsilon^{L_0}_{K''}(g)|U_j^+\big) = \mathrm{tr}\big(\Upsilon^{L_0}_{K'}(g)|U_j^+\big)$.

Case 4. The general case.

Let $K'' = K \cap K'$. Then $K'' \subseteq L_0^+ \cap (L_0')^+$, so cases 1 and 3 give

$$\mathrm{tr}\big(\Upsilon_K^{L_0}(g)|U_j^+\big) = \mathrm{tr}\big(\Upsilon_{K''}^{L_0}(g)|U_j^+\big) = \mathrm{tr}\big(\Upsilon_{K''}^{L_0'}(g)|U_j^+\big) = \mathrm{tr}\big(\Upsilon_{K'}^{L_0'}(g)|U_j^+\big). \qquad \square$$

We are now in a position to verify (3.5.2) and (3.5.3).

Proof of Corollary 3.5.5. Let $g \in H_{\mathrm{qr}}^i$. Then $\Theta_{\pi^+}(g) = \mathrm{tr}\big(\Upsilon_K^{L_0}(g)\big) = \sum_{j=1}^{M'} \mathrm{tr}\big(\Upsilon_K^{L_0}(g)|U_j^+\big)$, by Corollary 3.3.2. But from Lemmas 3.5.6 and 3.5.1, we have

$$\mathrm{tr}\big(\Upsilon_K^{L_0}(g)|U_j^+\big) = \begin{cases} \mathrm{tr}\big(\Upsilon_{M_j}^{L_j}(g)|U_j^+\big), & j \in J, \\ 0, & j \notin J. \end{cases}$$

Equation (3.5.2) then follows from (3.5.1).

Now assume ${}^{\theta}g = g$ and $\theta \in \bigcap_{j\in J} N_{H^+}(M_j)$. Then for each $j \in J$, we have

$$\begin{aligned}
\varphi_j(1)\,\mathrm{tr}\big(\Upsilon_{M_j}^{L_j}(g)|U_j^+\big) &= d(\pi_j^+) \int_{Z_0\backslash H^+} \int_{M_j} \varphi({}^{xm}g)\, d_j m\, d\dot{x} \\
&= d(\pi_j^+) \sum_{\ell=0}^{d_\theta - 1} \int_{Z_0\backslash H} \int_{M_j} \varphi({}^{x\theta^\ell m}g)\, d_j m\, d\dot{x} \\
&= d(\pi_j^+) \sum_{\ell=0}^{d_\theta - 1} \int_{Z_0\backslash H} \int_{M_j} \varphi({}^{x\theta^\ell m \theta^{-\ell}}g)\, d_j m\, d\dot{x} \\
&= d_\theta\, d(\pi_j^+) \int_{Z_0\backslash H} \int_{M_j} \varphi({}^{xk}g)\, d_j m\, d\dot{x},
\end{aligned}$$

where the last equation is achieved using the change of variables $m \mapsto {}^{\theta^\ell}m$. $\square$

We are really only interested in the values of Θ_{π^+} at points in H_{qr}^1, since at points in H_{qr} we have just the character of π, while, for $1 < i < d_\theta$, at points in H_{qr}^i we may restrict π^+ to $G \rtimes \langle \theta^i \rangle$, effectively replacing θ by θ^i.

Recall that we have ordered the constituents in the decomposition of π_o^+ so that $m_1 \le m_2 \le \cdots \le m_{M'}$. Let j_0 be the largest index such that $m_{j_0} = 1$. If j_0 does not exist, Corollary 3.5.2 says that $\Theta_{\pi^+} \equiv 0$ on H_{qr}^1. So suppose j_0 does exist. Note that the constituents $U_1^+, U_2^+, \ldots, U_{j_0}^+$ are precisely those such that $U_j^+ = U_j$. Using the notation of Corollary 3.5.5, we have $J(1) = \{1 \le j \le s \mid m_j = 1\}$, and so

$$\Theta_{\pi^+}(g) = \sum_{j=1}^{j_0} \frac{d(\pi_j^+)}{\varphi_j(1)} \int_{Z_0\backslash H^+} \int_{M_j} \varphi_j({}^{xm}g)\, d_j m\, d\dot{x}, \qquad \big(g \in H_{\mathrm{qr}}^1\big). \tag{3.5.4}$$

If $\theta \in \bigcap_{j\in J} N_{H^+}(M_j)$, then

$$\Theta_{\pi^+}(g) = |\langle\theta\rangle| \sum_{j=1}^{j_0} \frac{d(\pi_j^+)}{\varphi_j(1)} \int_{Z_0\backslash H} \int_{M_j} \varphi_j({}^{xm}g)\, d_j m\, d\dot{x}, \qquad \big(g \in (H_\theta^1)_{\mathrm{qr}}\big). \tag{3.5.5}$$

4. Depth-zero supercuspidal representations

In this section we analyze (3.5.4) and (3.5.5) in the case that π is induced from a representation σ of ZK_0 which is trivial on K_1 and cuspidal as a representation of K_0/K_1.

4.1 The inducing data

We are interested in those supercuspidal representations of G which are obtained using a special case of Howe's construction as follows. For a complete account of Howe's construction of the tame supercuspidal representations of G, see [24]. Let $Z = Z(G) \simeq F^\times$. Let E be a degree n, unramified extension of F, and let λ be an admissible quasi-character of $E^\times$ which is trivial on $1 + \mathscr{P}_E$. In this case, the property of admissibility is equivalent to the regularity condition that $(\lambda \circ \eta_i)|\mathscr{O}_E^\times$, $i = 1, 2$, are distinct for all pairs $\eta_1 \neq \eta_2$ of elements of $\mathrm{Gal}(E/F)$. Choose an F-basis $\mathcal{B}$ of E, and identify $E^\times$ with a subgroup of G via the embedding $\alpha \mapsto [\mathrm{mult}(\alpha, \cdot)]_{\mathcal{B}}$, $\alpha \in E^\times$. Let K be a maximal parahoric subgroup of G such that $E^\times \cap K = \mathscr{O}_E^\times$. Such a subgroup K exists, as we may take it to be the conjugate of K_0 afforded by the change of basis matrix between $\mathcal{B}$ and some fixed integral F-basis of E. Then $\mathsf{T} = (E^\times \cap K)/K'$ is a torus in $K/K' \simeq \mathbf{GL}_n(k_F)$, where K' is the pro-unipotent radical of K. The restriction of λ to $\mathscr{O}_E^\times$ induces a character of T. By the admissibility of λ, this character is in general position, as the Galois groups $\mathrm{Gal}(E/F)$ and $\mathrm{Gal}(k_E/k_F)$ are canonically isomorphic. Therefore, we may associate to λ a cuspidal representation (σ, W) of K/K' via Deligne-Lusztig induction. Inflate σ to K. Since $ZK = \langle \varpi_F \rangle K$, we may extend σ to an irreducible representation of $N_G(K) = ZK$ by letting ϖ_F act on W by the scalar operator associated to $\lambda(\varpi_F) \in \mathbb{C}^\times$. Note that (σ, W) is necessarily finite-dimensional. If we let $\pi = \text{c-Ind}_{ZK}^{G}\, \sigma$, then π is an irreducible, admissible, supercuspidal representation of G. The representation π is *depth-zero* in the sense that it has non-zero K'-fixed vectors. To make use of Corollary 1.5.3, we would like our inducing data σ and K to both be θ-stable (note that Z is always θ-stable).

Hypothesis H5. *There exists a θ-stable maximal parahoric subgroup $K \subset G$.*

Remark. Based on H1, in either of the cases of $\mathrm{val}\,\theta = \pm 1$, we have that the maximal parahoric K_0 is θ-stable if and only if $J \in N_G(K_0) = ZK_0$, since it is always $(\tau \circ \theta_0)$-stable. In general, H5 is satisfied if and only if there exists $g \in G$ such that $\tau\big(\theta_0(g)\big)^{-1} J g \in ZK_0$. For example, H5 fails to be true when θ_0 is trivial and $J = \mathrm{diag}(1, \ldots, 1, \varpi_F)$.

Let G act on $\mathrm{Aut}_{F_0}(\mathbf{G})$ by $(g \cdot \eta)(x) = {}^{g}\big(\eta(x^g)\big)$, for $\eta \in \mathrm{Aut}_{F_0}(\mathbf{G})$, $g \in G$, and $x \in \mathbf{G}$. Then, replacing θ by $g \cdot \theta$ for a suitable choice of $g \in G$, we may assume that $K = K_0$. Note that this allows us to choose $L_0 = K_0$ to satisfy H4, so that H4 is essentially replaced by H5. Since the pro-unipotent radical $K_1 \subset K_0$ is unique, we must also have $\theta(K_1) = K_1$. For any automorphism η of K_0, write $K_{0,\eta} = (K_0)_\eta$ and $K_{1,\eta} = (K_1)_\eta$.

If we further assume that σ is θ-stable, then Corollary 1.5.3 tells us that π is θ-stable as well. Use the notation of §1.5.2, and choose an intertwining operator $A_\sigma \in \mathrm{Hom}(\sigma, \sigma \circ \theta)$ with $A_\sigma^{d_\theta} = 1$. Let $A_\pi = \Phi(A_\sigma) \in \mathrm{Hom}(\pi, \pi \circ \theta)$, where Φ is as in Proposition 1.5.2. Then $A_\pi^{d_\theta} = 1$ as well. Use A_σ and A_π to define σ^+ and π^+, respectively.

4.2 Some integral formulas

We now analyze (3.5.4) and (3.5.5) for π as in §4.1.

4.2.1 In the case $\operatorname{val}\theta = 1$. The property that $Z(H)/Z_0$ is compact is essential for Lemma 3.5.3 and Theorem 3.5.4. Our choice of H is then guided by Lemma 1.4.5. For $\operatorname{val}\theta = 1$, we have chosen $H = G$, and there will be only one term in the expressions for Θ_{π^+} in Corollary 3.5.5. Let $\dot\chi_{\sigma^+}$ be the extension by zero of χ_{σ^+} to all of G^+. From Proposition 1.5.4, we have $\pi^+ \cong \text{c-Ind}_{(ZK_0)^+}^{G^+}\sigma^+$, and therefore $\dot\chi_{\sigma^+}$ is a sum of matrix coefficients of π^+. Using this in (3.5.4), we now have

$$\Theta_{\pi^+}(g) = \frac{d(\pi^+)}{\deg(\sigma)}\int_{Z_0\backslash G^+}\int_K \dot\chi_{\sigma^+}({}^{xk}g)\,dk\,d\dot x, \tag{4.2.1}$$

for any $g \in G^1_{\mathrm{qr}}$. If g commutes with θ, then (3.5.5) gives

$$\Theta_{\pi^+}(g) = |\langle\theta\rangle|\;\frac{d(\pi^+)}{\deg(\sigma)}\int_{Z_0\backslash G}\int_K \dot\chi_{\sigma^+}({}^{xk}g)\,dk\,d\dot x. \tag{4.2.2}$$

4.2.2 In the case $\operatorname{val}\theta = -1$. In this case, we have had to restrict to a large subgroup $H \subsetneq G$ to ensure convergence of our integral formulas (see §3.4). The subgroup H was chosen precisely so that $Z(H)/Z_0$ is compact and $\pi|H$ has finite length. To find suitable matrix coefficients to use in these formulas in this case, let us analyze the irreducible constituents of $\pi_o^+ = \pi^+|H^+$, given the above construction of π. As in §3.4, we should first decompose $\pi_o = \pi|H$. The following lemma will let us make use of [30, Lemma 3.4].

Lemma 4.2.1. *$ZH = E^\times H = E^\times\mathbf{SL}_n(F)$.*

Proof. Clearly $E^\times\mathbf{SL}_n(F)$ is contained in $E^\times H$. Now, $E^\times\mathbf{SL}_n(F) = \{g \in G \mid \det g \in N_{E/F}E^\times\}$. Let $e \in E^\times$ and $h \in H$. Then $\det e \in N_{E/F}E^\times$ and $\det h \in \mathcal{O}_F^\times$. But since E/F is unramified, $N_{E/F}\mathcal{O}_E^\times = \mathcal{O}_F^\times$, and so $\det eh \in N_{E/F}E^\times$. This proves the second equality.

Since $Z \cong F^\times$, to show the first equality it suffices to show that $E^\times$ is contained in ZH. Let $e \in E^\times$, and let $r = \operatorname{val}_F(\det e)$. Now, $\det e \in N_{E/F}E^\times$, and so since E/F is unramified, we have $r \equiv 0 \mod n$. If we set $z = \operatorname{diag}(\varpi_F^{r/n},\dots,\varpi_F^{r/n}) \in Z$, then $\det z^{-1}e \in \mathcal{O}_F^\times$, so that $z^{-1}e \in H$. Thus, $e = z(z^{-1}e)$ lies in ZH. ☐

Lemma 4.2.2 (Moy-Sally)**.** *$\pi|ZH$ has n irreducible, inequivalent components.*

Proof. This is [30, Lemma 3.4], noting that in the present unramified case

$$\left|F^\times/N_{E/F}E^\times\right| = [E:F] = n.$$ ☐

Remark. In [30], Moy and Sally restrict to the case that $(p,n) = 1$ to ensure tame ramification. However, the cited result holds even without this restriction.

Let V_σ be the image of W in V_π under the usual embedding (see §1.5.2). We may characterize V_σ as precisely those elements of V_π whose support is contained in ZK_0. We now show that each irreducible component of π_o is induced, and moreover is generated by the image of V_σ under a fixed element of G. Recall that $gH\cdot V_\sigma = \operatorname{Span}\{\pi(gh)v \mid h \in H,\ v \in V_\sigma\}$. Since H is normal in G, $gH\cdot V_\sigma$ is an H-invariant subspace of V_π.

Proposition 4.2.3. *For any $g \in G$, $gH \cdot V_\sigma \cong \text{c-Ind}_{{}^gK_0}^H({}^g\sigma_o)$ as H-spaces, where ${}^g\sigma_o({}^gx) = \sigma_o(x)$ for $x \in K_0$.*

Proof. Fix $g \in G$. Let $T\colon gH \cdot V_\sigma \to \text{c-Ind}_{{}^gK_0}^H({}^g\sigma_o)$ be the linear map defined by $(Tf)(h) = f(g^{-1}h)$, for $f \in gH \cdot V_\sigma$ and $h \in H$. First, we show that the image of T is contained in $\text{c-Ind}_{{}^gK_0}^H({}^g\sigma_o)$. Let $f \in gH \cdot V_\sigma$ be non-zero. Then $f = \pi(gh_1)f_1 + \cdots + \pi(gh_r)f_r$ for some $h_j \in H$, $f_j \in V_\sigma$, and $r \in \mathbb{Z}_{>0}$. For $h \in H$, we have $(Tf)(h) \neq 0$ only if $f_j(g^{-1}hgh_j) \neq 0$ for at least one j. Since $\operatorname{supp} f_j \subset ZK_0$ for each j, we have

$$\operatorname{supp} Tf \subset H \cap \left(\bigcup_{j=1}^r {}^g(K_0h_j^{-1})Z\right) = \bigcup_{j=1}^r {}^g(K_0h_j^{-1}),$$

where the set on the right is compact, hence compact modulo ${}^gK_0$. Furthermore, since f transforms appropriately under left multiplication by elements of K_0, we have

$$(Tf)({}^gkh) = f(kg^{-1}h) = \sigma_o(k)f(g^{-1}h) = {}^g\sigma_o({}^gk)(Tf)(h),$$

for any $k \in K_0$. Finally, if f is right K-invariant for some compact, open subgroup $K \subset G$, then Tf is right $(K \cap H)$-invariant. Thus, $Tf \in \text{c-Ind}_{{}^gK_0}^H({}^g\sigma_o)$.

Suppose $Tf \equiv 0$ for some $f \in gH \cdot V_\sigma$. This says $f|g^{-1}H \equiv 0$. Since H is normal in G, we have $f|Hg^{-1} \equiv 0$. Let $f = \sum_{j=1}^r \pi(gh_j)f_j$ as before. Then, $\operatorname{supp} f \subset \bigcup_{j=1}^r ZK_0h_j^{-1}g^{-1} \subset ZHg^{-1}$. But for $z \in Z$ and $h \in H$, $f(zhg^{-1}) = \sigma(z)f(hg^{-1}) = 0$, so in fact $f|ZHg^{-1} \equiv 0$. Therefore, $f \equiv 0$ and $\ker T = \{0\}$, so T is injective.

Finally, it is clear that T is an H-map, and thus embeds $gH \cdot V_\sigma$ in $\text{c-Ind}_{{}^gK_0}^H({}^g\sigma_o)$ as a non-zero H-invariant subspace. By the irreducibility of $\text{c-Ind}_{{}^gK_0}^H({}^g\sigma_o)$, we must have that this subspace is the whole space. □

Corollary 4.2.4. *Let $g_1, g_2, \ldots, g_n$ be a set of representatives for G/ZH. Then the irreducible components of π_o are $g_jH \cdot V_\sigma \cong \text{c-Ind}_{{}^{g_j}K_0}^H({}^{g_j}\sigma_o)$, $j = 1, 2, \ldots, n$.*

As in §3.4, to decompose π_o^+, we need only investigate the action of A_π on the constituents of π_o.

Lemma 4.2.5. *The subspace V_σ is A_π-invariant.*

Proof. Let $f \in V_\sigma$. Then $(A_\pi f)(x) \neq 0$ only if $\theta(x) \in \operatorname{supp} f \subset ZK_0$. Since ZK_0 is θ-stable, conclude that $\operatorname{supp} A_\pi f \subset ZK_0$, and so $A_\pi f \in V_\sigma$. □

Corollary 4.2.6. *Let $g \in G$. Then $gH \cdot V_\sigma$ is A_π-invariant if and only if $\theta(g) \equiv g \mod ZH$.*

Proof. For $g = 1$, the result follows from Lemma 4.2.5 and the fact that H is θ-stable. Otherwise, we have $A_\pi \cdot (gH \cdot V_\sigma) = \theta(g)H \cdot V_\sigma$. Suppose $\theta(g) = gzh$ for some $z \in Z$, $h \in H$. Then $\theta(g)H \cdot V_\sigma = gH \cdot V_\sigma$, since z acts as a scalar. On the other hand, if we assume $gH \cdot V_\sigma$ is A_π-invariant, then again we have $\theta(g)H \cdot V_\sigma = gH \cdot V_\sigma$. Therefore, $\theta(g) \equiv g \mod ZH$, by the inequivalence of the components in Corollary 4.2.4. □

From Corollary 3.5.2, the constituents of π_o in which we are interested are the A_π-invariant ones. However, we would also like the inducing data for such constituents to be θ-stable, a property not guaranteed by the previous result.

Proposition 4.2.7. *Let $g \in G$. Then ${}^gK_0$ is θ-stable if and only if $\theta(g) \equiv g \mod ZK_0$. In this case, ${}^g\sigma_o$ is also θ-stable, and $gH \cdot V_\sigma \cong \text{c-Ind}_{({}^gK_0)^+}^{H^+}({}^g\sigma_o)^+$ as H^+-spaces.*

Proof. Since $N_G(K_0) = ZK_0$, it is immediate that ${}^gK_0$ is θ-stable if and only if $\theta(g) \equiv g \mod ZK_0$. Write $\theta(g) = gzk_0$ for some $z \in Z$ and $k_0 \in K_0$. It is not hard to check that $A'_\sigma = \sigma(zk_0)A_\sigma \in \mathrm{Hom}_{{}^gK_0}({}^g\sigma_o, {}^g\sigma_o \circ \theta)$, so that ${}^g\sigma_o$ is θ-stable. Define $({}^g\sigma_o)^+$ via A'_σ. Let $\kappa = \text{c-Ind}_{{}^gK_0}^H {}^g\sigma_o$, and set $A_\kappa = \Phi(A'_\sigma) \in \mathrm{Hom}_H(\kappa, \kappa\circ\theta)$, where Φ is as in Proposition 1.5.2. Use A_κ to define κ^+ as usual. Then $\kappa^+ \cong \text{c-Ind}_{({}^gK_0)^+}^{H^+}({}^g\sigma_o)^+$, by Proposition 1.5.4. We will show $gH \cdot V_\sigma \cong \kappa^+$. Let V_κ be the space of κ (and also of κ^+), and let $T\colon gH\cdot V_\sigma \to V_\kappa$ be the H-isomorphism given in the proof of Proposition 4.2.3. An easy calculation shows that T commutes with the action of θ on the respective spaces, that is, $TA_\pi = A_\kappa T$. So T is in fact an H^+-isomorphism. □

Proposition 4.2.8. *Let $g_1, g_2, \ldots, g_n$ be a set of representatives for G/ZH, and set $U_j = g_jH\cdot V_\sigma$, $1 \le j \le n$. Then we may re-order this set such that, as a H^+-space, $V_\pi = U_1^+ \oplus U_2^+ \oplus \cdots \oplus U_{n'}^+$, for some $n' \le n$. Here, each U_j^+ is defined as in §3.4. Furthermore, $m_k \mid d_\theta$ and $m_k \le m_{k+1}$ for each k.*

Proof. Arrange the elements of G/ZH into $\langle\theta\rangle$-conjugacy classes. That is, consider $gZH \sim g'ZH$ if there exists an integer j such that ${}^{\theta^j}(gZH) = g'ZH$. Let n' be the number of such conjugacy classes. For each $1 \le k \le n'$, choose a ZH-representative g_k that represents each $\langle\theta\rangle$-conjugacy class in order of cardinality of the classes from smallest to largest. Let m_k be the corresponding cardinality. The class of g_k is $\{g_kZH, \theta(g_k)ZH, \ldots, \theta^{m_k-1}(g_k)ZH\}$. In this arrangement we have $m_k \le m_{k+1}$ for each k, and it is straightforward to show that each m_k must divide the order of θ. Choose $n - n'$ more ZH-representatives $g_{n'+1}, \ldots, g_n$ to round out the set. As an H-space, V_π decomposes as $V_\pi = \bigoplus_k U_k$, and for each k and any integer i we have $A_\pi^i \cdot (g_kH)\cdot V_\sigma = \theta^i(g_k)H\cdot V_\sigma$. The result now follows from Proposition 3.4.4. □

Let $g_1, \ldots, g_n$ and $U_1, \ldots, U_n$ be as in the proposition, in the given ordering. Corollaries 3.5.2 and 4.2.6 say that only the pieces which correspond to a representative g_k with $\theta(g_k) \equiv g_k \mod ZH$ make a contribution to Θ_{π^+}. We will always choose $g_1 = 1$. Using the determinant map to induce a bijection between G/ZH and $F^\times/\mathrm{N}_{E/F}E^\times$, we have the following easy lemma.

Lemma 4.2.9. *The only representatives in the collection $\{g_1, \ldots, g_n\}$ which satisfy $\theta(g_k) \equiv g_k \mod ZH$ are*

(i) g_1, if n is odd, or

(ii) g_1 and g_2, if n is even, with $\det g_2 \in \varpi_F^{n/2}\mathrm{N}_{E/F}E^\times$.

Proof. Fix $1 \le k \le n$. Suppose $\det g_k \in \varpi_F^i \mathrm{N}_{E/F}E^\times$, for some $0 \le i \le n-1$. Then $\det\theta(g_k) \in \varpi_F^{n-i}\mathrm{N}_{E/F}E^\times$, and the result follows. □

It remains to specify appropriate matrix coefficients to use in (3.5.4) and (3.5.5). Use the notation of §3.5, recalling that for $1 \le j \le n'$, we set $\pi_j^+ = \pi_o^+|U_j^+$. Having chosen $g_1 = 1$, we have $U_1 = H\cdot V_\sigma \cong \text{c-Ind}_{K_0^+}^{H^+}\sigma_o^+$. Take $K = K_0$. Let $\chi_{\sigma_o^+}$ be the character of $\sigma_o^+ = \sigma^+|K_0^+$, and let $\dot\chi_{\sigma_o^+}$ be its extension by zero to all of H^+. Then $\dot\chi_{\sigma_o^+}$ is a sum of matrix coefficients of π_1^+, with $\dot\chi_{\sigma_o^+}(1) = \deg(\sigma) \ne 0$. By Lemma 4.2.9, this is all we need in the case that n is odd.

If n is even, consider the contragredient spaces $\widetilde{U}_1, \widetilde{U}_2 \subset \widetilde{V}_{\pi_o^+} \subset V_\pi^*$. We have $\widetilde{U}_2 = \pi^*(g_2)\widetilde{U}_1$. Therefore, if φ_1 is any matrix coefficient of π_1^+, the function $\varphi_2\colon H^+ \to \mathbb{C}$ given by $\varphi_2(x) = \varphi_1(x^{g_2})$, $x \in H^+$, is a matrix coefficient of π_2^+. Therefore, the function $x \mapsto \dot\chi_{\sigma_o^+}(x^{g_2})$ is a sum of matrix coefficients of π_2^+. Collect all of this in the following result.

Corollary 4.2.10. *Let s be 1 if n is odd, and 2 if n is even. Then,*

$$\Theta_{\pi^+}(g) = \frac{1}{\deg(\sigma)} \sum_{j=1}^{s} d(\pi_j^+) \int_{Z_0\backslash H^+} \int_{K_0} \dot{\chi}_{\sigma_o^+}({}^{g_j^{-1}xk}g)\,dk\,d\dot{x}, \tag{4.2.3}$$

for $g \in H^1_{\mathrm{qr}}$. If g commutes with θ, then

$$\Theta_{\pi^+}(g) = \frac{|\langle\theta\rangle|}{\deg(\sigma)} \sum_{j=1}^{s} d(\pi_j^+) \int_{Z_0\backslash H} \int_{K_0} \dot{\chi}_{\sigma_o^+}({}^{g_j^{-1}xk}g)\,dk\,d\dot{x}. \tag{4.2.4}$$

Remark. Since $H = G$ for $\operatorname{val}\theta = 1$, in the notation of §3.5 we really have $\pi_1^+ = \pi^+$. Therefore, if we write $\sigma_o = \sigma$ and use $s = 1$ and $g_1 = 1$ in this case, then we may use (4.2.4) for both cases $\operatorname{val}\theta = \pm 1$.

4.3 Further hypotheses

Recall that an element $\gamma \in G$ is called *topologically unipotent* if $\gamma^{p^\ell} \to 1$ as $\ell \to \infty$. If $\gamma \in K_0$ is topologically unipotent, then its image in K_0/K_1 is unipotent. In light of this, we wish to use Chapter 2 to analyze (4.2.4) in the case that σ is Deligne-Lusztig as a representation of K_0/K_1, and g is of the form $\gamma\theta$, for some topologically unipotent element $\gamma \in K_{0,\theta}$. We now make some hypotheses on the structure of G and K_0/K_1 relative to θ to facilitate this analysis.

Recall that in the case that n is even and $\operatorname{val}\theta = -1$, the sum in (4.2.4) has two terms, with the second corresponding to a representative g_2 of G/ZH such that $\det g_2 \in \varpi_F^{n/2} \mathrm{N}_{E/F} E^\times$. We need some notation to be able to make statements which apply simultaneously to both cases. Set

$$I^\theta_{G/ZH} = \begin{cases} \{1,2\}, & n \text{ even and } \operatorname{val}\theta = -1, \\ \{1\}, & \text{otherwise.} \end{cases}$$

Hypothesis H6.

(1) *For each $i \in I^\theta_{G/ZH}$, we may choose the representative g_i so that $\theta(g_i) \equiv g_i \mod Z$.*

(2) *Fix $i \in I^\theta_{G/ZH}$, and suppose $x \in g_i^{-1}H$. If $(1-\theta)(x) \in ZK_0$, then $x \in K_0 g_i^{-1} G_\theta$.*

Remarks.

(1) To ensure that the inducing data of the irreducible subrepresentation of $\pi|H$ associated to the coset $g_i ZH$ is θ-stable, we only need to assume that each g_i satisfies $\theta(g_i) \equiv g_i \mod ZK_0$ (see Proposition 4.2.7). However, in the cases we consider, we may go further by assuming H6(1), significantly simplifying the general formulas to follow. For $i = 1$, H6(1) is always satisfied, since we always take $g_1 = 1$.

(2) Note that H6(1) can fail to be true. For example, suppose $n = 2$ and $[F : F_0] = 2$, with F/F_0 ramified. Let τ be the non-trivial element of $\mathrm{Gal}(F/F_0)$, and let $\theta(x) = {}^t\tau(x)^{-1}$, for $x \in G$. Let $g_2 = \mathrm{diag}(\varpi_F, 1)$, so that $\det g_2 \in \varpi_F \mathrm{N}_{E/F} E^\times$. If -1 is not a square in k_F, then there does not exist $h \in H$ such that $(gh)^{-1}\theta(gh) \in Z$.

Using H6(1), we may apply Proposition 4.2.7 to the case that n is even and $\operatorname{val}\theta = -1$ to see that $d(\pi_2^+) = d(\pi_1^+)$. Indeed, we have $\deg\big(({}^{g_2}\sigma_o)^+\big) = \deg(\sigma_o^+)$ and the measures of $Z_0\backslash({}^{g_2}K_0)^+$ and $Z_0\backslash K_0^+$ relative to

$d\dot{x}$ are equal as well. Also using H6(1), it is straightforward to check that each g_i, for $i \in I^{\theta}_{G/ZH}$, must normalize both $\mathbf{G}_\theta$ and G_θ.

The following lemma approximates H6(2) near θ.

Lemma 4.3.1. *Fix $i \in I^{\theta}_{G/ZH}$, and suppose $\gamma \in K_{0,\theta}$ is topologically unipotent. If $y \in g_i^{-1}H$ such that ${}^y(\gamma\theta) \in (ZK_0)^+$, then $y \in K_0 g_i^{-1} G_\theta$.*

Proof. We will show that $(1-\theta)(y) \in ZK_0$, from which H6(2) will give the result.

By H6(1), $(1-\theta)(g_i^{-1}) \in Z$. If we set $\ell_i = \mathrm{val}_F\big(g_i^{-1}\theta(g_i)\big)$, then $(1-\theta)(g_i^{-1}) \in \varpi_F^{\ell_i} K_0$. By assumption, $y\gamma\theta(y^{-1}) \in ZK_0$. Then, since $\det y\gamma\theta(y^{-1}) \in \varpi_F^{n\ell_i}\mathcal{O}_F^\times$, we have $w = \varpi_F^{-\ell_i}\big({}^y(\gamma\theta)\big) \in K_0^+$. By H2, there exists a positive integer ℓ_θ satisfying $p^{\ell_\theta} \equiv 1 \mod d_\theta$, so that $\theta^{p^{m\ell_\theta}} = \theta$ for any integer $m > 0$. Consider cases based on i, starting with $i = 1$. Now, $\ell_1 = 0$, and since γ is θ-fixed and topologically unipotent, we have

$$w^{p^{m\ell_\theta}} = {}^y\big(\gamma^{p^{m\ell_\theta}}\theta\big) \to {}^y\theta \quad \text{as} \quad m \to \infty.$$

Therefore, since K_0^+ is closed and each element of the sequence $\{w^{p^{m\ell_\theta}}\}_{m\in\mathbb{Z}_{>0}}$ lies therein, we must have ${}^y\theta \in K_0^+$ and $(1-\theta)(y) \in K_0$, and we have finished the proof in this case. Now suppose n is even, $\mathrm{val}\,\theta = -1$, and $i = 2$. Set $\beta = \mathrm{N}_\theta \varpi_F^{-1}$. Since $\mathrm{val}\,\theta = -1$, $\varpi_F\theta(\varpi_F) = \alpha_\theta$, for some $\alpha_\theta \in \mathcal{O}_F^\times$, and it follows that $\beta = \mathrm{N}_{\theta^2}(\alpha_\theta^{-1}) \in \mathcal{O}_F^\times$. Now, $w^{d_\theta} = \beta^{\ell_2}\,{}^y(\gamma\theta)^{d_\theta}$, so $w^{rd_\theta+1} = \varpi_F^{-\ell_2}\beta^{r\ell_2}\,{}^y(\gamma\theta)^{rd_\theta+1}$ for any integer $r \geq 0$. Therefore,

$$w^{p^{m\ell_\theta}} = \varpi_F^{-\ell_2}\,\beta^{r_m\ell_2}\,{}^y(\gamma^{p^{m\ell_\theta}}\theta), \qquad \big(m \in \mathbb{Z}_{>0}\big),$$

where $r_m = (p^{m\ell_\theta}-1)/d_\theta$. Since $\beta \in \mathcal{O}_F^\times$, the sequence $\{\beta^{r_m}\}_{m\in\mathbb{Z}_{>0}}$ has a convergent subsequence. If $\{m_j\}$ is a set of indices such that the sequence $\{\beta^{r_{m_j}}\}$ converges to $\beta_\infty \in \mathcal{O}_F^\times$, then

$$w^{p^{m_j\ell_\theta}} \to \varpi_F^{-\ell_2}\beta_\infty^{\ell_2}\,{}^y\theta \quad \text{as} \quad j \to \infty.$$

As in the previous case, we conclude that $\varpi_F^{-\ell_2}\beta_\infty^{\ell_2}\,{}^y\theta \in K_0^+$, hence $(1-\theta)(y) \in \varpi_F^{\ell_2}K_0$. □

We now assume that the necessary structure exists for it to be possible for σ to be Deligne-Lusztig and θ-stable. Recall that we have chosen a degree n, unramified extension E/F. Any choice of F-basis for E affords an embedding of the torus $\mathrm{R}_{E/F}\,\mathbb{G}_\mathrm{m}$ onto a maximal F-torus of $\mathbf{GL}_n$ (see the example of §1.2.3), and the images of all such embeddings are G-conjugate by the appropriate change of basis matrices. If $\mathbf{S} \subset \mathbf{GL}_n$ is the image of such an embedding, then $\mathbf{S}(F) \simeq E^\times$. As in §1.4.1, lift τ to an element of $\mathrm{Gal}(\bar{F}/F_0)$, and consider θ as an automorphism of the abstract group $\mathbf{GL}_n = \mathbf{GL}_n(\bar{F})$.

Hypothesis H7. *There exists an $\mathcal{O}_F$-basis for $\mathcal{O}_E$ such that the image $\mathbf{S}$ of the corresponding embedding $\mathrm{R}_{E/F}\,\mathbb{G}_\mathrm{m} \hookrightarrow \mathbf{GL}_n$ is θ-stable.*

Let $\mathbf{T}$ be the maximal F_0-torus $\mathrm{R}_{F/F_0}\mathbf{S} \simeq \mathrm{R}_{E/F_0}\,\mathbb{G}_\mathrm{m}$ of $\mathbf{G}$. By Lemma 1.4.3, $\mathbf{T}$ is θ-stable. Under the identification of G with $\mathbf{G}(F_0)$, we have $T = \mathbf{T}(F_0)$ identified with $\mathbf{S}(F)$.

Turn now to the structure over the residue fields. We have assumed that K_0 is θ-stable, so we may also assume that $J \in K_0$. Identify K_0/K_1 with $\mathrm{G} = \mathrm{GL}_n(k_F)$. We will also use θ to denote the automorphism of G which is induced by $\theta|K_0$. This automorphism is also of the form of (1.4.1), with χ trivial, J replaced by $\bar{J}$, and τ replaced by the induced element $\bar{\tau} \in \mathrm{Gal}(k_F/k_{F_0})$.

Let F^{un} be the unramified closure of F. Then $k_{(F^{\mathrm{un}})}$ is the algebraic closure $\bar{k}_F$. Lift $\bar{\tau}$ to the element of $\mathrm{Gal}(\bar{k}_F/k_{F_0})$ induced by $\tau|F^{\mathrm{un}}$. Then we may let θ also denote the automorphism of the abstract group $\mathbf{GL}_n = \mathbf{GL}_n(\bar{k}_F)$ of the form (1.4.1) with respect to $\bar{J}$ and $\bar{\tau}$, and with χ trivial. Let $\bar{\mathcal{B}}_{F/F_0}$ be the fixed k_{F_0}-basis for k_F from §3.1, and take $\{\mathrm{id}, \bar{\tau}, \ldots, \bar{\tau}^{e-1}\}$ as a set of representatives of $\bar{\Sigma} = \mathrm{Gal}(\bar{k}_F/k_{F_0})/\mathrm{Gal}(\bar{k}_F/k_F)$, where e is the ramification index of F/F_0. Let $\mathbf{G} = \mathrm{R}_{k_F/k_{F_0}} \mathbf{GL}_n$, constructed using $\bar{\Sigma}$ and $\bar{\mathcal{B}}_{F/F_0}$ (see §1.2). Then $\mathbf{G}$ is a connected, reductive algebraic group defined over k_{F_0} such that $\mathbf{G}(k_{F_0}) \simeq \mathbf{GL}_n(k_F)$. As over the local fields, we identify $\mathbf{G}(k_{F_0})$ and $\mathbf{GL}_n(k_F)$ via the isomorphism afforded by $\bar{\mathcal{B}}_{F/F_0}$, and also identify these two groups with K_0/K_1. Let θ also denote the semisimple element of $\mathrm{Aut}_{k_{F_0}}(\mathbf{G})$ provided by Proposition 1.4.2(2), from θ as an automorphism of $\mathbf{GL}_n$. Let $\bar{d}_\theta$ be the order of θ in $\mathrm{Aut}_{k_{F_0}}(\mathbf{G})$. Note that $\bar{d}_\theta$ divides d_θ.

Let $\mathsf{S} = \big(\mathbf{S}(F) \cap K_0\big)/K_1$. Then S is the set of k_F-rational points of a maximal, k_F-minisotropic torus $\mathbf{S}$ of $\mathbf{GL}_n$. The torus $\mathbf{S}$ is the image of the embedding $\mathrm{R}_{k_E/k_F}\mathbb{G}_{\mathrm{m}} \hookrightarrow \mathbf{GL}_n$ afforded by the k_F-basis of k_E induced by the basis from H7. Now, since $\mathsf{S} \simeq k_E^\times$, the eigenvalues of any element of S are precisely its $\mathrm{Gal}(k_E/k_F)$-conjugates. Therefore, S is non-degenerate in the sense of property (iii) of [8, Proposition 3.6.1]. By *loc. cit.*, we then have that $\mathbf{S}$ is the unique torus of $\mathbf{GL}_n$ containing S. But since S is θ-stable, it is also contained in $\theta(\mathbf{S})$. Therefore, $\mathbf{S}$ is θ-stable. Let $\mathbf{T}$ be the maximal k_{F_0}-torus $\mathrm{R}_{k_F/k_{F_0}}\mathbf{S}$ of $\mathbf{G}$. Again by Lemma 1.4.3, $\mathbf{T}$ is θ-stable. As well, Lemma 1.2.3(3) says that $\mathbf{T}$ is k_{F_0}-minisotropic. Under the identification of G with $\mathbf{G}(k_{F_0})$, we have $\mathsf{T} = \mathbf{T}(k_{F_0})$ identified with $\mathbf{S}(k_F) \simeq k_E^\times$, and $\mathsf{T} = (T \cap K_0)/K_1$.

The following sets up the necessary structure to use Chapter 2.

Hypothesis H8.

(1) There exists a θ-stable Borel subgroup $\mathbf{B}$ *of* $\mathbf{G}$ *which contains* $\mathbf{T}$.

(2) There exists a θ-stable character λ of T *in general position.*

Remark. We will be interested in values of the Deligne-Lusztig characters of G^+ examined in Chapter 2 on elements of the form $u\theta$, for u a θ-fixed, unipotent element of K_0/K_1. In light of Corollary 2.2.6, in order that these character values are not zero, we have assumed in H8(1) that we have an appropriate θ-stable pair, rather than just an appropriate θ-stable torus.

Since it is true in the cases we will consider, we make the following hypothesis to avoid overly burdensome complications in our general statements.

Hypothesis H9. *The subgroup* $\mathbf{G}_\theta$ *of θ-fixed points of* $\mathbf{G}$ *is connected.*

Remark. Again, $\mathbf{G}_\theta$ will fail to be connected, for example, when it is an orthogonal group.

Combining H8(1) with H9, we have that $\mathbf{T}_\theta$ is a maximal k_{F_0}-torus of $\mathbf{G}_\theta$. By Corollary 1.3.7, it is k_{F_0}-minisotropic. Write $\widetilde{\mathfrak{X}} = \widetilde{\mathfrak{X}}_{\mathbf{T}}(1,1)$. For each $x \in \widetilde{\mathfrak{X}}$, the maximal torus ${}^x\mathbf{T}_\theta$ of $\mathbf{G}_\theta$ is also k_{F_0}-minisotropic, as $x \in \mathsf{G}$ implies that ${}^x\mathbf{T}_\theta$ is isomorphic over k_{F_0} to $\mathbf{T}_\theta$. We would like to be able to construct irreducible, cuspidal representations of G_θ using Deligne-Lusztig induction from each ${}^x\mathsf{T}_\theta$.

Hypothesis H10. *For each $x \in \widetilde{\mathfrak{X}}$, there exists a character λ_x of ${}^x\mathsf{T}_\theta$ which is in general position with respect to $W_{\mathbf{G}_\theta}({}^x\mathbf{T}_\theta)^{k_{F_0}}$.*

We also need to be able to lift these cuspidal representations of G_θ to $K_{0,\theta}$, and induce from the normalizer of $K_{0,\theta}$ to obtain irreducible, supercuspidal representations of G_θ.

Hypothesis H11. *The subgroup $K_{0,\theta}$ is a maximal parahoric subgroup of G_θ whose normalizer in G_θ is $Z(G_\theta)K_{0,\theta}$, and $K_{0,\theta}/K_{1,\theta}$ is naturally isomorphic to G_θ.*

The following hypothesis will allow us to apply the general results of [21] to the supercuspidal representations of G_θ mentioned above.

Hypothesis H12. *The quotient $Z(G_\theta)/Z_0$ is compact.*

Remark. This quotient can fail to be compact, for example, in the case that θ is the inner automorphism of G corresponding to an element which is regular but not elliptic.

Finally, we would also like to use the results of [13, §12.4]. Let $(G_\theta)_{0^+}$ (resp. $(\mathfrak{g}_\theta)_{0^+}$) be the subset of topologically unipotent (resp. topologically nilpotent) elements of G_θ (resp. $\mathfrak{g}_\theta$).

Hypothesis H13.

(1) *For each $\mathsf{x} \in \widetilde{\mathfrak{X}}$, there exists an element $\bar{X}_{\mathsf{x}} \in \mathrm{Lie}({}^{\mathsf{x}}\mathsf{T}_\theta)$ whose centralizer in G_θ is precisely ${}^{\mathsf{x}}\mathsf{T}_\theta$.*

(2) *There exists a G_θ-equivariant map $\log\colon (G_\theta)_{0^+} \to (\mathfrak{g}_\theta)_{0^+}$ whose restriction to $K_0 \cap (G_\theta)_{0^+}$ induces a G_θ-equivariant map $\log\colon (\mathsf{G}_\theta)_{\mathrm{unip}} \to (\mathrm{Lie}(\mathsf{G})_\theta)_{\mathrm{nilp}}$.*

Remark. Both of these hold in the case that θ is Galois. In other cases, if we assume that $\mathbf{G}_\theta$ splits over the maximal unramified extension of F_0, then H13 holds whenever [13, Restrictions 12.4.1] hold (see [13, Lemma 12.4.2]).

4.4 Descent to G_θ

For the remainder, assume that

$$\chi_{\sigma_o^+}|K_0^+ = \varepsilon_+ \, \mathsf{R}_{\mathsf{T}^+}^{\mathsf{G}^+} \lambda^+,$$

where the righthand-side is lifted to K_0^+, λ^+ is some extension of λ to T^+, and we have adjusted the sign by $\varepsilon_+ = \varepsilon_{\mathsf{G}}\, \varepsilon_{\mathsf{T}}$ using [14, Corollary 2.5]. We now use Chapter 2 to express Θ_{π^+}, on elements of K_0^+ near θ, as a linear combination of characters of representations of G_θ. It is necessary here to restrict our attention to G-regular elements, rather than quasi-regular elements. Recall that every G-regular element of G^+ is quasi-regular (Lemma 1.3.14). We first analyze the inner integrals of (4.2.4).

For each pair (i, x), consisting of an integer $i \in I^\theta_{G/ZH}$ and an element $\mathsf{x} \in \widetilde{\mathfrak{X}}$, construct an irreducible, supercuspidal representation $\pi_\theta(i, \mathsf{x})$ of G_θ as follows. We have assumed in H10 that there exists a character λ_{x} of ${}^{\mathsf{x}}\mathsf{T}_\theta$ which is in general position with respect to $W_{\mathbf{G}_\theta}({}^{\mathsf{x}}\mathsf{T}_\theta)^{k_{F_0}}$. Choose such a character λ_{x}. Since ${}^{\mathsf{x}}\mathsf{T}_\theta$ and T_θ are G-conjugate, they have the same k_{F_0}-rank. Therefore, using $\varepsilon_\theta = \varepsilon_{\mathbf{G}_\theta} \cdot \varepsilon_{\mathsf{T}_\theta}$ to adjust the sign ([15, Proposition 12.9]), there exists an irreducible, cuspidal representation σ_{x} of G_θ with character $\chi_{\mathsf{x}} = \varepsilon_\theta \, \mathsf{R}_{{}^{\mathsf{x}}\mathsf{T}_\theta}^{\mathsf{G}_\theta} \lambda_{\mathsf{x}}$. Using the natural isomorphism $\mathsf{G}_\theta \simeq K_{0,\theta}/K_{1,\theta}$ of H11, lift σ_{x} to a representation of $K_{0,\theta}$, and extend this to a representation of $Z(G_\theta)K_{0,\theta}$. Let χ_{x} also denote the character of this representation. Let $M_i = Z(G_\theta)({}^{g_i}K_{0,\theta})$. Since g_i normalizes G_θ, H11 implies that $M_i = N_{G_\theta}({}^{g_i}K_{0,\theta})$, and so

$$\pi_\theta(i,\mathrm{x}) = \text{c-Ind}_{M_i}^{G_\theta}({}^{g_i}\sigma_{\mathrm{x}})$$

is an irreducible, supercuspidal representation of G_θ. Let $\dot\chi_\theta(i,\mathrm{x})$ be the extension of ${}^{g_i}\chi_{\mathrm{x}}$ by zero to G_θ. For certain $g \in (H^1_\theta)_{G\text{-reg}}$, we can use Chapter 2 to relate the inner integral of (4.2.4) to integrals of $\dot\chi_\theta(i,\mathrm{x})$ over $K_{0,\theta}$.

Lemma 4.4.1. *Let γ be a topologically unipotent element of $K_{0,\theta}$ such that $\gamma\theta \in (H^1_\theta)_{G\text{-reg}}$. Let dk' be normalized Haar measure on $K_{0,\theta}$. For $i \in I^\theta_{G/ZH}$, define*

$$f_{\gamma,i} : H \to \mathbb{C}, \qquad x \mapsto \int_{K_{0,\theta}} \dot\chi_{\sigma_o^+}({}^{g_i^{-1}xk'}(\gamma\theta))\,dk'.$$

Fix i.

(1) The function $f_{\gamma,i}$ is locally constant and invariant under left-translation by elements of ${}^{g_i}K_0$. Its support is contained in $({}^{g_i}K_0)G_\theta$.

(2) For $x \in G_\theta$,

$$f_{\gamma,i}(x) = \sigma\big(g_i^{-1}\theta(g_i)\big)\,\varepsilon_+\,\varepsilon_\theta\,|\mathsf{T}/\mathsf{T}_\theta|^{-1}\sum_{\mathrm{x}\in\tilde{\mathfrak{X}}} {}^{\mathrm{x}}\lambda^+(\theta)\int_{K_{0,\theta}}\big(\dot\chi_\theta(i,\mathrm{x})\big)({}^{xk'}\gamma)\,dk'.$$

(3) The support of $f_{\gamma,i}$ is compact modulo Z_0.

Remark. Recall that we have assumed $g_i^{-1}\theta(g_i) \in Z$ for each $i \in I^\theta_{G/ZH}$ (H6(1)). Since σ is irreducible, $\sigma\big(g_i^{-1}\theta(g_i)\big)$ is a scalar operator on W, by Schur's Lemma. In the formula of (2), and throughout the rest of this chapter, we abuse notation and use $\sigma\big(g_i^{-1}\theta(g_i)\big)$ to also denote the element of $\mathbb{C}^\times$ associated to the scalar operator $\sigma\big(g_i^{-1}\theta(g_i)\big)$.

Proof. The integrand in the definition of $f_{\gamma,i}$ is invariant under K_0^+-conjugation of its argument, so in particular $f_{\gamma,i}$ is invariant under left-translation by elements of ${}^{g_i}K_0$. This also implies that $f_{\gamma,i}$ is locally constant. Fix $x \in H$. For $k' \in K_{0,\theta}$, $\dot\chi_{\sigma_o^+}$ is zero on ${}^{g_i^{-1}xk'}(\gamma\theta)$ unless this element lies in $(ZK_0)^+$. Suppose there exists $k' \in K_{0,\theta}$ that satisfies this condition. Then applying Lemma 4.3.1, we have $g_i^{-1}xk' \in K_0 g_i^{-1} G_\theta$, hence $x \in ({}^{g_i}K_0)G_\theta$. This completes the proof of (1).

Now let $x \in G_\theta$. Let k' be any element of $K_{0,\theta}$, and set $w = g_i^{-1}xk'$. For convenience, write $z_i = g_i^{-1}\theta(g_i) \in Z$. Since x and k' are θ-fixed, we have ${}^w\theta = z_i\theta$. Therefore, ${}^w(\gamma\theta) \in (ZK_0)^+$ if and only if ${}^w\gamma \in ZK_0$. However, $\det {}^w\gamma = \det\gamma \in \mathcal{O}_F^\times$, so we have ${}^w\gamma \in ZK_0 = \langle\varpi_F\rangle K_0$ if and only if ${}^w\gamma \in K_0$. Thus,

$$\begin{aligned}\dot\chi_{\sigma_o^+}\big({}^{g_i^{-1}xk'}(\gamma\theta)\big) &= \dot\chi_{\sigma_o^+}\big(z_i({}^w\gamma)\theta\big)\\ &= \begin{cases}\sigma(z_i)\,\varepsilon_+\big(\mathsf{R}^{\mathsf{G}^+}_{\mathsf{T}^+}\lambda^+\big)\big(({}^w\gamma)\theta\big), & {}^w\gamma \in K_0,\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

Since g_i normalizes G_θ, we have ${}^w\gamma \in K_0$ if and only if ${}^w\gamma \in K_{0,\theta}$. Also, ${}^w\gamma$ is topologically unipotent, so if it lies in K_0 we may apply (2.2.2) with $n=1$ and $\vartheta = \theta$ to get

$$(\mathsf{R}^{\mathsf{G}^+}_{\mathsf{T}^+}\lambda^+)\big(({}^w\gamma)\theta\big) = |\mathsf{T}/\mathsf{T}_\theta|^{-1}\sum_{\mathrm{x}\in\tilde{\mathfrak{X}}} {}^{\mathrm{x}}\lambda^+(\theta)\,\mathsf{Q}^{\mathsf{G}_\theta}_{{}^{\mathrm{x}}\mathsf{T}_\theta}\Big(\overline{({}^w\gamma)}\Big).$$

However, for each $\chi \in \widetilde{\mathfrak{X}}$, we have $\mathsf{Q}^{\mathsf{G}_\theta}_{{}^{x}\mathsf{T}_\theta}(v) = \big(\mathsf{R}^{\mathsf{G}_\theta}_{{}^{x}\mathsf{T}_\theta}\lambda_\chi\big)(v)$, for any $v \in (\mathsf{G}_\theta)_{\mathrm{unip}}$ (recall from §2.2 that $\mathsf{G}_{\mathrm{unip}}$ is the set of unipotent elements of G and $(\mathsf{G}_\theta)_{\mathrm{unip}} = \mathsf{G}_\theta \cap \mathsf{G}_{\mathrm{unip}}$). Therefore, if ${}^{w}\gamma \in K_{0,\theta}$, then

$$\dot\chi_{\sigma_o^+}\big(z_i({}^{w}\gamma)\theta\big) = \sigma(z_i)\,\varepsilon_+\,\varepsilon_\theta\,|\mathsf{T}/\mathsf{T}_\theta|^{-1}\sum_{\chi\in\widetilde{\mathfrak{X}}} {}^{\chi}\lambda^+(\theta)\,{}^{g_i}\chi_\chi({}^{xk'}\gamma).$$

Now, ${}^{w}\gamma \in K_{0,\theta}$ if and only if ${}^{xk'}\gamma \in {}^{g_i}K_{0,\theta}$, hence we may write

$$\dot\chi_{\sigma_o^+}(z_i u\theta) = \sigma(z_i)\,\varepsilon_+\,\varepsilon_\theta\,|\mathsf{T}/\mathsf{T}_\theta|^{-1}\sum_{\chi\in\widetilde{\mathfrak{X}}} {}^{\chi}\lambda^+(\theta)\,\big(\dot\chi_\theta(i,\chi)\big)({}^{xk'}\gamma).$$

Substituting this expression as the integrand in the definition of $f_{\gamma,i}$ gives (2).

For (3), note that $K_{0,\theta}$ is a compact, open subgroup of G_θ, and each $\dot\chi_\theta(i,\chi)$ is a sum of matrix coefficients of the irreducible, supercuspidal representation $\pi_\theta(i,\chi)$ of G_θ. Furthermore, by Lemma 1.3.12, γ is regular in G, hence regular in G_θ by Lemma 1.3.10. Therefore, we may apply [21, Part V, §4, Lemma 23] to see that the functions

$$G_\theta \to \mathbb{C}, \qquad\qquad x \mapsto \int_{K_{0,\theta}} \big(\dot\chi_\theta(i,\chi)\big)({}^{xk'}\gamma)\,dk',$$

each have compact support mod $Z(G_\theta)$. By (2), $f_{\gamma,i}|G_\theta$ is a (finite) linear combination of such functions, so there exists a compact set $\omega \subset G_\theta$ such that $\operatorname{supp}\big(f_{\gamma,i}|G_\theta\big) \subseteq \omega Z(G_\theta)$. It follows from (1) that $\operatorname{supp} f_{\gamma,i} \subseteq ({}^{g_i}K_0)\omega Z(G_\theta)$, and statement (3) is now a consequence of H12. □

The lemma allows us to express $\Theta_{\pi^+}(\gamma\theta)$ as a linear combination of the values $\Theta_{\pi_\theta(i,\chi)}(\gamma)$, for $i \in I^\theta_{G/ZH}$ and $\chi \in \widetilde{\mathfrak{X}}$. Note that the representations in the collection $\{\sigma_\chi\}$ all have the same degree. Write $\deg(\sigma_\theta)$ for this common degree. As well, the representations in the collection $\{\pi_\theta(i,\chi)\}$ all have the same formal degree. Write $d(\pi_\theta)$ for this common formal degree.

Theorem 4.4.2. *Let γ be as in Lemma 4.4.1. Then,*

$$\Theta_{\pi^+}(\gamma\theta) = |\langle\theta\rangle|\,\varepsilon_+\,\varepsilon_\theta\,|\mathsf{T}/\mathsf{T}_\theta|^{-1}\,\frac{\deg(\sigma_\theta)}{\deg(\sigma)}\,\frac{d(\pi_1^+)}{d(\pi_\theta)}\sum_{i\in I^\theta_{G/ZH}}\sum_{\chi\in\widetilde{\mathfrak{X}}} {}^{\chi}\lambda^+(\theta)\,\sigma\big(g_i^{-1}\theta(g_i)\big)\,\Theta_{\pi_\theta(i,\chi)}(\gamma).$$

Proof. Let $i \in I^\theta_{G/ZH}$. The integral $\int_{Z_0\backslash H} f_{\gamma,i}(x)\,d\dot x$ converges, by Lemma 4.4.1(3). By invariance of the measure $d\dot x$, we have $\int_{Z_0\backslash H} f_{\gamma,i}(x)\,d\dot x = \int_{Z_0\backslash H} f_{\gamma,i}(xk)\,d\dot x$, for any $k \in K_0$. Thus, using normalized Haar measure dk on K_0, we have

$$\int_{Z_0\backslash H} f_{\gamma,i}(x)\,d\dot x = \int_{K_0}\int_{Z_0\backslash H} f_{\gamma,i}(xk)\,d\dot x\,dk = \int_{Z_0\backslash H}\int_{K_0} f_{\gamma,i}(xk)\,dk\,d\dot x$$
$$= \int_{Z_0\backslash H}\int_{K_0} \dot\chi_{\sigma_o^+}({}^{g_i^{-1}xk}(\gamma\theta))\,dk\,d\dot x, \tag{4.4.1}$$

where the last integral manipulation is achieved by absorbing the integral over $K_{0,\theta}$ from the definition of $f_{\gamma,i}$ into the integral over K_0, using invariance of dk. From (4.2.4), we have

$$\Theta_{\pi^+}(\gamma\theta) = \frac{|\langle\theta\rangle|}{\deg(\sigma)}\sum_{i\in I^\theta_{G/ZH}} d(\pi_i^+)\int_{Z_0\backslash H}\int_{K_0} \dot\chi_{\sigma_o^+}({}^{g_i^{-1}xk}(\gamma\theta))\,dk\,d\dot x.$$

Recall that for n even and $\operatorname{val}\theta = -1$, $d(\pi_2^+) = d(\pi_1^+)$. For all other cases, $I^\theta_{G/ZH} = \{1\}$. Substituting (4.4.1) into the above equation, we have

$$\Theta_{\pi^+}(\gamma\theta) = |\langle\theta\rangle|\,\frac{d(\pi_1^+)}{\deg(\sigma)}\sum_{i\in I^\theta_{G/ZH}}\int_{Z_0\backslash H} f_{\gamma,i}(x)\,d\dot x. \tag{4.4.2}$$

Assume that the invariant measure $d\dot{x}$ has been normalized so that the image of K_0 in $Z_0\backslash H$ has volume 1. Similarly, let $d\dot{x}'$ be invariant measure on $Z_0\backslash G_\theta$, normalized so that the image of $K_{0,\theta}$ in $Z_0\backslash G_\theta$ has volume 1. Then Lemma 4.4.1(1), the invariance of $f_{\gamma,i}$ under left-translation by elements of ${}^{g_i}K_0$, and our choice of normalization of measures $d\dot{x}$ and $d\dot{x}'$ allows us to write

$$\int_{Z_0\backslash H} f_{\gamma,i}(x)\,d\dot{x} = \int_{Z_0\backslash({}^{g_i}K_0)G_\theta} f_{\gamma,i}(x)\,d\dot{x} = \int_{Z_0\backslash G_\theta} f_{\gamma,i}(x')\,d\dot{x}'.$$

Note that in the proof of Lemma 4.4.1(3), we have shown that the support of $f_{\gamma,i}|G_\theta$ is compact modulo Z_0, so that this last integral converges. Now apply Lemma 4.4.1(2) to get

$$\begin{aligned}\int_{Z_0\backslash H} f_{\gamma,i}(x)\,d\dot{x} &= \sigma\big(g_i^{-1}\theta(g_i)\big)\,\varepsilon_+\,\varepsilon_\theta\,|\mathsf{T}/\mathsf{T}_\theta|^{-1}\sum_{\chi\in\tilde{\mathfrak{X}}} {}^{\chi}\lambda^+(\theta)\int_{Z_0\backslash G_\theta}\int_{K_{0,\theta}} \big(\dot{\chi}_\theta(i,\chi)\big)({}^{x_\theta k'}\gamma)\,dk'\,d\dot{x}_\theta\\ &= \sigma\big(g_i^{-1}\theta(g_i)\big)\,\varepsilon_+\,\varepsilon_\theta\,|\mathsf{T}/\mathsf{T}_\theta|^{-1}\sum_{\chi\in\tilde{\mathfrak{X}}} {}^{\chi}\lambda^+(\theta)\,\frac{\deg(\sigma_\chi)}{d\big(\pi_\theta(i,\chi)\big)}\,\Theta_{\pi_\theta(i,\chi)}(\gamma).\end{aligned}$$

Substituting this into (4.4.2) and pulling out the common degrees $\deg(\sigma_\theta)$ and $d(\pi_\theta)$ completes the proof. □

4.5 Transfer to the Lie algebra

We may use [13] to express each $\Theta_{\pi_\theta(i,\chi)}$ in terms of the Fourier transform of an orbital integral on $\mathfrak{g}_\theta$, giving us an expression for Θ_{π^+} as a linear combination of such Fourier transforms.

For $X\in\mathfrak{g}$, write $\mathrm{dN}_\theta(X) = X + \mathrm{d}\theta(X) + \cdots + \mathrm{d}\theta^{d-1}(X)$. Let $\mathfrak{g}_\theta' = \ker(\mathrm{dN}_\theta)$. Then $\mathfrak{g} = \mathfrak{g}_\theta \oplus \mathfrak{g}_\theta'$ as an F_0-space. Let b be as in (3.2.1). Since b is non-degenerate and $\mathrm{d}\theta$-invariant, it follows that $\mathfrak{g}_\theta'$ is the orthogonal complement of $\mathfrak{g}_\theta$ with respect to b. Therefore, the restriction of b to $\mathfrak{g}_\theta$ remains non-degenerate. Fix a non-trivial additive character Λ of F_0 with conductor $\mathscr{P}_{F_0}$. For $f\in C_c^\infty(\mathfrak{g}_\theta)$, the Fourier transform of f is defined by

$$\hat{f}(X) = \int_{\mathfrak{g}_\theta} f(Y)\Lambda\big(\mathrm{b}(X,Y)\big)\,dY, \qquad \big(X\in\mathfrak{g}_\theta\big).$$

Given an element $X\in\mathfrak{g}_\theta$, let $d\dot{x}_\theta$ be the unique (up to a constant) invariant measure on the homogeneous space $G_\theta/C_{G_\theta}(X)$. The orbital integral associated to the G_θ-orbit of X in $\mathfrak{g}_\theta$ is the distribution

$$\mu_X(f) = \int_{G_\theta/C_{G_\theta}(X)} f({}^{x}X)\,d\dot{x}_\theta, \qquad \big(f\in C_c^\infty(\mathfrak{g}_\theta)\big).$$

Define the Fourier transform of μ_X by $\hat{\mu}_X(f) = \mu_X(\hat{f})$. The distribution $\hat{\mu}_X$ is represented by a locally integrable function on $\mathfrak{g}_\theta$ ([20]), which we also denote $\hat{\mu}_X$.

Set $\mathfrak{k}_0 = \mathrm{M}_n(\mathscr{O}_F)$ and $\mathfrak{k}_1 = \mathrm{M}_n(\mathscr{P}_F)$, and let $\mathfrak{k}_{0,\theta}$ and $\mathfrak{k}_{1,\theta}$ be the respective subsets of $\mathrm{d}\theta$-fixed points.

Theorem 4.5.1. *For each $\chi\in\tilde{\mathfrak{X}}$, there exists a regular semisimple element $X_\chi\in\mathfrak{k}_{0,\theta}\smallsetminus\mathfrak{k}_{1,\theta}$ satisfying the following. The image of X_χ in* $\mathrm{Lie}(\mathsf{G})_\theta$ *under the* $\mathrm{mod}\,\mathscr{P}_F$ *map is the element $\bar{X}_\chi\in\mathrm{Lie}({}^{\chi}\mathsf{T}_\theta)$ of H13(1). Also, for each regular element $g\in(G_\theta)_{0^+}$, we have*

$$\Theta_{\pi_\theta(i,\chi)}(g) = d(\pi_\theta)\,\hat{\mu}_{({}^{g_i}X_\chi)}(\log g),$$

where log *is as in H13(2).*

Proof. This is [13, Lemma 12.4.3], noting that in *loc. cit.*, the measures have been normalized in such a way as to eliminate the $d(\pi_\theta)$ factor. □

We now combine Theorems 4.4.2 and 4.5.1.

Theorem 4.5.2. *For any $g \in G$ and any topologically unipotent element $\gamma \in K_{0,\theta}$ such that $\gamma\theta$ is G-regular in G^+, we have*

$$\Theta_{\pi^+}\big({}^g(\gamma\theta)\big) = |\langle\theta\rangle|\,\varepsilon_+\,\varepsilon_\chi\,|\mathsf{T}/\mathsf{T}_\theta|^{-1}\,d(\pi_1^+)\,\frac{\deg(\sigma_\theta)}{\deg(\sigma)}\sum_{i\in I^\theta_{G/ZH}}\sum_{\chi\in\widetilde{\mathfrak{X}}}{}^\chi\lambda^+(\theta)\,\sigma\big(g_i^{-1}\theta(g_i)\big)\,\hat{\mu}_{({}^{g_i}X_\chi)}(\log\gamma).$$

Note that the set of all elements of the form ${}^g(\gamma\theta)$, for g and γ as in the theorem, forms an open neighbourhood of θ in G^+.

4.6 A special case

In all of the cases we will consider, we will be able to apply Hilbert's Theorem 90 in one way or another to show that $(1-\theta)\colon \mathsf{T}\to\ker\mathrm{N}_\theta$ is surjective. We will now restate the results of §§4.4–4.5 in this case.

Theorem 2.2.9 says that the reduction formula for $\mathsf{R}^{\mathsf{G}^+}_{\mathsf{T}^+}\lambda^+$, on elements $u\theta$ with $u\in(\mathsf{G}_\theta)_{\mathrm{unip}}$, is now reduced to a single term. Therefore, we may simplify the construction of the supercuspidal representations $\{\pi_\theta(i,\chi)\mid i\in I_{G/ZH},\ \chi\in\widetilde{\mathfrak{X}}\}$ of G_θ in §4.4 as follows. First, we may replace H10 by the assumption that there exists an irreducible character λ_θ of T_θ which is in general position. Choose such a character λ_θ, and from it obtain a cuspidal representation σ_θ of G_θ. Then construct only one or two (depending on $\operatorname{val}\theta$ and n) representations $\{\pi_\theta(i)\mid i\in I_{G/ZH}\}$, similarly to §4.4. Again, write $d(\pi_\theta)$ for their common formal degree. Using (2.2.5) instead of (2.2.2), Lemma 4.4.1(2) then becomes

$$f_{\gamma,i}(x) = \varepsilon_+\,\varepsilon_\theta\,\lambda^+(\theta)\,\sigma\big(g_i^{-1}\theta(g_i)\big)\int_{K_{0,\theta}}\big(\dot\chi_\theta(i)\big)({}^{xk'}\gamma)\,dk', \qquad \big(i\in I^\theta_{G/ZH},\ x\in G_\theta\big),$$

where each $\dot\chi_\theta(i)$ is defined similarly to the functions $\dot\chi_\theta(i,\chi)$ in §4.4. Theorem 4.4.2 becomes

$$\Theta_{\pi^+}(\gamma\theta) = |\langle\theta\rangle|\,\varepsilon_+\,\varepsilon_\theta\,\lambda^+(\theta)\,\frac{\deg(\sigma_\theta)}{\deg(\sigma)}\,\frac{d(\pi_1^+)}{d(\pi)}\sum_{i\in I^\theta_{G/ZH}}\sigma\big(g_i^{-1}\theta(g_i)\big)\,\Theta_{\pi_\theta(i)}(\gamma).$$

We may also replace H13(1) by the assumption that there exists an element $\bar{X}_\theta\in\operatorname{Lie}(\mathsf{T}_\theta)$ whose centralizer in G_θ is precisely T_θ. Now Theorem 4.5.2 simplifies to

$$\Theta_{\pi^+}\big({}^g(\gamma\theta)\big) = |\langle\theta\rangle|\,\varepsilon_+\,\varepsilon_\theta\,\lambda^+(\theta)\,d(\pi_1^+)\,\frac{\deg(\sigma_\theta)}{\deg(\sigma)}\sum_{i\in I^\theta_{G/ZH}}\sigma\big(g_i^{-1}\theta(g_i)\big)\,\hat{\mu}_{({}^{g_i}X_\theta)}(\log\gamma), \tag{4.6.1}$$

choosing a single regular semisimple element $X_\theta\in\mathfrak{k}_{0,\theta}\smallsetminus\mathfrak{k}_{1,\theta}$ whose image in $\operatorname{Lie}(\mathsf{G})_\theta$ under the $\bmod\ \mathscr{P}_F$ map is $\bar{X}_\theta$.

5. EXAMINATION OF SOME CASES

We now apply the general results of the previous sections to several cases of particular interest.

5.1 The unramified Galois case

In this section, we examine the case of $\operatorname{val}\theta = 1$, $J = 1$ and F/F_0 unramified.

5.1.1 Definition of θ. Use the general setup of §3.1 and §4.3. Assume that F/F_0 is unramified, with $1 < d = [F : F_0] < \infty$ relatively prime to p. To ensure the existence of a stable elliptic torus, we also assume that d is relatively prime to n. As in §1.2.4, τ induces a semisimple F_0-automorphism $\theta = \eta_\tau$ of $\mathbf{G} = \mathrm{R}_{F/F_0}\,\mathbf{GL}_n$. Since F/F_0 is unramified, we have $\mathrm{Gal}(k_F/k_{F_0}) = \langle\bar\tau\rangle$. On $\mathbf{G} = \mathrm{R}_{k_F/k_{F_0}}\,\mathbf{GL}_n$, we also have $\theta = \eta_{\bar\tau}$.

5.1.2 Verification of hypotheses.

H1, page *31.* This is satisfied by the definition of θ above.

H2, page *32.* We have $d_\theta = d = [F : F_0]$, and we have assumed that $\gcd(d, p) = 1$.

H3, page *32.* The set $(\mathbf{GL}_n)^\Sigma_{\psi_\tau}$ of ψ_τ-fixed points of $(\mathbf{GL}_n)^\Sigma$ is just the image of $\mathbf{GL}_n$ under the diagonal embedding, and therefore is connected. Hence, $\mathbf{G}_\theta$ is also connected.

H4, page *32* / *H5,* page *44.* The maximal parahoric subgroup $K_0 = \mathrm{GL}_n(\mathscr{O}_F)$ is θ-stable.

H6(1), page *48.* There is nothing to verify, since $\operatorname{val}\theta = 1$ implies $I^\theta_{G/ZH} = \{1\}$, and $g_1 = 1$.

H9, page *50.* By the same argument as for H3, $\mathbf{G}_\theta$ is connected.

H11, page *51.* We have $K_{0,\theta} = \mathrm{GL}_n(\mathscr{O}_{F_0})$ and $K_{1,\theta} = 1 + \mathrm{M}_n(\mathscr{P}_{F_0})$, so $K_{0,\theta}/K_{1,\theta}$ is naturally isomorphic to $\mathsf{G}_\theta = \mathbf{GL}_n(k_{F_0})$. Also, $N_{G_\theta}(K_{0,\theta}) = F_0^\times K_{0,\theta} = Z(G_\theta)K_{0,\theta}$.

H12, page *51.* Since $\operatorname{val}\theta = 1$, we choose $H = G$, and so $Z_0 = Z(G)_\theta = F_0^\times$. Since $G_\theta = \mathbf{GL}_n(F_0)$, the quotient $Z(G_\theta)/Z_0$ is trivial.

H13(2), page *51.* Let $\mathfrak{b}_1$ be the subset of $\mathfrak{k}_0$ consisting of elements whose $(i,j)^{\text{th}}$ entries lie in $\mathscr{P}_F$ for $i \geq j$, and let $B_1 = 1 + \mathfrak{b}_1$. Write $\mathfrak{b}_{1,\theta} = (\mathfrak{b}_1)_\theta$ and $B_{1,\theta} = (B_1)_\theta$. Then $(\mathfrak{g}_\theta)_{0^+} = {}^{G_\theta}(\mathfrak{b}_{1,\theta})$ and $(G_\theta)_{0^+} = {}^{G_\theta}(B_{1,\theta})$. Clearly, it suffices to take $\log\colon x \mapsto x - 1$, the inverse of the truncated exponential map $\exp\colon X \mapsto 1 + X$.

Remaining hypotheses. We leave H6(2), H7, H8, H10, and H13(1) to be verified below.

5.1.3 A factorization result. In this section (and this section only), we may drop the assumption that d and n are relatively prime. Our aim is to prove the following factorization result, which is equivalent to H6(2) (page 48) for the present case.

Lemma 5.1.1. *For any $g \in G$ such that $g\theta(g^{-1}) \in K_0$, there exists an element $k \in K_0$ such that $g\theta(g^{-1}) = k\theta(k^{-1})$.*

First, we verify a variant of the additive version of Hilbert's Theorem 90.

Lemma 5.1.2. *For any $\alpha \in \mathcal{O}_F$ such that $\mathrm{Tr}_{F/F_0}\alpha = 0$, there exists an element $\beta \in \mathcal{O}_F$ such that $\alpha = \beta - \tau(\beta)$.*

Proof. Since $\gcd(d,p) = 1$, d lies in $\mathcal{O}_F^\times$. Take $\beta = d^{-1}\sum_{j=0}^{d-2}(d-j-1)\tau^j(\alpha)$. □

Note that any $\alpha \in \mathcal{O}_F \cap (1-\tau)F$ satisfies the hypothesis of Lemma 5.1.2. Considering θ acting on F^n coordinatewise, we may extend this result to elements of $\mathcal{O}_F^n \cap (1-\theta)F^n$.

Proof of Lemma 5.1.1. Suppose $g \in G$ with $g\theta(g^{-1}) \in K_0$. Let U be the unipotent radical of the upper-triangular Borel subgroup of G. We first consider the case that $g \in U$, proceeding by induction on n. If $n = 1$, then take $k = g = 1$. For $n > 1$, write $g = \left(\begin{smallmatrix} g_0 & C \\ 0 & 1 \end{smallmatrix}\right)$, for some $C \in F^{n-1}$ and some upper-triangular unipotent $g_0 \in \mathbf{GL}_{n-1}(F)$. Then

$$g\theta(g^{-1}) = \begin{pmatrix} g_0\tau(g_0^{-1}) & C - g_0\tau(g_0^{-1})\tau(C) \\ 0 & 1 \end{pmatrix}.$$

Now, $g_0\tau(g_0^{-1})$ must lie in $\mathrm{GL}_{n-1}(\mathcal{O}_F)$, so by induction there exists $k_0 \in \mathrm{GL}_{n-1}(\mathcal{O}_F)$ such that $g_0\tau(g_0^{-1}) = k_0\tau(k_0^{-1})$. We must also have $Y - \tau(Y) \in \mathcal{O}_F^{n-1}$, for $Y = k_0^{-1}C$, so by Lemma 5.1.2 there exists $Y' \in \mathcal{O}_F^{n-1}$ such that $Y' - \tau(Y') = Y - \tau(Y)$. Therefore, we may take $k = \left(\begin{smallmatrix} k_0 & k_0Y' \\ 0 & 1 \end{smallmatrix}\right)$.

Now consider any $g \in G$ such that $g\theta(g^{-1}) \in K_0$. By multiplying g on the left by elementary matrices in K_0, we may row-reduce g to an upper-triangular matrix which has the same diagonal entries as $x = \mathrm{diag}(\varpi^{j_1}, \ldots, \varpi^{j_n})$, for some integers j_i. Let $k_1 \in K_0$ be the product of these elementary matrices, so that $y = k_1gx^{-1} \in U$. But $x \in G_\theta$, so $y\theta(y^{-1}) = k_1g\theta(g^{-1})\theta(k_1^{-1}) \in K_0$. By the previous case, there exists $k_y \in K_0$ with $k_y\theta(k_y^{-1}) = y\theta(y^{-1})$, so take $k = k_1^{-1}k_y$. □

5.1.4 A θ-stable elliptic torus.

We now provide an $\mathcal{O}_F$-basis for $\mathcal{O}_E$ to satisfy H7 (page 49), for E a degree n unramified extension of F. As well, we will show that the resulting torus T of G satisfies the hypothesis of §4.6, and verify H13(1). Let E_0 be a degree n, unramified extension of F_0, and let E be the composite extension FE_0 of F_0. Since $\gcd(n,d) = 1$, we have $[E : E_0] = d$ and $[E : F] = n$, so that we have associated diagrams of p-adic fields (with all extensions unramified) and residue fields as below, where also k_E is the composite $k_Fk_{E_0}$.

$$\begin{tikzcd} E \arrow[d, dash, "d"'] \arrow[dr, dash, "n"] & \\ E_0 \arrow[dr, dash, "n"'] & F \arrow[d, dash, "d"] \\ & F_0 \end{tikzcd} \qquad \begin{tikzcd} k_E \arrow[d, dash, "d"'] \arrow[dr, dash, "n"] & \\ k_{E_0} \arrow[dr, dash, "n"'] & k_F \arrow[d, dash, "d"] \\ & k_{F_0} \end{tikzcd}$$

Since restriction to F defines an isomorphism of $\mathrm{Gal}(E/E_0)$ onto $\mathrm{Gal}(F/F_0)$, there exists a lift of τ to an element of $\mathrm{Gal}(E/F_0)$ such that E_0 is the fixed field of $\langle\tau\rangle$. Assume that we have chosen our lift of τ

into $\mathrm{Gal}(\bar{F}/F_0)$, as in §1.4.1 and §4.3, so that $\tau|E$ is such a lift into $\mathrm{Gal}(E/F_0)$. Choose any $\mathcal{O}_{F_0}$-basis $\mathcal{B} = \{v_1,\ldots,v_n\}$ for $\mathcal{O}_{E_0}$. Then $\mathcal{B}$ is also an $\mathcal{O}_F$-basis for $\mathcal{O}_E$, and we claim that this basis will satisfy H7. To simplify some technical details further on, choose $v_1 = 1$. Take $\{\mathrm{id},\tau^d,\tau^{2d},\ldots,\tau^{(n-1)d}\}$ as a set of representatives for $\Sigma' = \mathrm{Gal}(\bar{F}/F)/\mathrm{Gal}(\bar{F}/E)$. Using these representatives and F-basis $\mathcal{B}$ of E, follow the example of §1.2.3 to form the F-torus $\mathbf{S} = \mathrm{R}_{E/F}\mathbb{G}_{\mathrm{m}}$. But $\{\tau^{(i-1)d}\}$ is also a set of representatives of $\Sigma'_0 = \mathrm{Gal}(\bar{F}/F_0)/\mathrm{Gal}(\bar{F}/E_0)$, so in fact $\mathrm{R}_{E_0/F_0}\mathbb{G}_{\mathrm{m}}$, constructed using $\mathcal{B}$ as an F_0-basis for E_0, is equal to $\mathbf{S}$. Therefore, $\mathbf{S}$ is an F_0-torus. Identify $\mathbf{S}$ with its image in $\mathbf{GL}_n$ under the isomorphism given in the cited example, using constants $\{c_{ijk}\} \subset F$ such that $v_i v_j = \sum_k c_{ijk} v_k$. Then elements of $\mathbf{S}$ have the form $\left(\sum_k x_k c_{kji}\right)$ for some $(x_k) \in \bar{F}^n$. Now, our constants $\{c_{ijk}\}$ actually lie in F_0, so the chosen embedding $\mathbf{S} \hookrightarrow \mathbf{GL}_n$ is defined over F_0. Therefore, $\mathbf{S}$ is τ-stable, and we have verified H7. As in §4.3, the torus $\mathbf{T} = \mathrm{R}_{F/F_0}\mathbf{S} \subset \mathbf{G}$ is θ-stable, and $T = \mathbf{T}(F_0) \simeq \mathbf{S}(F) \simeq E^\times$. The isomorphism $\tilde{f}_{\mathcal{B}}$ induces an action of θ on $E^\times$, given by

$$\theta(\alpha_1 v_1 + \cdots + \alpha_n v_n) = \tau(\alpha_1)v_1 + \cdots + \tau(\alpha_n)v_n, \qquad \big((\alpha_i) \in F^n\big).$$

However, $\mathcal{B}$ is contained in E_0, and so this induced action of θ on $E^\times$ is given by τ.

Similarly, for T as in §4.3, we have $\mathsf{T} = \mathsf{T}(k_{F_0}) \simeq \mathsf{S}(k_F) \simeq k_E^\times$, and the induced action of θ on $k_E^\times$ is given by $\bar{\tau}$. Identify T and $k_E^\times$, and note that restricted to T, we have $\mathrm{N}_\theta = \mathrm{N}_{k_E/k_{E_0}}$. Therefore, by Hilbert's Theorem 90 we have $(1-\theta)(\mathsf{T}) = \ker \mathrm{N}_\theta$. This puts us in the situation of §4.6. As discussed in *loc. cit.*, to satisfy H13(1), we only need to show that there exists an element $\bar{X}_\theta \in \mathrm{Lie}(\mathsf{T}_\theta)$ whose centralizer in G_θ is precisely T_θ. For this, it suffices to show that T_θ is non-degenerate in G_θ in the sense of [8, Proposition 3.6.1]. However, G_θ is isomorphic over k_F to GL_n, via the composition of the map $\mathsf{G} \to \mathsf{GL}_n^\Sigma$ which defines G with projection $\mathsf{GL}_n^\Sigma \to \mathsf{GL}_n$ onto the first component of GL_n^Σ. Since this same map also induces the isomorphism $\mathsf{G}(k_{F_0}) \simeq \mathsf{GL}_n(k_F)$, to verify (iii) of [8, Proposition 3.6.1] it suffices to show that T_θ contains an element with distinct eigenvalues as an element of $\mathsf{GL}_n(k_F)$. Using the identification $\mathsf{T} \simeq k_E^\times$, it suffices to show that $(k_E^\times)_\theta$ contains an element whose $\mathrm{Gal}(k_E/k_F)$-conjugates are all distinct. Now, $(k_E^\times)_\theta \simeq k_{E_0}^\times$. Write $q = q_{F_0}$. If ζ is a generator of $k_E^\times$, then ζ^m is a generator of $k_{E_0}^\times$, for $m = (q^{dn}-1)/(q^n-1)$. Since $\gcd(n,d) = 1$, it follows that ζ^m is the required element of $(k_E^\times)_\theta$.

5.1.5 A θ-stable Borel subgroup. In this section, we verify H8(1) (page 50). First, we will construct a θ-stable Borel subgroup of $\mathbf{G}$ which contains our θ-stable torus $\mathbf{T}$. Fortunately, such subgroups are plentiful. Note that it suffices to find a ψ_τ-stable Borel subgroup of $(\mathbf{GL}_n)^\Sigma$ which contains $\mathbf{S}^\Sigma$.

Let $\mathbf{C} \supset \mathbf{D}$ be the pair in $\mathbf{GL}_n$ consisting of the Borel subgroup of upper-triangular matrices and the torus of diagonal matrices. Both are defined over F_0, so $\mathbf{C}^\Sigma$ and $\mathbf{D}^\Sigma$ are just the direct products of d copies of $\mathbf{C}$ and $\mathbf{D}$, respectively, and hence are ψ_τ-stable. As well, $N_{(\mathbf{GL}_n)^\Sigma}(\mathbf{D}^\Sigma)$ is just the direct product of d copies of $N_{\mathbf{GL}_n}(\mathbf{D})$, and so any Borel subgroup of $(\mathbf{GL}_n)^\Sigma$ containing $\mathbf{D}^\Sigma$ is of the form $(\mathbf{C}^\Sigma)^x = \prod_j \mathbf{C}^{x_j}$, where $x = (x_k)$ for some $\{x_k\} \subset N_{\mathbf{GL}_n}(\mathbf{D})$. Let $A_{\mathcal{B}}$ be the invertible $n \times n$ matrix $\big(\tau^{(i-1)d}(v_j)\big)$, and let $A_{\mathcal{B}}^\Sigma$ be the element of $(\mathbf{GL}_n)^\Sigma_{\psi_\theta}$ obtained by taking d copies of $A_{\mathcal{B}}$. Then $\mathbf{S} = \mathbf{D}^{A_{\mathcal{B}}}$, as in the example of §1.2.3, and so $\mathbf{S}^\Sigma = (\mathbf{D}^\Sigma)^{A_{\mathcal{B}}^\Sigma}$. Therefore, any Borel subgroup $\mathbf{B} \subset (\mathbf{GL}_n)^\Sigma$ containing $\mathbf{S}^\Sigma$ is of the form $(\mathbf{C}^\Sigma)^{xA_{\mathcal{B}}^\Sigma}$, for some $x \in N_{(\mathbf{GL}_n)^\Sigma}(\mathbf{D}^\Sigma)$. In particular, if we take x to be in the image of the diagonal embedding of $N_{\mathbf{GL}_n}(\mathbf{D})$ into $N_{(\mathbf{GL}_n)^\Sigma}(\mathbf{D}^\Sigma)$, then x is ψ_τ-fixed, and both $(\mathbf{C}^\Sigma)^x$ and $\mathbf{B}$ are ψ_τ-stable.

Again, since all of our local field extensions are unramified, the Galois groups between corresponding local and residue field extensions are isomorphic. Thus, the above argument may be repeated in $\mathbf{GL}_n$ and $\mathbf{G}$, implying the existence of a θ-stable Borel subgroup of $\mathbf{G}$ which contains $\mathbf{T}$. This completes the verification of H8(1).

5.1.6 Characters of T and T_θ. Here we verify H8(2) (page 50) and H10 (page 50). Write $q = q_{F_0}$. Fix a primitive $(q^{dn}-1)^{\text{th}}$ root of unity $\eta \in \mathbb{C}^\times$ and a generator ζ of $k_E^\times$. Any character of $k_E^\times \simeq \mathsf{T}$ is of the form $\lambda_\ell(\zeta) = \eta^\ell$ for some integer $0 \le \ell \le q^{dn}-2$. On the other hand, $\bar{\tau}$ is a generator of $\mathrm{Gal}(k_E/k_{E_0})$, so $\bar{\tau}(\zeta) = \zeta^{q^{mn}}$ for some integer m, with $0 \le m \le d-1$ and $\gcd(m,d) = 1$. Therefore, since $\gcd(q^{mn}-1, q^{dn}-1) = q^n - 1$, a character λ_ℓ is θ-stable if and only if $\ell = \ell_0(q^{dn}-1)/(q^n-1)$ for some $0 \le \ell_0 \le q^n - 2$. To verify H8(2), we would like to find such a θ-stable character λ_ℓ which is in general position.

In particular, consider $\lambda = \lambda_\ell$, for $\ell = \sum_{j=0}^{d-1} q^{jn}$. Then ℓ is of the above form, with $\ell_0 = 1$, and so λ is θ-stable. Now, $\mathrm{Gal}(k_E/k_F) \simeq W_{\mathrm{G}}(\mathsf{T})$ is generated by the Frobenius automorphism $\zeta \mapsto \zeta^{q^d}$. Suppose we have $\lambda(\zeta^{q^{jd}}) = \lambda(\zeta)$ for some $1 \le j \le n$. Then $1 = \lambda(\zeta^{-1})\lambda(\zeta^{q^{jd}}) = \eta^{\ell(q^{jd}-1)}$, so $q^{dn}-1$ divides $\ell(q^{jd}-1)$. But then $q^n - 1$ divides $q^{jd}-1$, which is only possible if $n|jd$. Since n and d are relatively prime, we can only have $j = n$, and thus λ is in general position.

As mentioned in §5.1.4, we are in the situation of §4.6. Therefore, instead of H10, we need only show that there exists a character of T_θ which is in general position with respect to $W_{\mathbf{G}_\theta}(\mathsf{T}_\theta)^{k_{F_0}}$. In §5.1.4, we also showed that T_θ was non-degenerate in G_θ. Therefore, by [8, Corollary 3.6.5], $W_{\mathbf{G}_\theta}(\mathsf{T}_\theta)^{k_{F_0}} \simeq N_{\mathrm{G}_\theta}(\mathsf{T}_\theta)/\mathsf{T}_\theta \simeq \mathrm{Gal}(k_{E_0}/k_{F_0})$. If $\eta_0 \in \mathbb{C}^\times$ is a primitive $(q^n-1)^{\text{th}}$ root of unity and ζ_0 is a generator of $k_{E_0}^\times$, then the character $\zeta_0 \mapsto \eta_0$ is in general position.

5.1.7 The result. As noted in §5.1.4, we are in the situation of §4.6. From (4.6.1), we have

$$\Theta_{\pi^+}\big({}^g(\gamma\theta)\big) = [F : F_0]\, \varepsilon_+\, \varepsilon_\theta\, \lambda^+(\theta)\, d(\pi^+)\, \frac{\deg(\sigma_\theta)}{\deg(\sigma)}\, \hat{\mu}_{X_\theta}(\gamma - 1),$$

for any $g \in G$ and any $\gamma \in B_{1,\theta}$ such that $\gamma\theta$ is G-regular in G^+. Finally, from [15, Proposition 12.9], we have

$$\frac{\deg(\sigma_\theta)}{\deg(\sigma)} = \left\| \frac{\mathrm{G}}{\mathrm{G}_\theta} \right\|_{p'}^{-1} \left| \frac{\mathsf{T}}{\mathsf{T}_\theta} \right| = \left(\prod_{i=1}^{n-1} S_{\mathrm{geom}}([F:F_0], q_{F_0}^i) \right)^{-1}$$

where $S_{\mathrm{geom}}(\ell, x) = (x^\ell - 1)/(x-1)$ for $\ell \in \mathbb{Z}_{>0}$, $x \in \mathbb{R}$.

5.2 The symplectic case

In this section, we examine the case of $\operatorname{val}\theta = -1$, $F_0 = F$, $n = 2m$ for some $m \in \mathbb{Z}_{>0}$, and J a skew-symmetric element of $\mathrm{GL}_n(\mathcal{O}_F)$.

5.2.1 Skew-symmetric bilinear forms on $\mathcal{O}_F$-modules. First we will verify that the familiar result of equivalence of skew-symmetric bilinear forms on finite-dimensional vector spaces holds for the case of finitely generated $\mathcal{O}_F$-modules. For the basic facts concerning forms on modules over rings, see [6].

Let M be a torsion-free, finitely generated $\mathcal{O}_F$-module. By [9, Ch. 7, Lemma 5.3], M is in fact a free

$\mathscr{O}_F$-module. Let $\mathsf{f}\colon M \times M \to \mathscr{O}_F$ be a non-singular, skew-symmetric bilinear form on M. By [6, §5, n° 1, Théorème 1], M must be of even rank, say $2m$.

Theorem 5.2.1. *There exists a basis $\mathcal{B}$ of M such that*

$$[\mathsf{f}]_{\mathcal{B}} = \begin{pmatrix} & 1 & & & \\ -1 & & & & \\ & & \ddots & & \\ & & & & 1 \\ & & & -1 & \end{pmatrix}.$$

Proof. Choose any basis $\mathcal{B}' = \{x'_1, \dots, x'_{2m}\}$ of M and set $x_1 = x'_1$. Let $J = [\mathsf{f}]_{\mathcal{B}'}$. We have $\det J \in \mathscr{O}_F^\times$, since f is non-singular. Therefore, there exists an index j_0 such that $J_{1j_0} = \mathsf{f}(x_1, x'_{j_0}) \in \mathscr{O}_F^\times$. Note that $J_{11} = \mathsf{f}(x_1, x_1) = 0$, so $j_0 > 1$. Let $x_2 = \mathsf{f}(x_1, x'_{j_0})^{-1} x'_{j_0}$. Then $\mathsf{f}(x_1, x_2) = 1$.

If $m = 1$, then we must have $j_0 = 2$, and $\mathcal{B} = \{x_1, x_2\}$ is the required basis. For $m \geq 2$, proceed by induction on m. Let N be the $\mathscr{O}_F$-submodule of M generated by $\{x_1, x_2\}$ and let $N^\perp$ be the orthogonal complement to N with respect to f. Suppose $x \in N \cap N^\perp$. Write $x = a_1 x_1 + a_2 x_2$ for some $a_1, a_2 \in \mathscr{O}_F$. Then $\mathsf{f}(x, x_1) = a_2$ and $\mathsf{f}(x, x_2) = a_1$. But then $x \in N^\perp$ implies $a_1 = a_2 = 0$, so $x = 0$. Thus $N \cap N^\perp = \{0\}$. Now let x be any element of M, and set $x' = \mathsf{f}(x, x_1)x_2 - \mathsf{f}(x, x_2)x_1$ and $x'' = x - x'$. Then $x' \in N$, $x'' \in N^\perp$, and clearly $x = x' + x''$. Therefore $M = N \oplus N^\perp$.

Clearly N is free of rank 2. Since $\mathscr{O}_F$ is a principal ideal domain, $N^\perp$ is also free and must have rank $\leq 2n$. Since the ranks of N and $N^\perp$ must sum to $2n$, we have that $N^\perp$ is of rank $2(n-1)$. Choose a basis $\mathcal{D}$ of $N^\perp$ and let $\mathcal{B}''$ be the basis $\{x_1, x_2\} \cup \mathcal{D}$ of M. Then

$$[\mathsf{f}]_{\mathcal{B}''} = \begin{pmatrix} & 1 & \\ -1 & & \\ & & [\mathsf{f}|N^\perp]_{\mathcal{D}} \end{pmatrix},$$

and so since $[\mathsf{f}]_{\mathcal{B}''}$ is skew-symmetric and invertible, $\left[\mathsf{f}|N^\perp\right]_{\mathcal{D}}$ must be as well. Therefore, $\mathsf{f}|N^\perp$ is non-singular and skew-symmetric, and so applying our induction hypothesis we obtain a basis $\mathcal{D}'$ of $N^\perp$ such that $\left[\mathsf{f}|N^\perp\right]_{\mathcal{D}'}$ has the required form. Taking $\mathcal{B} = \{x_1, x_2\} \cup \mathcal{D}'$ gives the required basis of M. □

Corollary 5.2.2. *For any non-singular, skew-symmetric bilinear form g on M there exists an invertible element $A \in \mathrm{End}_{\mathscr{O}_F}(M)$ such that $\mathsf{g}(x, y) = \mathsf{f}(Ax, Ay)$ for all $x, y \in M$.*

Proof. By the theorem, there exist bases $\mathcal{B} = \{x_1, \dots, x_{2m}\}$ and $\mathcal{B}' = \{x'_1, \dots, x'_{2m}\}$ of M such that

$$\mathsf{f}(x_i, x_j) = \mathsf{g}(x'_i, x'_j) = \begin{cases} 1, & 1 \leq i \leq 2m-1,\ j = i+1, \\ 0, & 1 \leq i \leq j \leq 2m,\ j \neq i+1. \end{cases}$$

So take A defined by $Ax'_i = x_i$, for $i = 1, 2, \dots, 2m$, and extend linearly. □

We now prove a factorization result for linear operators on M which are self-adjoint with respect to f, to be used in verifying H6(2). Recall that given an element $A \in \mathrm{End}_{\mathscr{O}_F}(M)$, there exists a unique adjoint operator ${}^{\mathsf{f}}A \in \mathrm{End}_{\mathscr{O}_F}(M)$ such that $\mathsf{f}(Ax, y) = \mathsf{f}(x, {}^{\mathsf{f}}Ay)$, for all $x, y \in M$.

Corollary 5.2.3. *Let $S \in \mathrm{End}_{\mathcal{O}_F}(M)$ be invertible and self-adjoint with respect to f. Then there exists an invertible element $A \in \mathrm{End}_{\mathcal{O}_F}(M)$ such that $S = {}^{\mathsf{f}}AA$.*

Proof. Let $\mathsf{g}\colon M \times M \to \mathcal{O}_F$ by $\mathsf{g}(x,y) = \mathsf{f}(Sx,y)$. Then g is skew-symmetric and non-singular. By the preceding corollary, there exists an invertible element $A \in \mathrm{End}_{\mathcal{O}_F}(M)$ such that $\mathsf{g}(x,y) = \mathsf{f}(Ax,Ay)$ for all $x,y \in M$. Therefore, $\mathsf{f}(Ax,y) = \mathsf{g}(x,A^{-1}y) = \mathsf{f}(x,SA^{-1}y)$, hence ${}^{\mathsf{f}}A = SA^{-1}$. □

5.2.2 Definition of θ.

Let $n = 2m$ for some $m \in \mathbb{Z}_{>0}$, and set $\mathbf{G} = \mathbf{GL}_n$ and $G = \mathbf{G}(F)$. Let L be a degree m unramified extension of F, and let $\mathsf{f}\colon L \times L \to F$ be the trace form, *i.e.*,

$$\mathsf{f}(\alpha,\beta) = \mathrm{Tr}_{L/F}\,\alpha\beta, \qquad (\alpha,\beta \in L).$$

Then f is a non-singular, symmetric F-bilinear form on L. Clearly $\mathsf{f}(\mathcal{O}_L \times \mathcal{O}_L) \subseteq \mathcal{O}_F$, and so the restriction of f is a symmetric $\mathcal{O}_F$-bilinear form on $\mathcal{O}_L$. Choose an $\mathcal{O}_F$-basis $\mathcal{B} = \{\xi_1,\ldots,\xi_m\}$ of $\mathcal{O}_L$, and let J_0 be the symmetric matrix $[\mathsf{f}]_{\mathcal{B}} \in \mathrm{M}_m(\mathcal{O}_F)$. By [9, Ch. 7, Theorem 6.1], $|\det J_0| = 1$, so $J_0 \in \mathrm{GL}_m(\mathcal{O}_F)$. Hence the restriction of f to $\mathcal{O}_L$ is non-singular, and by [9, Ch. 7, Lemma 6.3], it induces a non-singular, symmetric k_F-bilinear form $\bar{\mathsf{f}}$ on k_L. Note that $\bar{\mathsf{f}}$ is in fact just the trace form over k_F.

Let θ be the involution of $\mathbf{G}$ given by

$$\theta(x) = J^{-1}\,{}^{\mathrm{t}}x^{-1}J, \qquad (x \in \mathbf{G}),$$

where J is the skew-symmetric element of $\mathrm{GL}_n(\mathcal{O}_F)$ defined in block form as

$$J = \begin{pmatrix} 0_m & J_0 \\ -J_0 & 0_m \end{pmatrix}.$$

Since $J \in \mathrm{GL}_n(\mathcal{O}_F)$, we may proceed as in §4.3 to produce an automorphism of $\mathsf{G} = \mathsf{GL}_n = \mathbf{GL}_n(\bar{k}_F)$ induced by θ. Clearly this map will be given by $\theta(x) = \bar{J}^{-1}\,{}^{\mathrm{t}}x^{-1}\bar{J}$, where

$$\bar{J} = \begin{pmatrix} 0_m & \bar{J}_0 \\ -\bar{J}_0 & 0_m \end{pmatrix}.$$

Note that the projection $\bar{\mathcal{B}}$ is a k_F-basis of k_L, and so since $\bar{\mathsf{f}}(\bar{\alpha},\bar{\beta}) = \overline{\mathsf{f}(\alpha,\beta)}$ for any $\alpha,\beta \in \mathcal{O}_L$, we have $\bar{J}_0 = [\bar{\mathsf{f}}]_{\bar{\mathcal{B}}}$.

5.2.3 Verification of Hypotheses.

H1, page *31.* This is satisfied by the definition of θ above.

H2, page *32.* We have $d_\theta = 2$, and we have assumed that p is odd.

H3, page *32.* The group $\mathbf{G}_\theta$ is isomorphic over F to the connected group $\mathbf{Sp}_n$.

H4, page *32* / *H5,* page *44.* The maximal parahoric subgroup $K_0 = \mathrm{GL}_n(\mathcal{O}_F)$ is θ-stable.

H6(1), page *48.* Since $\mathrm{val}\,\theta = -1$ and n is even, we have $I^{\theta}_{G/ZH} = \{1,2\}$. Choose $g_1 = 1$ and

$$g_2 = \begin{pmatrix} \varpi_F 1_m & 0_m \\ 0_m & 1_m \end{pmatrix}.$$

Then $\theta(g_2) = \varpi_F^{-1} g_2$.

H9, page *50.* We also have $\mathbf{G}_\theta \simeq \mathbf{Sp}_n$ over k_F, so that $\mathbf{G}_\theta$ is connected.

H11, page *51.* We have $K_{0,\theta} \simeq \mathrm{Sp}_n(\mathcal{O}_{F_0})$ and $K_{1,\theta}$ is its pro-unipotent radical. Since the natural projection $K_{0,\theta} \to K_{0,\theta}/K_{1,\theta}$ coincides with the mod $\mathcal{P}_F$ map applied to the coefficients of the elements of $K_{0,\theta}$, the quotient $K_{0,\theta}/K_{1,\theta}$ is naturally isomorphic to G_θ. Also, $N_{G_\theta}(K_{0,\theta}) = K_{0,\theta} = Z(G_\theta)K_{0,\theta}$.

H12, page *51.* Since $\mathrm{val}\,\theta = -1$, we choose H as in §3.4. In this case, the groups $Z(G_\theta)$ and Z_0 coincide and are finite, and the quotient $Z(G_\theta)/Z_0$ is trivial.

H13, page *51.* The group $\mathbf{Sp}_n$ is split over the prime subfield, hence $\mathbf{G}_\theta$ is split over F. For this case, we assume [13, Restrictions 12.4.1], so that [13, Lemma 12.4.2] provides H13(1) and H13(2).

Remaining hypotheses. We leave H6(2), H7, H8 and H10 to be verified below.

5.2.4 A factorization result. We now verify H6(2) (page 48). First, we prove a factorization result in K_0.

Lemma 5.2.4. *For any element $k \in K_0$ such that $\theta(k) = k^{-1}$, there exists $k_1 \in K_0$ such that $k = k_1\theta(k_1^{-1})$.*

Proof. Let $M = \mathcal{O}_F^n$ and let $\mathcal{E}$ be the standard basis of M. Let $\mathfrak{g}$ be the $\mathcal{O}_F$-bilinear form on M such that $[\mathfrak{g}]_{\mathcal{E}} = J$. Then $\mathfrak{g}$ is non-singular and skew-symmetric. For any $k \in K_0$, it is easy to check that ${}^{\mathfrak{g}}k = \theta(k^{-1})$. The condition $\theta(k) = k^{-1}$ is equivalent to k being self-adjoint with respect to $\mathfrak{g}$. By Corollary 5.2.3, for any self-adjoint $k \in K_0$ there exists $k_2 \in K_0$ such that $k = {}^{\mathfrak{g}}k_2 k_2 = \theta(k_2^{-1})k_2$. Take $k_1 = \theta(k_2^{-1})$. □

Verification of H6(2). Write $z_1 = 1$ and $z_2 = \varpi_F$, so that $(1-\theta)(g_i) = z_i$ for each $i \in I^\theta_{G/ZH}$. Fix i, and suppose $x \in g_i^{-1}H$ such that $y = (1-\theta)(x) \in ZK_0$, as in H6(2). By considering $\det y$, we see that $k = z_i y \in K_0$. Moreover, since $\theta(y) = y^{-1}$, we also have $\theta(k) = k^{-1}$. Therefore, by the lemma there exists an element $k_1 \in K_0$ such that $k = (1-\theta)(k_1)$. This says that $(1-\theta)(k_1^{-1}x) = z_i^{-1}$. However, $g_i^{-1}G_\theta$ is the unique coset of G_θ in G consisting of all elements $w \in G$ satisfying $(1-\theta)(w) = z_i^{-1}$. Therefore, $k_1^{-1}x \in g_i^{-1}G_\theta$, hence $x \in K_0 g_i^{-1}G_\theta$. □

5.2.5 A θ-stable elliptic torus. We now provide an $\mathcal{O}_F$-basis for $\mathcal{O}_E$ to satisfy H7 (page 49), for E a degree n unramified extension of F. As well, we will show that the resulting torus T of G satisfies the hypothesis of §4.6. Note that since $F_0 = F$ in this case, we will have $\mathbf{T} = \mathbf{S}$. Fix an element $\epsilon \in \mathcal{O}_L^\times$ such that $\bar\epsilon$ is non-square in k_L. Then ϵ is non-square in L and the quadratic extension $E = L(\sqrt{\epsilon})$ is unramified. We may use the basis $\mathcal{B}$ from §5.2.2 to form an $\mathcal{O}_F$-basis $\mathcal{C} = \{\xi_1, \dots, \xi_m, \xi_1\sqrt{\epsilon}, \dots, \xi_m\sqrt{\epsilon}\}$ of $\mathcal{O}_E$. Let ν be the Frobenius automorphism of E/F (see §1.1.1). Then $\mathrm{Gal}(E/L) = \langle \nu^m \rangle$ and $\mathrm{Gal}(L/F) = \langle \nu|L \rangle$. Lifting ν to an element of $\mathrm{Gal}(\bar F/F)$, we may take $\{\mathrm{id}, \nu, \nu^2, \dots, \nu^{n-1}\}$ as a set of representatives of the quotient $\Sigma = \mathrm{Gal}(\bar F/F)/\mathrm{Gal}(\bar F/E)$, and $\{\mathrm{id}, \nu, \nu^2, \dots, \nu^{m-1}\}$ as a set of representatives of the quotient $\Sigma' = \mathrm{Gal}(\bar F/F)/\mathrm{Gal}(\bar F/L)$. Let $A_{\mathcal{B}}$ be the $m \times m$ matrix $\left(\nu^{i-1}(\xi_j)\right)$. Then,

$$ {}^{\mathrm{t}}A_{\mathcal{B}}A_{\mathcal{B}} = \left(\textstyle\sum_h \nu^{h-1}(\xi_i)\,\nu^{h-1}(\xi_j)\right) = (\mathrm{Tr}_{L/F}\,\alpha_i\alpha_j) = J_0. $$

Similarly, let $A_{\mathcal{C}}$ be the $n \times n$ matrix whose $(i,j)^{\text{th}}$ coordinate is

$$(A_{\mathcal{C}})_{ij} = \begin{cases} \nu^{i-1}(\xi_j), & 1 \le j \le m, \\ \nu^{i-1}(\xi_{j-m}\sqrt{\epsilon}), & m < j \le n. \end{cases}$$

Then we may express $A_{\mathcal{C}}$ in block form as

$$A_{\mathcal{C}} = \begin{pmatrix} A_{\mathcal{B}} & A_L^E A_{\mathcal{B}} \\ A_{\mathcal{B}} & -A_L^E A_{\mathcal{B}} \end{pmatrix},$$

where A_L^E is the $m \times m$ diagonal matrix with $\nu^{i-1}(\sqrt{\epsilon})$ in the i^{th} diagonal position. Let $\mathbf{D}$ be the diagonal torus in $\mathbf{G}$, and let w_θ be the element

$$w_\theta = \begin{pmatrix} 0 & 2A_L^E \\ -2A_L^E & 0 \end{pmatrix} \in N_{\mathbf{G}}(\mathbf{D}).$$

Then, in fact, we have $J = {}^{\mathrm{t}}A_{\mathcal{C}} w_\theta^{-1} A_{\mathcal{C}}$. Consider the torus $\mathbf{T} = \mathbf{D}^{A_{\mathcal{C}}} \simeq \mathrm{R}_{E/F}\,\mathbb{G}_{\mathrm{m}}$, as in the example of §1.2.3. Then for $t \in \mathbf{T}$, we have $x = {}^{w_\theta}\big({}^{\mathrm{t}}({}^{A_{\mathcal{C}}}t)^{-1}\big) \in \mathbf{D}$, and so $\theta(t) = x^{A_{\mathcal{C}}}$ is back in $\mathbf{T}$. Hence, $\mathbf{T}$ is θ-stable, and we have verified H7. Notice that under the identification $\mathbf{T} \simeq \mathrm{R}_{E/F}\,\mathbb{G}_{\mathrm{m}}$, $\theta|\mathbf{T}$ is given by $\mathrm{R}_{E/F}\,\iota \circ \eta_{\nu^m}$, where ι is inversion on $\mathbb{G}_{\mathrm{m}}$ and η_{ν^m} is as in §1.2.4. Thus, the action of θ on $E^\times$ induced by the isomorphism $\tilde{f}_{\mathcal{C}}\colon T \to \mathbb{G}_{\mathrm{m}}(E)$ is given by

$$\theta(\alpha) = \nu^m(\alpha)^{-1}, \qquad \big(\alpha \in E^\times\big).$$

Recall that $\bar{\epsilon}$ is a non-square in k_L. Therefore, $k_E = k_L(\sqrt{\bar{\epsilon}})$ and $\bar{\mathcal{C}}$ is a k_F-basis for k_E. For T as in §4.3, we have $\mathsf{T} = \mathsf{T}(k_F) \simeq k_E^\times$, and the induced action of θ on $k_E^\times$ is given by

$$\theta(\alpha) = \bar{\nu}^m(\alpha)^{-1} = \alpha^{-q^m}, \qquad \big(\alpha \in k_E^\times\big),$$

where $q = q_F$. Consider $1-\theta$ and N_θ restricted to T. Since $\bar{\nu}^m$ generates $\mathrm{Gal}(k_E/k_L)$, from above we have $1-\theta = \mathrm{N}_{k_E/k_L}$ and $\mathrm{N}_\theta = 1-\bar{\nu}^m$. By Hilbert's Theorem 90, $\mathrm{N}_\theta(\mathsf{T}) = \ker(1-\theta)$, so that $|\mathsf{T}| = |\ker \mathrm{N}_\theta|\,|\ker(1-\theta)|$. This implies that $(1-\theta)(\mathsf{T}) = \ker \mathrm{N}_\theta$ as well, so that we are now in the situation of §4.6.

5.2.6 A θ-stable Borel subgroup. In this section, we verify H8(1) (page 50). First, we will find a θ-stable Borel subgroup of $\mathbf{G}$ which contains our θ-stable torus $\mathbf{T}$. Let $X = X^*(\mathbf{D})$ be the space of algebraic characters on $\mathbf{D}$. Denote the group of automorphisms of $\mathbf{G}$ which stabilize $\mathbf{D}$ by $\mathrm{Aut}(\mathbf{G}/\mathbf{D})$, and let this group act on X by

$$\mu\alpha = \alpha \circ \mu^{-1}, \qquad \big(\alpha \in X,\ \mu \in \mathrm{Aut}(\mathbf{G}/\mathbf{D})\big).$$

This action preserves the set $\Phi \subset X$ of roots of $\mathbf{D}$. Indeed, take $\alpha \in \Phi$ and non-zero $Y \in \mathfrak{g}_\alpha$, for $\mathfrak{g}_\alpha$ the root space of α in $\mathfrak{g} = \mathrm{Lie}(\mathbf{G})$. For any $x \in \mathbf{G}$, we have $\mathrm{d}\mu \circ \mathrm{Ad}\,x = \mathrm{Ad}\,\mu(x) \circ \mathrm{d}\mu$ ([38, p. 73]), hence

$${}^{x}\big(\mathrm{d}\mu(Y)\big) = \mathrm{d}\mu\big({}^{\mu^{-1}(x)}Y\big) = \mathrm{d}\mu\big(\alpha(\mu^{-1}(x))Y\big) = (\mu\alpha)(x)\,\mathrm{d}\mu(Y), \qquad \big(x \in \mathbf{D}\big).$$

Therefore, $\mathrm{d}\mu(Y) \neq 0$ lies in $\mathfrak{g}_{\mu\alpha}$, and so μ can be considered as an element of $\mathrm{Aut}(\Phi)$. We have $\mathrm{Aut}(\Phi) = W_{\mathbf{G}}(\mathbf{D}) \rtimes \langle \gamma_o \rangle$, where γ_o is the element of $\mathrm{Aut}(\Phi)$ that sends each root to its negative. Let Δ be the standard base of simple roots in Φ; that is, Δ consists of all characters of $\mathbf{D}$ of the form $(d_i) \mapsto d_k/d_{k+1}$, $1 \le k \le n-1$. Let $w_o \in W_{\mathbf{G}}(\mathbf{D})$ be the element which sends $(d_1,\dots,d_n) \in \mathbf{D}$ to $(d_n,\dots,d_1)$. Then $\gamma = w_o\gamma_o$ stabilizes Δ, and any other base of simple roots is of the form $w\Delta$ for some $w \in W_{\mathbf{G}}(\mathbf{D})$, with stabilizer $\langle w\gamma w^{-1}\rangle$.

Let $\theta' = \mathrm{Int}_{\mathrm{L}} A_{\mathcal{C}} \circ \theta \circ \mathrm{Int}_{\mathrm{R}} A_{\mathcal{C}}$, so that $\theta'|\mathbf{D}$ is induced by $\theta|\mathbf{T}$ via conjugation by $A_{\mathcal{C}}$ in $\mathbf{G}$. For $x \in \mathbf{D}$, we have $\theta'(x) = {}^{w_\theta}x^{-1}$. Since $w_\theta^2 \in \mathbf{D}$, $\theta'|\mathbf{D}$ is an involution.

Proposition 5.2.5. *Φ has a base of simple roots which is stable under the action of θ'.*

Proof. We need to find an element $w \in W_{\mathbf{G}}(\mathbf{D})$ such that $\theta' = w\gamma w^{-1}$ as elements of $\mathrm{Aut}(\Phi)$. Now, the actions of γ and θ' on $\mathbf{D}$ are given by

$$\begin{aligned}\gamma &: (d_1,\dots,d_n) \mapsto (d_n^{-1},\dots,d_1^{-1}),\\ \theta' &: (d_1,\dots,d_n) \mapsto (d_{m+1}^{-1},\dots,d_n^{-1},d_1^{-1},\dots,d_m^{-1}).\end{aligned}$$

Therefore,

$$w : (d_1,\dots,d_n) \mapsto (d_m,\dots,d_1,d_{m+1},\dots,d_n)$$

is the required element of $W_{\mathbf{G}}(\mathbf{D})$. □

Corollary 5.2.6. *There exists a θ'-stable Borel subgroup $\mathbf{B}'$ of $\mathbf{G}$ which contains $\mathbf{D}$.*

Proof. Let $\mathbf{B}' = \langle \mathbf{D}, 1+\mathfrak{g}_\alpha\rangle_{\alpha\in w\Delta}$, with w as in the proof of the proposition. For any $\alpha \in w\Delta$, we have

$$\theta'\left(1+\mathfrak{g}_\alpha\right) = 1 + \mathrm{d}\theta'\left(\mathfrak{g}_\alpha\right) = 1 + \mathfrak{g}_{\theta'\alpha},$$

where $\theta'\alpha$ lies in $w\Delta$ as well. Thus $\mathbf{B}'$ is θ'-stable. □

Corollary 5.2.7. *There exists a θ-stable Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ which contains $\mathbf{T}$.*

Proof. Take $\mathbf{B} = (\mathbf{B}')^{A_{\mathcal{C}}}$. □

We may repeat the above arguments for T and $\mathsf{D} = {}^{A_{\bar{\mathcal{C}}}}\mathsf{T}$ in G to obtain a θ-stable Borel subgroup $\mathsf{B} \subset \mathsf{G}$ which contains T. This completes the verification of H8(1).

5.2.7 Characters of T and T_θ.

Here we verify H8(2) (page 50) and H10 (page 50). Again, write $q = q_F$. Fix a primitive $(q^n-1)^{\text{th}}$ root of unity $\eta \in \mathbb{C}^\times$ and a generator ζ of $k_E^\times$. Any character of $\mathsf{T} \simeq k_E^\times$ is of the form $\lambda_\ell(\zeta) = \eta^\ell$ for some integer $0 \le \ell \le q^n-2$. Since q^n-1 factors as $(q^m+1)(q^m-1)$, λ_ℓ is θ-stable if and only if $\ell = \ell_0(q^m-1)$, for some $0 \le \ell_0 \le q^m$. To verify H8(2), we would like to find such a θ-stable character λ_ℓ which is in general position. Consider the case of $\ell_0 = 1$, so that $\ell = q^m-1$. Write $\lambda = \lambda_\ell$, and suppose that j is an integer such that $1 \le j \le n$ and $\lambda \circ \bar{\nu}^j = \lambda$. Then,

$$1 = \lambda(\zeta)^{-1}\lambda(\bar{\nu}^j(\zeta)) = \lambda(\zeta^{q^j-1}) = \eta^{(q^j-1)(q^m-1)},$$

so we must have $(q^n-1)|(q^j-1)(q^m-1)$. Then $(q^m+1)|(q^j-1)$, which is only possible for $j = n$. Therefore, λ is in general position.

To verify H10, we have the following.

Lemma 5.2.8. *Let λ be a θ-stable element of* $\mathrm{Irr}(\mathsf{T})$*. If λ is in general position, then* $\mathrm{Res}^{\mathsf{T}}_{\mathsf{T}_\theta}\lambda$ *is in general position.*

Proof. We shall prove the contrapositive instead. Write $\lambda = \lambda_\ell$ with $\ell = \ell_0(q^m - 1)$, as above, and suppose $\chi_\ell = \mathrm{Res}^{\mathsf{T}}_{\mathsf{T}_\theta}\lambda_\ell$ is not in general position. We may identify $W_{\mathbf{G}_\theta}(\mathsf{T}_\theta)$ with the subgroup

$$\langle\{\,(i\ j)(n-i\ n-j) \mid 1 \le i < j \le m\,\} \cup \{\,(i\ n-i) \mid 1 \le i \le m\,\}\rangle$$

of the symmetric group S_n. The only non-trivial element of this subgroup which commutes with $(1\ 2\ \cdots\ n)$ is $(1\ m+1)(2\ m+2)\cdots(m\ n)$, so we may identify $W_{\mathbf{G}_\theta}(\mathsf{T}_\theta)^{k_F}$ with $\langle \bar\nu^m\rangle$. Now, $(k_E^\times)_\theta = \langle \zeta^{q^m-1}\rangle$. Therefore, if χ_ℓ is not in general position, we must have $({}^{\bar\nu^m}\lambda_\ell)(\zeta^{q^m-1}) = \lambda_\ell(\zeta^{q^m-1})$, which can only occur if ℓ_0 is 0 or $(q^m+1)/2$. Clearly λ_0 is not in general position, and it is easy to show that every element of $\mathrm{Gal}(k_E/k_F)$ stabilizes λ_ℓ for $\ell = (q^n-1)/2$. □

The following will be needed in the appendix.

Lemma 5.2.9. *If* $\lambda \in \mathrm{Irr}(\mathsf{T})$ *is θ-stable, then* $\lambda | k_L^\times \equiv 1$.

Proof. The element $\zeta^{q^m+1} \in k_E^\times$ is a generator of $k_L^\times$, and $\lambda(\zeta^{q^m+1}) = \lambda(\zeta)/\lambda(\theta(\zeta))^{-1} = 1$. □

5.2.8 The result.

We may now apply the results of §4.6 to the present case. From (4.6.1), we have

$$\Theta_{\pi^+}\big({}^g(\gamma\theta)\big) = |\langle\theta\rangle|\,\varepsilon_+\,\varepsilon_\theta\,\lambda^+(\theta)\,d(\pi_1^+)\,\frac{\deg(\sigma_\theta)}{\deg(\sigma)}\,\big(\hat\mu_{X_\theta}(\log\gamma) + \sigma(\varpi_F)^{-1}\,\hat\mu_{({}^{g_2}X_\theta)}(\log\gamma)\big),$$

for any $g \in G$ and any topologically unipotent element $\gamma \in K_{0,\theta}$ such that $\gamma\theta$ is G-regular in G^+. Now, $\mathbf{G}$ is k_F-split and T is k_F-minisotropic, so $\mathrm{rk}_{k_F}\mathbf{G} = \mathrm{rk}\,\mathbf{G} = n$ and $\mathrm{rk}_{k_F}\mathsf{T} = \mathrm{rk}\,Z(\mathbf{G}) = 1$. Since n is even, we have $\varepsilon_+ = -1$. Similarly, $\mathbf{G}_\theta$ also splits over k_F and T_θ is also k_F-minisotropic in $\mathbf{G}_\theta$. So $\mathrm{rk}_{k_F}\mathbf{G}_\theta = \mathrm{rk}\,\mathbf{G}_\theta = m$ and $\mathrm{rk}_{k_F}\mathsf{T}_\theta = \mathrm{rk}\,Z(\mathbf{G}_\theta)^0$. However, $Z(\mathbf{G}_\theta)^0$ is trivial, so $\mathrm{rk}_{k_F}\mathsf{T}_\theta = 0$. Therefore, we have $\varepsilon_\theta = (-1)^m$. Substituting these and $|\langle\theta\rangle| = 2$ into the above expression, we have

$$\Theta_{\pi^+}\big({}^g(\gamma\theta)\big) = 2\,(-1)^{m+1}\,\lambda^+(\theta)\,d(\pi_1^+)\,\frac{\deg(\sigma_\theta)}{\deg(\sigma)}\,\big(\hat\mu_{X_\theta}(\log\gamma) + \sigma(\varpi_F)^{-1}\,\hat\mu_{({}^{g_2}X_\theta)}(\log\gamma)\big).$$

Finally, from [15, Proposition 12.9], we have

$$\frac{\deg(\sigma_\theta)}{\deg(\sigma)} = \left\|\frac{\mathsf{G}}{\mathsf{G}_\theta}\right\|_{p'}^{-1}\left|\frac{\mathsf{T}}{\mathsf{T}_\theta}\right| = \frac{q^m-1}{\prod_{\ell=1}^m (q^{2\ell-1}-1)},$$

where $q = q_F$.

We will now show that both terms in the above formula are necessary. That is, we will show that $\hat\mu_{X_\theta}$ and $\hat\mu_{({}^{g_2}X_\theta)}$ differ on $(\mathfrak{g}_\theta)_{0^+}$. To do this, we will find a function $f \in C_c^\infty(\mathfrak{g}_\theta)$ with $\mathrm{supp}\,f \subseteq (\mathfrak{g}_\theta)_{0^+}$ such that $\hat\mu_{({}^{g_i}X_\theta)}(f)$ differ for $i = 1,2$ (recall that $g_1 = 1$). For each i, let $\mathcal{O}_i = {}^{G_\theta}({}^{g_i}X_\theta)$. Since $\hat\mu_{({}^{g_i}X_\theta)}(f) = \mu_{({}^{g_i}X_\theta)}(\hat f) \ne 0$ if and only if $\mathrm{supp}\,\hat f \cap \mathcal{O}_i \ne \emptyset$, it suffices to find such an f such that $\mathrm{supp}\,\hat f \cap \mathcal{O}_1 \ne \emptyset$ but $\mathrm{supp}\,\hat f \cap \mathcal{O}_2 = \emptyset$. In particular, consider $f = \widehat{\mathrm{ch}}_{\mathfrak{k}_{0,\theta}}$. Then $\mathrm{supp}\,f \subseteq \mathfrak{k}_{1,\theta} \subset (\mathfrak{g}_\theta)_{0^+}$. Since we may assume that dY (see §4.5) is normalized so that $\hat{\hat g}(X) = g(-X)$ for any $g \in C_c^\infty(\mathfrak{g}_\theta)$ and any $X \in \mathfrak{g}_\theta$, we have $\hat f = \mathrm{ch}_{\mathfrak{k}_{0,\theta}}$. We have now reduced to showing that $\mathfrak{k}_{0,\theta} \cap \mathcal{O}_1 \ne \emptyset$ and $\mathfrak{k}_{0,\theta} \cap \mathcal{O}_2 = \emptyset$.

Because of the way that we have constructed $\mathbf{T}$, we may embed $\mathfrak{t} = \mathrm{Lie}(T) \simeq E$ in $\mathbf{M}_2(L)$ by

$$\alpha + \beta\sqrt{\epsilon} \mapsto \begin{pmatrix} \alpha & \epsilon\beta \\ \beta & \alpha \end{pmatrix}, \qquad (\alpha, \beta \in L,).$$

To see how $\mathrm{d}\theta$ acts on such an element, note that for $\alpha \in L^\times$, we have $\theta(\alpha) = \alpha^{-1}$. From this, we may conclude that $J_0^{-1}{}^{\mathrm{t}}\alpha J_0 = \alpha$ for any $\alpha \in L$, where we identify L with its image in $M_m(F)$ afforded by the F-basis $\mathcal{B}$ of L. Therefore,

$$\mathrm{d}\theta\colon \begin{pmatrix} \alpha & \epsilon\beta \\ \beta & \alpha \end{pmatrix} \mapsto \begin{pmatrix} -\alpha & \epsilon\beta \\ \beta & -\alpha \end{pmatrix},$$

and so $\mathfrak{t}_\theta$ may be identified with the F-subspace $L\sqrt{\epsilon}$ of E. Choose $\beta \in \mathcal{O}_L^\times$ such that $\bar{\beta}$ generates $k_L^\times$, and let $X_\theta \in \mathfrak{t}_\theta$ be the element corresponding to $\beta\sqrt{\epsilon} \in L\sqrt{\epsilon}$.

To proceed, we will need the language of buildings. In the interest of not burdening ourselves with a full background discussion on the building of a connected, reductive, p-adic group, we refer the reader to the survey in [39]. For details on Moy-Prasad filtrations, see [29]. Let $\mathfrak{B} = \mathfrak{B}(\mathbf{G}, F)$ and $\mathfrak{B}_\theta = \mathfrak{B}(\mathbf{G}_\theta, F)$ be the Bruhat-Tits buildings of G and G_θ, respectively. For $x \in \mathfrak{B}$, Moy and Prasad define a filtration $\{\mathfrak{g}_{x,r}\}_{r\in\mathbb{R}}$ of $\mathfrak{g}$, indexed by the real numbers. Each subset in the filtration is a lattice in $\mathfrak{g}$. For $r \in \mathbb{R}$ and $x \in \mathfrak{B}$, define

$$\mathfrak{g}_{x,r^+} = \bigcup\nolimits_{s>r} \mathfrak{g}_{x,s}, \qquad \mathfrak{g}_r = \bigcup\nolimits_{x\in\mathfrak{B}} \mathfrak{g}_{x,r}, \qquad \mathfrak{g}_{r^+} = \bigcup\nolimits_{s>r} \mathfrak{g}_s.$$

We have similar definitions for $\mathfrak{g}_\theta$ relative to $\mathfrak{B}_\theta$. As in [36], θ induces an action on $\mathfrak{B}$, and we may identify $\mathfrak{B}_\theta$ with the fixed points in $\mathfrak{B}$ under this action. According to the definition in [4, §5], our given choice of X_θ is a $\mathbf{T}_\theta$-*good element of depth* 0 in $\mathfrak{t}_\theta$. Since X_θ is regular, and the building of a torus is a point, [26, Theorem 2.3.1] implies that there exists a unique $x_\theta \in \mathfrak{B}_\theta$ such that $X_\theta \in (\mathfrak{g}_\theta)_{x_\theta,0} \smallsetminus (\mathfrak{g}_\theta)_{x_\theta,0^+}$. Because of our choice of β, we have $(\mathfrak{g}_\theta)_{x_\theta,0} = \mathfrak{k}_{0,\theta}$ and $(\mathfrak{g}_\theta)_{x_\theta,0^+} = \mathfrak{k}_{1,\theta}$. For any $g \in G_\theta$, $x \in \mathfrak{B}_\theta$, and $r \in \mathbb{R}$, we have ${}^g(\mathfrak{g}_\theta)_{x,r} = (\mathfrak{g}_\theta)_{g\cdot x,r}$. The uniqueness of x_θ implies that

$$G_\theta \cdot x_\theta = \left\{ x \in \mathfrak{B}_\theta \mid (\mathfrak{g}_\theta)_{x,0} \cap \mathcal{O}_1 \neq \emptyset \right\}. \tag{5.2.1}$$

In particular, $x_\theta \in G_\theta \cdot x_\theta$, so that $(\mathfrak{g}_\theta)_{x_\theta,0} \cap \mathcal{O}_1 \neq \emptyset$. Now, ${}^{g_2}X_\theta$ is a ${}^{g_2}\mathbf{T}_\theta$-good element of depth 0 in $\mathrm{Lie}({}^{g_2}\mathbf{T}_\theta) = {}^{g_2}\mathfrak{t}_\theta$, and $g_2 \cdot x_\theta$ is the unique element of $\mathfrak{B}_\theta$ such that ${}^{g_2}X_\theta \in (\mathfrak{g}_\theta)_{g_2\cdot x_\theta,0} \smallsetminus (\mathfrak{g}_\theta)_{g_2\cdot x_\theta,0^+}$. Since X_θ and ${}^{g_2}X_\theta$ lie in distinct G_θ-orbits, $g_2 \cdot x_\theta$ cannot lie in $G_\theta \cdot x_\theta$. From (5.2.1), we may conclude that $(\mathfrak{g}_\theta)_{x_\theta,0} \cap \mathcal{O}_2 = \emptyset$.

5.3 The unitary case

In this section, we examine the case of $\operatorname{val}\theta = -1$, $[F : F_0] = 2$, and J an element of $\mathbf{GL}_n(F)$ which is Hermitian with respect to the non-trivial element τ of $\mathrm{Gal}(F/F_0)$.

5.3.1 Restriction to a sub-case. The following result allows us to restrict our attention to the case that n is odd and F/F_0 is unramified. Let θ be as above, and write $\mathrm{N} = \mathrm{N}_{F/F_0}$. For any matrix X with entries in F, let ${}^*X = \tau({}^{\mathrm{t}}X)$. Recall that a square matrix X with entries in F is called Hermitian (with respect to τ) if ${}^*X = X$.

Proposition 5.3.1. *If F/F_0 is ramified or n is even, then the depth-zero supercuspidal representations of G constructed in §4.1 cannot be θ-stable.*

Proof. Let E, λ, and π be as in §4.1. Suppose that π is θ-stable. By Corollary 1.5.6 and the proof of Lemma 1.5.5, π must be unitary. Since conjugation by J preserves equivalence, we also have that π is stable under the involution $x \mapsto {}^*x^{-1}$. By [35, Lemma 2.1], there exists an automorphism τ' of E which fixes F_0, stabilizes but does not fix F, has order 2, and satisfies $\lambda \circ \tau' = \lambda^{-1}$. Therefore, we may apply the results of [18, §2]. Let $E_{\tau'}$ be the fixed field of τ' in E. A Howe factorization of λ consists of quasi-characters λ_0 and λ_1 of F and E, respectively, such that λ_1 is generic over F and $\lambda = \lambda_1(\lambda_0 \circ \mathrm{N}_{E/F})$. In the notation of *loc. cit.*, we have $E' = E_{\tau'}$, $r = 1$, $E_0 = F$, and $E_1 = E$. Since E/F is unramified and λ_1 is generic, the conductoral exponent f_1 of λ_1 must be 1. First suppose F/F_0 is ramified. Then since E/F is unramified, we must also have E/E' ramified. But then [18, Lemma 2.1(i)] says that we must have $f_1 > 1$, a contradiction. On the other hand, if F/F_0 is unramified, then the hypotheses of [18, Lemma 2.1(iv)] are satisfied, and so $n = [E : F]$ is odd. $\square$

In light of this result, we will assume for the remainder of this examination of the unitary case that F/F_0 is unramified. As usual, we take a common uniformizer $\varpi = \varpi_F = \varpi_{F_0}$.

For $g \in G$, replacing θ by $g^{-1} \cdot \theta$ (see §4.1) is equivalent to replacing J by ${}^*g\,J\,g$. The orbit of the Hermitian matrix J under the action $J \mapsto {}^*g\,J\,g$ is uniquely determined by the class of $\det J$ in $F_0^\times/\mathrm{N}F^\times$ ([25]). The following result allows us to restrict our attention to Hermitian J with $\det J \in \mathrm{N}F^\times$. Since F/F_0 is unramified, we may take $\{1, \varpi\}$ as a set of representatives for $F_0^\times/\mathrm{N}F^\times$.

Proposition 5.3.2. *Suppose n is odd. For $i = 0, 1$, choose a Hermitian matrix $J_i \in G$ such that* $\det J_i \in \varpi^i \mathrm{N}F^\times$*, and define $\theta_i : x \mapsto J_i\, {}^*x^{-1} J_i^{-1}$. Then there exists $g \in G$ such that $g \cdot \theta_1 \equiv \theta_0$.*

Proof. There exists $g_0 \in G$ such that ${}^*g_0\,J_0\,g_0 = 1$. Since n is odd and F/F_0 is unramified, there exists $g_1 \in G$ such that ${}^*g_1\,J_1\,g_1 = \varpi$. Then both $g_0^{-1} \cdot \theta_0$ and $g_1^{-1} \cdot \theta_1$ are given by $x \mapsto {}^*x^{-1}$. Take $g = g_0 g_1^{-1}$. $\square$

5.3.2 τ-Hermitian forms on $\mathcal{O}_F$-modules.

In this section, we prove a factorization result for linear operators on an $\mathcal{O}_F$-module M, similar to the symplectic case (§5.2.1). Let M be a torsion-free, finitely generated $\mathcal{O}_F$-module. As in §5.2.1, M is in fact a free $\mathcal{O}_F$-module. Suppose that M is of rank n. For this discussion, we do not need to restrict to the case that n is odd. View τ as an automorphism of the ring $\mathcal{O}_F$. A form $\mathsf{f} : M \times M \to \mathcal{O}_F$ on M is called *τ-sesquilinear* (henceforth *sesquilinear*) if it is linear in the first term, respects addition in the second term, and satisfies $\mathsf{f}(x, \alpha y) = \tau(\alpha)\mathsf{f}(x,y)$ for all $x, y \in M$ and $\alpha \in \mathcal{O}_F$. A sesquilinear form f is called *τ-Hermitian* (henceforth *Hermitian*) if $\mathsf{f}(y,x) = \tau\big(\mathsf{f}(x,y)\big)$ for all $x, y \in M$. If f is Hermitian and $\mathcal{B}$ is an basis of M, then $[\mathsf{f}]_{\mathcal{B}} \in \mathrm{M}_n(\mathcal{O}_F)$ is necessarily Hermitian. Again, we refer the reader to [6] for the basic facts concerning forms on modules over rings. We now show that the characterization of $\mathrm{GL}_n(\mathcal{O}_F)$-orbits of Hermitian elements in $\mathbf{GL}_n(F)$ given in [19, §3] implies that all non-singular, Hermitian forms on M are equivalent. Fix such a form f.

Theorem 5.3.3. *There exists a basis $\mathcal{B}$ of M such that* $[\mathsf{f}]_{\mathcal{B}} = 1_n$.

Proof. Let $\mathcal{C}$ be any basis of M, and let $J = [\mathsf{f}]_{\mathcal{C}} \in \mathrm{GL}_n(\mathcal{O}_F)$. When F/F_0 is unramified, [19] says that the $\mathrm{GL}_n(\mathcal{O}_F)$-orbit of J in $\mathbf{GL}_n(F)$, under the action $k \cdot J = {}^*kJk$, contains an element of block form $\mathrm{diag}(\varpi^{a_1} 1_{m_1}, \ldots, \varpi^{a_s} 1_{m_s})$, where $m_1 + \cdots + m_s$ is a partition of n by integers, and $a_1 > \cdots > a_s$ is a decreasing sequence of integers. However, since $\mathrm{GL}_n(\mathcal{O}_F)$ is stable under the map $k \mapsto {}^*k$, the orbit of J is entirely contained in $\mathrm{GL}_n(\mathcal{O}_F)$. The only element of the above form which lies in $\mathrm{GL}_n(\mathcal{O}_F)$ is 1_n. Therefore, there exists $k \in \mathrm{GL}_n(\mathcal{O}_F)$ such that ${}^*kJk = 1_n$. Let $\mathcal{B}$ be the basis of M such that k is the change of basis matrix $[\mathrm{id}]_{\mathcal{C}\mathcal{B}}$. □

Corollary 5.3.4. *For any non-singular, Hermitian form* g *on* M*, there exists an invertible element* $A \in \mathrm{End}_{\mathcal{O}_F}(M)$ *such that* $\mathsf{g}(x,y) = \mathsf{f}(Ax, Ay)$ *for all* $x, y \in M$*.*

Proof. By the theorem, there exist bases $\mathcal{B} = \{x_1, \dots, x_n\}$ and $\mathcal{B}' = \{x'_1, \dots, x'_n\}$ of M such that

$$\mathsf{f}(x_i, x_j) = \mathsf{g}(x'_i, x'_j) = \begin{cases} 1, & i = j, \\ 0, & \text{otherwise.} \end{cases}$$

So take A defined by $Ax'_i = x_i$ for $i = 1, 2, \dots, n$, and extend linearly. □

As in the symplectic case, we use this result to prove a factorization result for self-adjoint, invertible linear operators on M, which will in turn be used to verify H6(2). Recall that given an element $A \in \mathrm{End}_{\mathcal{O}_F}(M)$, there exists a unique adjoint operator ${}^{\mathsf{f}}A \in \mathrm{End}_{\mathcal{O}_F}(M)$ such that $\mathsf{f}(Ax, y) = \mathsf{f}(x, {}^{\mathsf{f}}Ay)$, for all $x, y \in M$.

Corollary 5.3.5. *Let* $S \in \mathrm{End}_{\mathcal{O}_F}(M)$ *be invertible and self-adjoint with respect to* f*. Then there exists an invertible element* $A \in \mathrm{End}_{\mathcal{O}_F}(M)$ *such that* $S = {}^{\mathsf{f}}AA$*.*

Proof. Argue precisely as in the proof of Corollary 5.2.3, replacing the term "skew-symmetric" by "Hermitian". □

5.3.3 Definition of θ.

In light of Proposition 5.3.1, assume that n is odd and F/F_0 is unramified. Let E_0 be an unramified, degree n extension of F_0. Similarly to §5.2.2, let f be the trace form on E_0. Choose an $\mathcal{O}_{F_0}$-basis $\mathcal{B} = \{\xi_1, \dots, \xi_n\}$ of $\mathcal{O}_{E_0}$, and let J be the symmetric matrix $[\mathsf{f}]_{\mathcal{B}} \in \mathrm{M}_n(\mathcal{O}_{F_0})$. Then, in fact, $J \in \mathrm{GL}_n(\mathcal{O}_{F_0})$, and J is Hermitian with $\det J \in \mathrm{N}F^\times$. Let θ be the F_0-automorphism of $\mathbf{G} = \mathrm{R}_{F/F_0}\mathbf{GL}_n$ given in the proof of Proposition 1.4.2(2). Since the automorphism $\mathrm{Int}_{\mathrm{R}} J$ of $\mathbf{GL}_n$ is defined over F_0, the automorphisms $(\mathrm{Int}_{\mathrm{R}} J)^\Sigma$ and ψ_τ of $(\mathbf{GL}_n)^\Sigma$ commute, and we may write $\theta = \eta_\tau \circ \mathrm{R}_{F/F_0}\theta_1$, where θ_1 is the F_0-involution $x \mapsto J^{-1}{}^{\mathrm{t}}x^{-1}J$ of $\mathbf{GL}_n$. As well, η_τ and $\mathrm{R}_{F/F_0}\theta_1$ commute and are both of order 2, so we have $d_\theta = 2$. Write θ_2 for the involution $\psi_\tau \circ \theta_1^\Sigma$ of $\mathbf{GL}_n^\Sigma$. Since $\mathbf{GL}_n$ is defined over F_0, we have $(\mathbf{GL}_n)^\Sigma = \mathbf{GL}_n \times \mathbf{GL}_n$, and θ_2 is given by $\theta_2(x,y) = \big(\theta_1(y), \theta_1(x)\big)$, for $x, y \in \mathbf{GL}_n$.

The restriction of f to $\mathcal{O}_{E_0}$ induces a non-singular, symmetric k_{F_0}-bilinear form $\bar{\mathsf{f}}$ on k_{E_0}. In fact, $\bar{\mathsf{f}}$ is just the trace form over k_{F_0}, and $[\bar{\mathsf{f}}]_{\bar{\mathcal{B}}} = \bar{J}$. Therefore, if we also write θ_1 for the k_{F_0}-automorphism $x \mapsto \bar{J}^{-1}{}^{\mathrm{t}}x^{-1}\bar{J}$ of GL_n, the automorphism θ of $\mathbf{G}$ is given by $\theta = \eta_{\bar{\tau}} \circ \mathrm{R}_{k_F/k_{F_0}}\theta_1$.

5.3.4 Verification of hypotheses.

H1, page *31.* This is satisfied by the definition of θ above.

H2, page *32.* We have $d_\theta = 2$, and we have assumed that p is odd.

H3, page *32.* The subgroup of θ_2-fixed points of $\mathbf{GL}_n^\Sigma$ is $(\mathbf{GL}_n)^\Sigma_{\theta_2} = \{\,(x, \theta_1(x)) \mid x \in \mathbf{GL}_n\,\}$. Therefore, $(\mathbf{GL}_n)^\Sigma_{\theta_2} \simeq \mathbf{GL}_n$, and so both $(\mathbf{GL}_n)^\Sigma_{\theta_2}$ and $\mathbf{G}_\theta$ are connected.

H4, page *32* / *H5,* page *44.* The maximal parahoric subgroup $K_0 = \mathrm{GL}_n(\mathcal{O}_F)$ is θ-stable.

H6(1), page *48.* Since we have assumed that n is odd, we have $I^{\theta}_{G/ZH} = \{1\}$ and $g_1 = 1$, so there is nothing to verify.

H9, page *50.* The same argument which showed that $\mathbf{G}_\theta$ is connected may be applied to show that G_θ is connected.

H11, page *51.* Let $U \subset G$ be the subgroup of matrices $g \in G$ such that ${}^*gg = 1$. By [19, Proposition 1(a)], the quotient $\bar{U} = (U \cap K_0)/(U \cap K_1)$ may also be described as the subgroup of G consisting of elements $g \in \mathsf{G}$ such that ${}^*gg = 1$, where now the map $g \mapsto {}^*g$ is defined using $\bar{\tau}$. By Theorem 5.3.3, there exists an element $k \in K_0$ such that ${}^*kJk = 1$. But then $G_\theta = {}^kU$, and $K_{0,\theta}/K_{1,\theta} = ({}^kU \cap K_0)/({}^kU \cap K_1) = {}^{\bar{k}}\bar{U}$. Since also ${}^*\bar{k}\bar{J}\bar{k} = 1$, we have ${}^{\bar{k}}\bar{U} = \mathsf{G}_\theta$.

H12, page *51.* Since $\operatorname{val}\theta = -1$, we choose H as in §3.4. The group G_θ is a unitary group, and in this case the groups $Z_0 = Z(H)_\theta$, $Z(G_\theta)$, and $Z(G)_\theta$ all coincide and are equal to $(\ker \mathrm{N}_{F/F_0}) \cap \mathcal{O}_F^\times$. In particular, the quotient $Z(G_\theta)/Z_0$ is trivial.

H13, page *51.* The group $\mathbf{G}_\theta$ is isomorphic over F to $\mathbf{GL}_n$, hence split over F. For this case, we assume [13, Restrictions 12.4.1], so that [13, Lemma 12.4.2] provides H13(1) and H13(2).

Remaining hypotheses. We leave H6(2), H7, H8 and H10 to be verified below.

5.3.5 A factorization result. To verify H6(2) (page 48), we may argue exactly as in the symplectic case (§5.2.4). First, we prove a factorization result in K_0.

Lemma 5.3.6. *For any element $k \in K_0$ such that $\theta(k) = k^{-1}$, there exists $k_1 \in K_0$ such that $k = k_1\theta(k_1^{-1})$.*

Proof. Argue exactly as in the proof of Lemma 5.2.4, but replace the terms "$\mathcal{O}_F$-bilinear" and "skew-symmetric" by "sesquilinear" and "Hermitian", respectively, and appeal to Corollary 5.3.5 instead of Corollary 5.2.3. ☐

The verification of H6(2) now proceeds exactly as in the symplectic case (§5.2.4), though here we need only consider $g_1 = z_1 = 1$, since $I^{\theta}_{G/ZH} = \{1\}$.

5.3.6 A θ-stable elliptic torus. We now provide an $\mathcal{O}_F$-basis for $\mathcal{O}_E$ to satisfy H7 (page 49), for E a degree n unramified extension of F. As well, we will show that the resulting torus T of G satisfies the hypothesis of §4.6.

Let E be the composite field FE_0. Then we have the same towers of p-adic and residue fields as in the unramified Galois case with $d = 2$ (see the diagram of §5.1.4), where the p-adic extensions are all unramified. We may regard the generator τ of $\mathrm{Gal}(F/F_0)$ as the restriction to F of the unique element of $\mathrm{Gal}(E/F_0)$ of order 2. In this case, since E_0 is the unique unramified extension of F_0 of degree n inside E, E_0 must be the fixed field of τ. Our construction will be a mix of the arguments of the Galois and symplectic cases. In §5.3.3, we chose an $\mathcal{O}_{F_0}$-basis $\mathcal{B}$ of $\mathcal{O}_{E_0}$. Since E_0/F_0 is unramified, it is also an F_0-basis for E_0. Moreover, since n is odd, it is also an F-basis for E. Let ν be the Frobenius automorphism of

$\mathrm{Gal}(E/F_0)$ (see §1.1.1). Then $\tau = \nu^n$, $\mathrm{Gal}(E/F) = \langle \nu^2 \rangle$, and $\mathrm{Gal}(E_0/F_0) = \langle \nu^2|E_0 \rangle$. Similarly to the symplectic case (§5.2.2), if we set $A_{\mathcal{B}}$ to be the $n \times n$ matrix $\left(\nu^{2(i-1)}(\xi_j)\right)$, then ${}^{\mathrm{t}}A_{\mathcal{B}}A_{\mathcal{B}} = J$. Let $\mathbf{D}$ be the diagonal torus in $\mathbf{GL}_n$, and let $\mathbf{S} = \mathbf{D}^{A_{\mathcal{B}}} \simeq \mathrm{R}_{E/F}\,\mathbb{G}_{\mathrm{m}}$. Here, $\mathbf{S}$ is precisely the torus (also denoted $\mathbf{S}$) constructed in the Galois case (§5.1.4), with $d = 2$. By the same argument, $\mathbf{S}$ is defined over F_0, and thus is τ-stable. An easy calculation using $J = {}^{\mathrm{t}}A_{\mathcal{B}}A_{\mathcal{B}}$ shows that for $x \in \mathbf{D}$, $\theta_1(x^{A_{\mathcal{B}}}) = (x^{A_{\mathcal{B}}})^{-1}$, so that $\mathbf{S}$ is not only θ_1-stable, but θ_1-*split*. We have now verified H7. As in §4.3, the torus $\mathbf{T} = \mathrm{R}_{F/F_0}\,\mathbf{S} \subset \mathbf{G}$ is θ-stable. Under the identification of $\mathbf{G}(F_0)$ with G, the torus $T = \mathbf{T}(F_0)$ is identified with the θ-stable torus $\mathbf{S}(F) \subset G$. Since $\mathbf{S}(F)$ is isomorphic to $E^\times = \mathbb{G}_{\mathrm{m}}(E)$ via $\tilde{f}_{\mathcal{B}}$, we also have $T \simeq E^\times$. Let ι be the inversion homomorphism of $\mathbf{S}$ ($\mathbf{S}$ is abelian). Since $\theta_1|\mathbf{S} = \iota$, we have $\theta|\mathbf{T} = \eta_\tau \circ \mathrm{R}_{F/F_0}\,\iota$. Then, just as in the symplectic case (§5.2.2), the induced action of θ on $E^\times$ is given by

$$\theta(\alpha) = \tau(\alpha)^{-1}, \qquad \left(\alpha \in E^\times\right).$$

In contrast to the symplectic case, however, note that here τ is not an element of $\mathrm{Gal}(E/F)$, but rather an element of $\mathrm{Gal}(E/F_0)$.

For T as in §4.3, we have $\mathsf{T} = \mathsf{T}(k_F) \simeq k_E^\times$, and the induced action of θ on $k_E^\times$ is given by

$$\theta(\alpha) = \bar{\tau}(\alpha)^{-1} = \alpha^{-q^n}, \qquad \left(\alpha \in k_E^\times\right),$$

where $q = q_{F_0}$. The identical argument as in the symplectic case (§5.2.2), with k_{E_0} playing the role of k_L, shows that $(1-\theta)(\mathsf{T}) = \ker \mathrm{N}_\theta$. This will put us in the situation of §4.6.

5.3.7 A θ-stable Borel subgroup. In this section, we verify H8(1) (page 50). First, we construct a θ-stable Borel subgroup of $\mathbf{G}$ which contains our θ-stable torus $\mathbf{T}$. As in the Galois case, this is easy to do, and it suffices to find a θ_2-stable Borel subgroup of $(\mathbf{GL}_n)^\Sigma$ which contains $\mathbf{S}^\Sigma$.

Let $\mathbf{C} \supset \mathbf{D}$ be the pair in $\mathbf{X}$ consisting of the Borel subgroup of upper-triangular matrices and the diagonal torus. Arguing precisely as in the Galois case (§5.1.5), any Borel subgroup $\mathbf{B} \subset (\mathbf{GL}_n)^\Sigma$ containing $\mathbf{S}^\Sigma$ is of the form $(\mathbf{C}^\Sigma)^{xA_{\mathcal{B}}^\Sigma} = \mathbf{C}^{x_1A_{\mathcal{B}}} \times \mathbf{C}^{x_2A_{\mathcal{B}}}$, for some $x = (x_1, x_2) \in N_{(\mathbf{GL}_n)^\Sigma}(\mathbf{D}^\Sigma)$. Here, $A_{\mathcal{B}}^\Sigma$ is the element $(A_{\mathcal{B}}, A_{\mathcal{B}}) \in (\mathbf{GL}_n)^\Sigma$. Choose $x_1 = 1$. Since $\mathbf{S}$ is θ_1-stable, $\mathbf{D}$ is stable under the automorphism $\theta_1' = \mathrm{Int}_{\mathrm{L}} A_{\mathcal{B}} \circ \theta_1 \circ \mathrm{Int}_{\mathrm{R}} A_{\mathcal{B}}$. Let $x_2 \in N_{\mathbf{GL}_n}(\mathbf{D})$ such that $\theta_1(\mathbf{C}) = \mathbf{C}^{x_2}$. Then the resulting Borel subgroup is $\mathbf{B} = \mathbf{C}^{A_{\mathcal{B}}} \times \theta_1(\mathbf{C}^{A_{\mathcal{B}}})$. This is clearly θ_2-stable, since θ_1 has order 2.

Once more, the identical argument over the residue fields yields a θ-stable Borel subgroup of $\mathbf{G}$ which contains T, verifying H8(1).

5.3.8 Characters of T and T_θ. Here we verify H8(2) (page 50) and H10 (page 50). Again, write $q = q_{F_0}$. Fix a primitive $\left(q^{2n}-1\right)^{\mathrm{th}}$ root of unity $\eta \in \mathbb{C}^\times$ and a generator ζ of $k_E^\times$. Any character of $\mathsf{T} \simeq k_E^\times$ is of the form $\lambda_\ell(\zeta) = \eta^\ell$, for some integer $0 \le \ell \le q^{2n} - 2$. Comparing with the symplectic case (§5.2.7), we have the same situation, where here the tower of fields $k_{F_0} \subset k_{E_0} \subset k_E$ takes the place of the tower of fields $k_F \subset k_L \subset k_E$ in the symplectic case. Therefore, λ_ℓ is θ-stable if and only if $\ell = \ell_0(q^n - 1)$, for some $0 \le \ell_0 \le q^n$. To verify H8(2), we note that there exists at least one such θ-stable character which is also in general position. In particular, as in the symplectic case, we may choose $\ell_0 = 1$.

Before we verify H10, we will show that T_θ is non-degenerate in G_θ in the sense of [8, Proposition 3.6.1]. Now, $\mathbf{G}_\theta$ is isomorphic over k_F to $\mathbf{GL}_n$, via the composition of the map $\mathbf{G} \to \mathbf{GL}_n^\Sigma$ which defines $\mathbf{G}$ with projection $\mathbf{GL}_n^\Sigma \to \mathbf{GL}_n$ onto the first component of $\mathbf{GL}_n^\Sigma$. Since this same map also induces the isomorphism $\mathbf{G}(k_{F_0}) \simeq \mathbf{GL}_n(k_F)$, to verify (iii) of [8, Proposition 3.6.1] it suffices to show that T_θ contains an element with distinct eigenvalues as an element of $\mathbf{GL}_n(k_F)$. Under the identification $\mathsf{T} \simeq k_E^\times$, it suffices to show that $(k_E^\times)_\theta$ contains an element whose $\mathrm{Gal}(k_E/k_F)$-conjugates are all distinct. Given our generator ζ of $k_E^\times$, ζ^{q^n-1} is a generator of $(k_E^\times)_\theta$. Since $\gcd(n,2)=1$, it follows that ζ^{q^n-1} is the required element of T_θ.

As mentioned in §5.3.6, we are in the situation of §4.6. Therefore, instead of H10, we need only show that there exists a character of T_θ which is in general position with respect to $W_{\mathbf{G}_\theta}(\mathsf{T}_\theta)^{k_{F_0}}$. Since T_θ is non-degenerate in G_θ, by [8, Corollary 3.6.5] we have $W_{\mathbf{G}_\theta}(\mathsf{T}_\theta)^{k_{F_0}} \simeq N_{\mathsf{G}_\theta}(\mathsf{T}_\theta)/\mathsf{T}_\theta$. However, this last group is trivial, hence every character of T_θ is in general position.

5.3.9 The result. We are now in the situation of §4.6. Therefore, from (4.6.1) we have

$$\Theta_{\pi^+}\left({}^g(\gamma\theta)\right) = 2\,\varepsilon_+\,\varepsilon_\theta\,\lambda^+(\theta)\,d(\pi_1^+)\,\frac{\deg(\sigma_\theta)}{\deg(\sigma)}\,\hat{\mu}_{X_\theta}(\log\gamma)$$

for any $g \in G$ and any topologically unipotent element $\gamma \in K_{0,\theta}$ such that $\gamma\theta$ is G-regular in G^+. From [15, Proposition 12.9], we have

$$\frac{\deg(\sigma_\theta)}{\deg(\sigma)} = \left\| \frac{\mathsf{G}}{\mathsf{G}_\theta} \right\|_{p'}^{-1} \left| \frac{\mathsf{T}}{\mathsf{T}_\theta} \right| = \left(\frac{q^n-1}{\prod_{\ell=1}^n (q^{n+i}-1)} \right) \left(\prod_{\ell=1}^{(n+1)/2} \frac{q^{2\ell-1}+1}{q^{2\ell-1}-1} \right),$$

where $q = q_{F_0}$.

Appendix

A. A convergence counterexample

As discussed in §3.4, there exist cases where

$$\Theta(\varphi,g)=\int_{Z'\backslash G^+}\int_K \varphi\big({}^{xk}g\big)\,dk\,d\dot{x} \tag{A.1}$$

fails to converge for certain quasi-regular (even regular) $g\in G^+$, where φ is a sum of matrix coefficients of the extension to G^+ of an irreducible, supercuspidal representation π of G, Z' is an appropriate closed subgroup of $Z(G^+)$, and K is a compact, open subgroup of G^+ with normalized Haar measure dk. In this appendix we provide such an example.

A.1 Definition of θ. Let p be a prime such that $p\equiv 3 \mod 4$, and let F be a finite extension of $\mathbb{Q}_p$. Assume that -1 is non-square in $k_F^\times$ (for example, we could take $F=\mathbb{Q}_p$). Write $q=q_F=|k_F|$. Let $\mathbf{G}=\mathbf{GL}_2$, $G=\mathbf{G}(F)$, and $Z=Z(G)$. Use the setup and notation of §5.2.2. Since $L=F$ in the present case, we may choose $\mathcal{B}=\{1\}$, so that for $x\in\mathbf{G}$, $\theta(x)=J^{-1}\,{}^{\mathrm{t}}x^{-1}J$, where J is the skew-symmetric matrix $\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$. Notice that $\theta(x)=(\det x)^{-1}x$ for any $x\in\mathbf{G}$. It follows that $\mathbf{G}_\theta=\mathbf{SL}_2$ and $Z(G^+)=\{\pm1\}$. It will make no difference to the convergence (or lack thereof) of (A.1) whether we choose $Z'=Z(G^+)$ or $Z'=\{1\}$, so take the latter. Recall that $K_0=\mathrm{GL}_2(\mathcal{O}_F)$ and $K_1=1+\mathrm{M}_2(\mathscr{P}_F)$. These are θ-stable subgroups of G, and the automorphism of $\mathsf{G}=K_0/K_1\simeq\mathbf{GL}_2(k_F)=\mathrm{GL}(2,q)$ induced by θ (also denoted θ) has the same form as above. Identify G and $\mathrm{GL}(2,q)$.

A.2 An elliptic θ-stable torus. Continuing as in §5.2.5, we may choose $\epsilon=-1$. Then $E=F(\sqrt{-1})$, and

$$\mathbf{T}=\left\{\ \gamma(a,b)=\begin{pmatrix}a&-b\\b&a\end{pmatrix}\ \middle|\ a,b\in\bar{F},\quad \det\gamma(a,b)\neq 0\right\}$$

is θ-stable. Let $a,b\in F$ such that $\gamma(a,b)\in T$. Note that $\gamma(a,b)$ is θ-fixed if and only if $\det\gamma(a,b)=a^2+b^2=1$. The isomorphism $\tilde{f}_{\mathcal{C}}\colon T\to E^\times$ is given by $\gamma(a,b)\mapsto a+b\sqrt{-1}$, so that the induced action of θ on $E^\times$ is

$$\theta(a+b\sqrt{-1})=(a-b\sqrt{-1})^{-1}.$$

We identify T and $E^\times$ via $\tilde{f}_{\mathcal{C}}$. Now, $\bar{\epsilon}=-1$ and $k_E\simeq k_F(\sqrt{-1})$, and we have analogous definitions and properties for k_E and T over k_F. Again, we identify T and $k_E^\times$ via the map $\tilde{f}_{\bar{\mathcal{C}}}$.

A.3 Regular elements. We refer to §1.3.2. Let $\mathfrak{g}=\mathbf{M}_2(F)$. An element $g\in G$ is regular if and only if its eigenvalues are distinct. Consider the notion of regular elements in G^+ determined by the discriminant functions D_0,D_1 of [10]. We have

$$D_1(g\theta)=2D_0(g),\qquad D_0(g\theta(g))=(\operatorname{tr}g)^2(\det g)^{-1}D_0(g),\qquad \big(g\in G\big).$$

Therefore, for $g\in G$, $g\theta$ is regular if and only if g is regular, and G-regular if and only if g is regular and has non-zero trace. We will be interested in certain regular elements of our torus T. Suppose $g=\gamma(a,b)\in T$

for some $a,b \in F$. The eigenvalues of g are $a \pm b\sqrt{-1}$, so g is regular if and only if $b \neq 0$. Suppose $ab \neq 0$. Then $\operatorname{tr} g$ is also non-zero, and so $g\theta$ is regular and G-regular. By Lemma 1.3.14, $g\theta$ is then also quasi-regular. Without appealing to this lemma, one can calculate directly that $\ker(\mathrm{Ad}(g\theta)-1)$ is spanned over F by J in this case.

A.4 Conjugacy classes in G**.** There are $q-1$ central conjugacy classes in G. We may choose representatives for the non-central conjugacy classes of G as follows ([5]). There are $q-1$ classes represented by the elements in the set $\left\{ \left(\begin{smallmatrix} \alpha & 1 \\ 0 & \alpha \end{smallmatrix}\right) \mid \alpha \in k_F^\times \right\}$, $(q-1)(q-2)/2$ classes doubly represented by the elements in the set $\{\mathrm{diag}(\alpha,\beta) \mid \alpha,\beta \in k_F^\times, \alpha \neq \beta\}$, where $\mathrm{diag}(\beta,\alpha)$ is in the same class as $\mathrm{diag}(\alpha,\beta)$, and $q(q-1)/2$ classes doubly represented by the non-central elements of T, where $\gamma(\alpha,\beta)$ is in the same class as $\gamma(\alpha,-\beta)$ for $\alpha,\beta \in k_F$ with $\beta \neq 0$. In total, there are q^2-1 classes.

We will also find it useful to parameterize non-central classes through the following easily verified proposition.

Proposition A.7. *The class of a non-central element $x \in \mathsf{G}$ is uniquely determined by the pair* $(\det x, \operatorname{tr} x)$.

A.5 A character of T **and Deligne-Lusztig induction.** Moving now to §5.2.7, let $\iota = \sqrt{-1} \in \mathbb{C}^\times$, and take $\eta = \exp\big(2\pi\iota/(q^2-1)\big)$. Let ζ be a generator of $k_E^\times$, and let $\lambda = \lambda_{q-1}$, so that $\lambda(\zeta) = \eta^{q-1}$. Then λ is both θ-stable and in general position. Let χ be the irreducible character $-\mathsf{R}_{\mathsf{T}}^{\mathsf{G}}\lambda$ of G. We will be interested in the values of χ on non-central elements of G with square determinant and zero trace. Examining our list of conjugacy class representatives in §A.4, we see that the only classes of such elements are those containing $\gamma(0,\beta)$ for some $\beta \in k_F^\times$. Under the identification $\mathsf{T} \simeq k_E^\times$, we have $\gamma(0,\pm\beta) \leftrightarrow \pm\beta\sqrt{-1}$. Let $c_0 = \lambda(\sqrt{-1})$. Then by Lemma 5.2.9, $\lambda(-1) = 1$, and so $c_0 = \pm 1$. Moreover, $\lambda(\pm\beta\sqrt{-1}) = c_0$ for any $\beta \in k_F^\times$. From the character table given in [5, Chapter 16, Exercise 18], we now have $\chi\big(\gamma(0,\pm\beta)\big) = -2c_0$ for any $\beta \in k_F^\times$.

A.6 Definition of π**.** Let σ be a representation of G with character χ. Using Lemma 5.2.9, from the character table given in [5, Chapter 16, Exercise 18] we see that $Z(\mathsf{G}) \subseteq \ker\sigma$. Since $\theta(x) = (\det x)^{-1}x$ for any $x \in \mathsf{G}$, it follows that $\sigma \circ \theta = \sigma$. Inflate σ to K_0 and extend to ZK_0 by setting $\sigma(\varpi_F) = 1$. This extension also satisfies $\sigma \circ \theta = \sigma$, so we may extend σ to a representation σ^+ of $(ZK_0)^+$ by setting $\sigma^+(\theta) = 1$. Let χ_{σ^+} be the character of σ^+, and let $\dot\chi_{\sigma^+}$ be its extension by zero to G^+.

Let $\pi = \text{c-Ind}_{ZK_0}^{G}\,\sigma$, an irreducible, supercuspidal representation of G. Define π^+ using $A_\pi = \Phi(\sigma^+(\theta))$, for Φ as in Proposition 1.5.2. Then $\pi^+ \simeq \text{c-Ind}_{(ZK_0)^+}^{G^+}\,\sigma^+$ (Proposition 1.5.4), and so $\dot\chi_{\sigma^+}$ is a sum of matrix coefficients of π^+.

A.7 The inner integral. In (A.1), we will take $K = K_0$, and consider $g \mapsto \Theta(\dot\chi_{\sigma^+}, g\theta)$ as a function on G. Fix $g \in G$, and define

$$f_g : G^+ \to \mathbb{C}, \qquad\qquad x \mapsto \int_{K_0} \dot\chi_{\sigma^+}\big({}^{xk}(g\theta)\big)\,dk.$$

Notice that f_g is invariant under left-translation by elements of $(ZK_0)^+$ and under right-translation by elements of K_0. Since K_0 is θ-stable, through a change of variables in the integral above we obtain $f_g(x\theta) = f_{\theta(g)}(x)$ for any $x \in G$, so we will concentrate on the properties of $f_g|G$. Let G act on itself by

θ-twisted conjugation. That is, for $x, y \in G$, let $x \cdot y = xy\theta(x^{-1})$. Since $\sigma^+(h\theta) = \sigma(h)$ for any $h \in ZK_0$, we have

$$f_g(x) = \int_{K_0} \dot{\chi}_\sigma\big((xk)\cdot g\big)\, dk, \qquad \big(x \in G\big). \tag{A.2}$$

The integrand above is zero unless its argument lies in ZK_0. Let $\mathcal{K}_g$ be the set-valued function on G given by

$$\mathcal{K}_g(x) = \{\, k \in K_0 \mid (xk)\cdot g \in ZK_0 \,\}, \qquad \big(x \in G\big),$$

and write

$$\operatorname{supp}\mathcal{K}_g = \{x \in G \mid \mathcal{K}_g(x) \neq \emptyset\}.$$

Clearly, $f_g(x) = 0$ for $x \in G \smallsetminus \operatorname{supp}\mathcal{K}_g$, and f_g is identically zero on G if $\operatorname{supp}\mathcal{K}_g$ is empty. So suppose it is not empty. The following formulas will allow us to determine $f_g|G$ completely in some cases. We have

$$\begin{aligned} \det\big((xk)\cdot g\big) &= (\det k)^2(\det x)^2(\det g) \\ &= \varpi_F^{2\operatorname{val}_F(\det x)}\,\varpi_F^{\operatorname{val}_F(\det g)}\,(\det k)^2\big(\operatorname{int}_F(\det x)\big)^2\big(\operatorname{int}_F(\det g)\big), \end{aligned} \tag{A.3}$$

$$\begin{aligned} \operatorname{tr}\big((xk)\cdot g\big) &= (\det x)(\det k)(\operatorname{tr} g) \\ &= \varpi_F^{\operatorname{val}_F(\det x)}\,(\det k)\big(\operatorname{int}_F(\det x)\big)(\operatorname{tr} g). \end{aligned} \tag{A.4}$$

Lemma A.1. *Let $x \in \operatorname{supp}\mathcal{K}_g$, $k \in \mathcal{K}_g(x)$, and write $y = (xk)\cdot g$. Then,*

(1) $\operatorname{val}_F(\det g)$ *is even;*

(2) $y_0 = \varpi_F^{\ell} y \in K_0$ *for* $\ell = -\operatorname{val}_F(\det x) - \frac{1}{2}\operatorname{val}_F(\det g)$*;*

(3) the image of $(\det y_0)\big(\operatorname{int}_F(\det g)\big)^{-1}$ *in* $k_F^\times$ *is square;*

(4) $\varpi_F^{-\operatorname{val}_F(\det g)/2}\operatorname{tr} g \in \mathcal{O}_F$*; and*

(5) $\operatorname{tr} y_0 \equiv 0 \mod \mathcal{P}_F$ *if and only if* $\varpi_F^{-\operatorname{val}_F(\det g)/2}\operatorname{tr} g \equiv 0 \mod \mathcal{P}_F$.

Proof. We prove (1) and (2) simultaneously. Since $y \in ZK_0$, $y_0 = \varpi_F^{\ell} y \in K_0$ for some integer ℓ. But from (A.3) we must have

$$2\ell + 2\operatorname{val}_F(\det x) + \operatorname{val}_F(\det g) = 0,$$

which is only possible if $\operatorname{val}_F(\det g)$ is even and $\ell = -\operatorname{val}_F(\det x) - \frac{1}{2}\operatorname{val}_F(\det g)$. Again from (A.3), we now have that $(\det y_0)\big(\operatorname{int}_F(\det g)\big)^{-1} = (\det k)^2\big(\operatorname{int}_F(\det x)\big)^2$, and (3) follows. Statements (4) and (5) are obvious from (A.4) and the formula for ℓ in (2). ☐

Proposition A.2. *If the image of* $\operatorname{int}_F(\det g)$ *is square in* $k_F^\times$ *and* $\varpi_F^{-\operatorname{val}_F(\det g)/2}(\operatorname{tr} g) \equiv 0 \mod \mathcal{P}_F$*, then*

$$f_g(x) = -2\,c_0 \operatorname{meas}(\mathcal{K}_g(x), dk), \qquad \big(x \in \operatorname{supp}\mathcal{K}_g\big).$$

Proof. Let x, k, y and y_0 be as in Lemma A.1. Combining our hypotheses on g with statements (3) and (5) of Lemma A.1, we see that the image of y_0 in G has square determinant and zero trace. From the discussion in §A.5, we have $\chi_\sigma(y) = -2c_0$, and the result now follows. ☐

A.8 Divergence. Finally, we will exploit Proposition A.2 to make $\Theta(\dot{\chi}_{\sigma^+}, g\theta)$ diverge for certain $g \in G$. For the following, use the convention $\mathrm{val}_F(0) = \infty$.

Lemma A.3. *Suppose a, b are elements of F such that $b \neq 0$ and $\mathrm{val}_F(a) \geq \mathrm{val}_F(b)$. Then for $g = \gamma(a,b)$, $\mathcal{K}_g(1) = K_0$.*

Proof. We have $\det g = a^2 + b^2$, and claim that $\mathrm{val}_F(\det g) = 2\,\mathrm{val}_F(b)$. If $\mathrm{val}_F(a) > \mathrm{val}_F(b)$, this is immediate. If $\mathrm{val}_F(a) = \mathrm{val}_F(b) \neq \infty$, then since -1 is non-square in $k_F^\times$, we must have $\mathrm{int}_F(a)^2 + \mathrm{int}_F(b)^2 \not\equiv 0 \mod \mathscr{P}_F$. Therefore, there is no cancellation in the sum $a^2 + b^2$, so that $\mathrm{val}_F(\det g) = 2\,\mathrm{val}_F(b)$. Suppose $k \in K_0$, and set $y_0 = \varpi_F^{-\mathrm{val}_F(b)}(k \cdot g)$. Then one may check directly that $y_0 \in \mathrm{M}_2(\mathscr{O}_F)$, and from (A.3) we have $\det y_0 \in \mathscr{O}_F^\times$. □

Proposition A.4. *Let a, b, and g be as in Lemma A.3, but assume that $\mathrm{val}_F(a) \gneqq \mathrm{val}_F(b)$. Then $\Theta(\dot{\chi}_{\sigma^+}, g\theta)$ diverges. In particular, there exists a regular element $g \in G$ for which $g\theta$ is regular, G-regular, and quasi-regular in G^+, and $\Theta(\dot{\chi}_{\sigma^+}, g\theta)$ diverges.*

Proof. As in the proof of Lemma A.3, $\mathrm{val}_F(\det g) = 2\,\mathrm{val}_F(b)$. With the added restriction $\mathrm{val}_F(a) \gneqq \mathrm{val}_F(b)$, we have $\mathrm{int}_F(\det g) \equiv (\mathrm{int}_F(b))^2 \mod \mathscr{P}_F$, so that the image of $\mathrm{int}_F(\det g)$ in $k_F^\times$ is square. We also have

$$\mathrm{val}_F(\varpi_F^{-\mathrm{val}_F(\det g)/2}\,\mathrm{tr}\, g) = \mathrm{val}_F(a) - \mathrm{val}_F(b) > 0.$$

Since $f_g(x\theta) = f_{\theta(g)}(x) = f_g(x)$ for any $x \in G$, f_g is completely determined by Proposition A.2. In particular, f_g is real-valued and either non-negative or non-positive. It follows that $\Theta(\dot{\chi}_{\sigma^+}, g\theta)$ is also real-valued. From Lemma A.3, $\mathcal{K}_g(1) = K_0$, so that $f_g(1) = -2c_0$. But f_g is invariant under left-translation by elements of $(ZK_0)^+$, so $f_g(x) = -2c_0$ for any $x \in (ZK_0)^+$. Recall that we have chosen $Z' = \{1\}$. Let dx be Haar measure on G^+, normalized so that $\mathrm{meas}(K_0, dx) = 1$. Then,

$$\left|\Theta(\dot{\chi}_{\sigma^+}, g\theta)\right|_\infty = \left|\int_{G^+} f_g(x)\,dx\right|_\infty \geq \left|\int_{G^+} \mathrm{ch}_{(ZK_0)^+}(x)\, f_g(x)\,dx\right|_\infty = 2\,\mathrm{meas}((ZK_0)^+, dx),$$

where $|\cdot|_\infty$ is absolute value on $\mathbb{R}$. However, by the invariance of dx we have

$$\mathrm{meas}((ZK_0)^+, dx) = 2\,\mathrm{meas}(ZK_0, dx) = 2\sum\nolimits_{j\in\mathbb{Z}} \mathrm{meas}(\varpi_F^j K_0, dx) = 2\sum\nolimits_{j\in\mathbb{Z}} 1,$$

which proves the first assertion. Concerning the second assertion on existence, we only need add the restriction that $a \neq 0$. □

Remark. If we are interested in finding an element $g \in G$ as in Proposition A.4 that is also θ-fixed, we may appeal to Hensel's Lemma. Suppose a is a non-zero element of $\mathscr{P}_F$, and let $f(X) = X^2 + a^2 - 1$. Then

$$\left|f(1+a')\right|_F < 1 = \left|f'(1+a')\right|_F^2, \qquad (a' \in \mathscr{P}_F),$$

where f' is the formal derivative of f. Therefore, by Hensel's Lemma there exists a root b of f in $\mathscr{O}_F$. Note that b must in fact lie in $\mathscr{O}_F^\times$, so that $\mathrm{val}_F(a) \gneqq \mathrm{val}_F(b)$. Now, $f(b) = 0$ says that $\det\gamma(a,b) = a^2 + b^2 = 1$, so that $g = \gamma(a,b)$ is the required θ-fixed element of G.

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