INCOMPRESSIBILITY OF GENERIC TORSORS OF NORM TORI

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To Nikolai Aleksandrovich,
my First Teacher in Algebraic Groups

Abstract. Let p be a prime integer, F a field of characteristic not p, T the norm torus of a degree p^n extension field of F, and E a T-torsor over F such that the degree of each closed point on E is divisible by p^n (a generic T-torsor has this property). We prove that E is p-incompressible. Moreover, all smooth compactifications of E (including those given by toric varieties) are p-incompressible. The main requisites of the proof are: (1) A. Merkurjev's degree formula (requiring the characteristic assumption), generalizing M. Rost's degree formula, and (2) combinatorial construction of a smooth projective fan invariant under an action of a finite group on the ambient lattice due to J.-L. Colliot-Thélène - D. Harari - A.N. Skorobogatov, produced by refinement of J.-L. Brylinski's method with a help of an idea of K. Künnemann.

Let F be a field, p a prime integer. We say that an F-variety (by which we mean just a separated F-scheme of finite type) is *p-incompressible* (resp., *incompressible*), if its *canonical p-dimension* (resp., *canonical dimension*), defined as in [13, §4b], is equal to its usual dimension. An integral variety X is incompressible if and only if any rational map $X \dashrightarrow X$ is dominant, [13, Proposition 4.3]; p-incompressibility is a p-local version of incompressibility implying the incompressibility.

Given an arbitrary F-variety V, we write n_V for the greatest common divisor of the degrees of the closed points on V. Usually, we are only interested in $v_p(n_V)$, where v_p is the p-adic valuation.

By a *compactification* of an F-variety V we mean a complete F-variety X containing a dense open subvariety isomorphic to V.

Given a finite separable extension field (or, more generally, an étale algebra) K/F, its *norm torus* $T = T_{K/F}$, also called *norm one torus* and usually denoted by $\mathcal{R}^{(1)}_{K/F}(\mathbb{G}_\mathrm{m})$, is the algebraic torus defined as the kernel of the norm map of algebraic tori

$$N_{K/F} : \mathcal{R}_{K/F}(\mathbb{G}_\mathrm{m}) \to \mathbb{G}_{\mathrm{m},F},$$

where $\mathcal{R}_{K/F}$ is the Weil transfer with respect to K/F. The group of F-points of T is the subgroup of norm 1 elements in $K^\times$. As a variety, T is the hypersurface in the affine F-space K given by the equation $N_{K/F} = 1$.

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We consider T-torsors over F (i.e., the principal homogeneous spaces of T) and call them simply *T-torsors*. Any element $a \in F^\times$ produces a T-torsor E_a with the set of F-points being the set of norm a elements in $K^\times$. As a variety, E_a is the hypersurface in the affine F-space K given by the equation $N_{K/F} = a$. The isomorphism class of E_a corresponds to the image of a under the connecting homomorphism $H^0(F, \mathbb{G}_\mathrm{m}) \to H^1(F, T)$ of the long exact sequence in Galois cohomology, arising from the short exact sequence

$$0 \to T \to \mathcal{R}_{K/F}(\mathbb{G}_\mathrm{m}) \to \mathbb{G}_{\mathrm{m},F} \to 0$$

of the definition of T. This connecting homomorphism is surjective (and its kernel is the norm subgroup). Thus every T-torsor E is isomorphic to E_a for some element $a \in F^\times$ (whose class modulo $N_{K/F}(K^\times)$ is uniquely determined by E).

Note that for any E, the integer n_E divides $\dim_F K$. Moreover, if K is the product of étale F-algebras $K_1, \dots, K_r$, then n_E divides $\dim_F K_i$ for each $i = 1, \dots, r$. In particular, $n_E = \dim_F K$ is possible only if K is a field.

The main result of this note is the following theorem known for cyclic K/F (see [14, Question 0.9]):

Theorem 1. *Assume that* $\operatorname{char} F \neq p$. *For some integer $n \geq 0$, let K/F be a (separable) extension field of degree p^n, T its norm torus, and E a T-torsor such that $n_E = p^n$. Then the F-variety E is p-incompressible. Any smooth compactification of the variety E is also p-incompressible.*

Example 2. Let t be an indeterminate, K/F an arbitrary finite separable field extension, and E_t the $T_{K(t)/F(t)}$-torsor of norm t elements (E_t is the *generic principal homogeneous space of* T, or *generic T-torsor* in the sense of [12, §3], produced out of the imbedding of T into the special algebraic group $\mathcal{R}_{K/F}(\mathbb{G}_\mathrm{m})$). Then the degree of every closed point on E_t is divisible by $d := [K : F]$. Proving this, one may replace the base field $F(t)$ by $F((t))$. If for a finite extension field $L/F((t))$, the element $t \in L$ is the norm for the d-dimensional étale L-algebra $K((t)) \otimes_{F((t))} L$, then $v(t)$ is divisible by d (see [4, Theorem in §(2.5) of Chapter 2]), where v is the extension to L of the t-adic discrete valuation on $F((t))$; therefore d divides $[L : F((t))]$ (see [4, Exercise 1c in §2 of Chapter 2]).

Proof of Theorem 1. According to [2, Corollaire 1], there exists a smooth projective (toric) F-variety X containing E as an open subvariety. Clearly, E is p-incompressible if X is so. (Actually, by [18, Proposition 4], for any extension field L/F, one has $X(L) \neq \emptyset$ if and *only if* $E(L) \neq \emptyset$ so that E is p-incompressible if and *only if* X is so.) Since the property of being p-incompressible is birationally invariant on connected smooth complete varieties (see e.g. [10, Remark 4.13] or [9, Lemma 3.6]), all smooth compactifications of E are p-incompressible, provided that X is so.

A connected smooth complete variety V over a field of characteristic $\neq p$ is called *(p, n)-rigid* here, if it is R^p-rigid in the sense of [15, §7] for the infinite sequence

$$R := (0, \dots, 0, 1, 0, \dots)$$

of 0 and 1 with precisely one 1 staying on the nth position. By definition, (p, n)-rigidity of V means that $v_p(n_V) = v_p(\deg c_R(-T_V))$, where T_V is the tangent bundle of V and c_R is the Chern class corresponding to the sequence R and the prime p in the sense of [15, §4]. Note that the Chern class c_R (for our particular choice of R) is additive (and known under the name the *additive Chern class*, cf. [15, Remark 4.1]) so that the minus sign in the above definition of (p,n)-rigidity can be omitted (but it is important to keep it in the definition of other types of rigidity corresponding to different choices of the sequence R, cf. [15, §7]).

The property of being (p,n)-rigid is a birationally invariant, cf. [15, Remark 7.5]. Moreover, we have the following

Lemma 3. *Let V and V' be connected smooth complete varieties of dimension p^n-1 over a field F of characteristic $\neq p$. Assume that each of the varieties $V_{F(V')}$ and $V'_{F(V)}$ has a closed point of degree prime to p. Then $v_p(n_V)=v_p(n_{V'})$. The variety V is (p,n)-rigid if and only if V' is so.*

Proof. The conditions on V and V' ensure that for any extension field K/F, the K-variety V_K has a closed point of prime to p degree if and only if V'_K does. It follows that $v_p(n_V)=v_p(n_{V'})$. With this equality in hand, the second statement follows by [15, Theorem 7.2(2b)]. □

We finish our proof of Theorem 1. By Theorem 4 right below, the variety X is (p,n)-rigid. A (p,n)-rigid variety is p-incompressible by [15, Corollary 7.3] (the converse is not true). This statement is the place where the degree formula of [15] is used and where the characteristic assumption is needed. The projectivity assumption made in [15] is superfluous because of [1, §10]. □

Theorem 4. *For E as in Theorem 1, any smooth compactification of E is (p,n)-rigid.*

Proof. Since the property of being (p,n)-rigid is birationally invariant, e.g., by Lemma 3, it suffices to construct one (p,n)-rigid smooth compactification of E. For this, let Γ be the Galois group of the normalization L of K/F and let $\mathfrak{X}$ be the Γ-set corresponding to the étale F-algebra K in the sense of [11, §18]. The cardinality of the set $\mathfrak{X}$ is equal to $p^n=[K:F]$.

The cocharacter lattice N of the split torus T_L is the lattice of the elements in the free abelian group $\mathbb{Z}[\mathfrak{X}]$ on $\mathfrak{X}$ with the sum of coordinates $=0$. There exists a smooth projective fan A of the lattice N (we do not require that A is invariant under the action of Γ on N yet), for instance, a fan producing the toric variety given by the projective space (see [6, Exercise of §1.4]).

The symmetric group S of all permutations of the set $\mathfrak{X}$ acts on N by permutations of the coordinates. By [2, Theorem 1], there exists a smooth projective S-invariant fan B of N which is a subdivision of A. This produces a smooth projective toric variety X_k (over any given field k) endowed with an action of S (see [3, §5.5]) as well as with an action of the split k-torus with the cocharacter lattice N. Actually, there is a scheme X (over the integers $\mathbb{Z}$) such that for any field k, the k-variety X_k is obtained from X by the base change $\mathbb{Z}\to k$.

In particular, the subgroup $\Gamma\hookrightarrow S$ acts on X_F. Twisting X_F by the principal homogenous space $\operatorname{Spec} L$ of the constant algebraic group Γ (quasi-projectivity of X_F is needed for existence of the twisting, see [5, Proposition 2.12] or [17, V.20]) we get a smooth projective T-equivariant compactification X of T (cf. [2, Preuve du Corollaire 1 á partir du Théorème 1]). Twisting afterwards X by the T-torsor E as in [5, Proposition 2.12] (using (quasi-)projectivity once again), we get a smooth compactification Y of E. We claim that the variety Y is (p,n)-rigid.

First of all, by [18, Proposition 4], we have $v_p(n_Y) = v_p(n_E) = n$. Therefore, to check (p,n)-rigidity of Y we have to check that $v_p(\deg c_R(-T_Y)) = n$, where R is the sequence introduced above. The integer $\deg c_R(-T_Y)$ does not depend on the base field anymore so that we may replace Y by X_k with an arbitrary chosen field $k \supset F$.

Let us choose a field $k \supset F$ possessing a degree p^n cyclic extension field l such that for its norm torus $T' = T_{l/k}$ there exists a T'-torsor E' with $v_p(n_{E'}) = n$ (we can find such k with a help of Example 2). Fixing an arbitrary bijection of the (order p^n cyclic) Galois group Γ' of l/k with the set $\mathfrak{X}$, we get an action of Γ' on $\mathfrak{X}$ and therefore on X_k. Twisting X_k by the principal homogeneous space $\operatorname{Spec} l$ of Γ' and then by the T'-torsor E', we get a smooth compactification Y' of E'. The torsor E' is the variety of norm a elements in l for some $a \in k^\times$. Let S be the Severi-Brauer variety of the cyclic central simple algebra $(l/k, a)$. By [11, Proposition 30.6], there exist rational maps in both directions between the varieties S and E' (one can actually show that E' is isomorphic to an open subset of S). It follows that $v_p(n_{Y'}) = v_p(n_S)$ (actually, $n_{Y'} = n_S$, cf. [15, Remark 6.6]). In particular, $(l/k, a)$ is a division algebra, and so the Severi-Brauer variety S is (p,n)-rigid by [15, §7.2] (this is only about a computation of $c_R(-T)$ for a projective space). Therefore, the variety Y' is (p,n)-rigid by Lemma 3. Since $v_p(n_{Y'}) = n$, it follows that $v_p(\deg c_R(-T_{Y'})) = n$. Summarizing, the integer $v_p(\deg c_R(-T))$ for Y is the same as for X_k, as well as for Y' and is therefore equal to n. □

A (p,n)-rigid variety is actually *strongly p-incompressible* in the sense of [8, §2] as well as of [13, §4d]. Therefore, for E as in Theorem 4, any smooth compactification of E is strongly p-incompressible. This statement is stronger than the part of the statement of Theorem 1 saying that any smooth compactification of E is p-incompressible. It can be formulated in terms of E alone (without mentioning its compactification) as follows:

Corollary 5. *Let E be as in Theorem 1 and let Y be an integral complete (not necessarily smooth) F-variety such that $v_p(n_Y) \geq n$ $(= v_p(n_E))$ and $v_p(n_{Y_{F(E)}}) = 0$ (i.e., $Y_{F(E)}$ has a closed point of a prime to p degree). Then*

(1) $\dim Y \geq \dim E$;
(2) *if* $\dim Y = \dim E$ *then* $v_p(Y) = n$ *and* $v_p(n_{E_{F(Y)}}) = 0$.

Proof. Apply the strong incompressibility of a smooth compactification of E given by a toric variety X, taking into account [18, Proposition 4] saying that for any extension field L/F, $X(L) = \emptyset$ provided that $E(L) = \emptyset$. □

Here is an application of Theorem 1 suggested in [13, §11d]:

Corollary 6. *For any p-primary (separable) field extension K/F in characteristic $\neq p$, the essential p-dimension as well as the essential dimension of the functor of non-zero norms of K/F (defined as in* [13, Example 11.11]*) is equal to the degree* $[K : F]$.

Proof. See [13, Corollary 11.5]. □

Remark 7. In the case of cyclic K/F, the statement of Corollary 6 as well as the statement of Theorem 1 holds also in characteristic p due to existence of a proof of

p-incompressibility for Severi-Brauer varieties avoiding a use of the degree formula (see [8, Examples 2.4 and 3.3]). On the other hand, neither Theorem 4 nor Corollary 5 are known in characteristic p even for cyclic field extensions. One may expect that Theorems 1, 4 and Corollaries 5, 6 hold in characteristic p for general (separable and p-primary) K/F. This is so in the case of $[K : F] = p$ (i.e., in the case of $n = 1$) due to results of O. Haution, [7, Corollary 10.2] (for Theorem 1 and Corollary 6 alone it suffices to use [16, Proposition 1.5(2)]=[13, Proposition 2.5(2)]).

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