ISOTROPY OF 8-DIMENSIONAL QUADRATIC FORMS OVER FUNCTION FIELDS OF QUADRICS

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Abstract. Let F be a field of characteristic different from 2 and let ϕ be an anisotropic 8-dimensional quadratic form over F with trivial determinant. We study the remaining open cases in the problem of describing the quadratic forms ψ such that ϕ becomes isotropic over the function field $F(\psi)$.

0. Introduction

Let F be a field of characteristic different from 2 and let ϕ and ψ be two anisotropic quadratic forms over F. An important problem in the algebraic theory of quadratic forms is to find conditions on ϕ and ψ so that $\phi_{F(\psi)}$ is isotropic. More precisely, one studies the question whether the isotropy of ϕ over $F(\psi)$ is standard in a sense. In this paper we consider the case where ϕ is an 8-dimensional anisotropic quadratic form with trivial determinant. Necessity of certain sufficient conditions for isotropy of ϕ over $F(\psi)$ was studied by A. Laghribi; we call the isotropy, caused by one of these conditions (slightly modified in fact), *L-standard*:

Definition. Let ϕ and ψ be anisotropic quadratic forms such that $\phi_{F(\psi)}$ is isotropic. Moreover, suppose that $\dim\phi = 8$ and $\det\phi = 1$. We say that the isotropy of $\phi_{F(\psi)}$ is *L-standard*, if at least one of the following conditions holds:

(L1) there exists a half-neighbor ϕ^* of ϕ such that $\psi \subset \phi^*$;

(L2) there exists a subform $\phi_0 \subset \phi$ and an anisotropic form $\pi \in W(F(\psi)/F)$ such that $\phi_0 \subset \pi$ and $\dim\phi_0 > \frac{1}{2}\dim\pi$.

The main purpose of this paper is to study the question of whether the isotropy of $\phi_{F(\psi)}$ is always L-standard.

There are plenty of examples of L-standard isotropy. If ψ is similar to a subform of ϕ, i.e. $\psi \subset k\phi$ for some $k \in F^*$, then the form $\phi_{F(\psi)}$ is isotropic. Setting $\phi^* = k\phi$, we see that condition (L1) holds and hence the isotropy of $\phi_{F(\psi)}$ is L-standard. Another simple example of L-standard isotropy occurs if condition (L2) holds and we suppose additionally that π is similar to a Pfister form. Then the conditions $\phi_0 \subset \pi$ and $\dim\phi_0 > \frac{1}{2}\dim\pi$ simply mean that ϕ_0 is a Pfister neighbor of π. Moreover, the condition $\pi \in W(F(\psi)/F)$ is equivalent to the isotropy of $(\phi_0)_{F(\psi)}$. Thus we see that the isotropy of $\phi_{F(\psi)}$ is always L-standard in the following case:

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(L2′) there exists a Pfister neighbor $\phi_0 \subset \phi$ such that $(\phi_0)_{F(\psi)}$ is isotropic.

It is easy to verify (see Corollaries 5.4 and 5.5) that if a quadratic form ψ has dimension ≥ 5 or if $\dim \psi = 4$ and $\det \psi \neq 1$, then condition (L2) is equivalent to condition (L2′).

In the case $\dim \psi = 2$ the isotropy over $F(\psi)$ is always L-standard because $\phi_{F(\psi)}$ isotropic implies ψ is similar to a subform of ϕ ([35, Lemma 5.1 of Chap. 2]).

Another simple case is when $\operatorname{ind} C(\phi) = 1$. Indeed, in this case ϕ is similar to a Pfister form and the form $\phi_{F(\psi)}$ is isotropic if and only if ψ is similar to a subform of ϕ. Hence, condition (L1) holds and therefore the isotropy of $\phi_{F(\psi)}$ is L-standard.

In the case where $\dim \psi = 3$ (or $\dim \psi = 4$ and $\det \psi = 1$) the isotropy of an arbitrary anisotropic form ϕ over $F(\psi)$ was studied in many papers ([3, App. II], [33], [26], [8], [9], [5]). The excellence property of $F(\psi)/F$ (see [3, App. II] or [33, Cor.]) easily shows that condition (L2) holds (see Corollary 5.2). In particular, the isotropy of $\phi_{F(\psi)}$ is L-standard.

The case of a quadratic form ϕ with $\operatorname{ind} C(\phi) = 2$ was studied by Laghribi in [24]. He proved that if the form $\phi_{F(\psi)}$ is isotropic, then ψ is similar to a subform of ϕ or there exists a 3-fold Pfister neighbor $\phi_0 \subset \phi$ such that $(\phi_0)_{F(\psi)}$ is isotropic (see also [7, Cor. 5.5] and [6, Th. 4.1]). In particular, the isotropy is L-standard.

In [24] and [23] Laghribi showed that in the case where $\dim \psi \geq 5$, the isotropy of $\phi_{F(\psi)}$ is always L-standard (see also [14]). Thus, it suffices to consider only the case where $\dim \psi = 4$ and $\det \psi \neq 1$. In [12] the authors proved that the isotropy is L-standard in this case if $\operatorname{ind} C(\phi) = 8$.

Thus one can say that the isotropy is L-standard except (possibly) the following case:

ψ is a 4-dimensional form with nontrivial determinant and $\operatorname{ind} C(\phi) = 4$.

Since $\operatorname{ind} C(\phi) = 4$, there exists a division biquaternion algebra D such that $c(\phi) = [D]$. The case under consideration naturally splits in four cases: $\operatorname{ind} C_0(\psi) \otimes D = \operatorname{ind} C_0(\psi) \otimes C(\phi) = 1$, 2, 4, or 8. The main result of our paper asserts that if $\operatorname{ind} C_0(\psi) \otimes C(\phi) \neq 4$, then the isotropy is L-standard. Taking into account the results mentioned above, we get the following

Theorem. *Let ϕ be an anisotropic 8-dimensional quadratic form with* $\det \phi = 1$ *and ψ be a quadratic form such that $\phi_{F(\psi)}$ is isotropic. Let the case where* $\dim \psi = 4$, $\det \psi \neq 1$, *and* $\operatorname{ind} C(\phi) = \operatorname{ind}(C(\phi) \otimes_F C_0(\psi)) = 4$ *be excluded. Then the isotropy of $\phi_{F(\psi)}$ is L-standard.*

Moreover, we show that presence of the exception in the theorem is essential (see Corollary 8.4): *there exist a field F, an 8-dimensional quadratic form $\phi \in I^2(F)$, and a 4-dimensional F-form ψ with nontrivial determinant such that $\phi_{F(\psi)}$ is isotropic, but the isotropy is not L-standard.*

The proof of the theorem is based on the computation of the second Chow group of the variety $X_\psi \times X_D$ where X_ψ is a projective quadric given by the equation $\psi = 0$ and X_D is Severi–Brauer variety of D. In this paper we prove the following assertion (see Theorem 4.1, Proposition 4.3, and Lemma 7.7):

Theorem. *Let ψ be a 4-dimensional quadratic form with nontrivial determinant and D be a biquaternion F-algebra (here we do not assume that D is a division algebra). Then the group* $\operatorname{Tors}\operatorname{CH}^2(X_\psi \times X_D)$ *is zero or isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Moreover,*

- *if the group* $\operatorname{CH}^2(X_\psi \times X_D)$ *is nontrivial, then*
$$\operatorname{ind} C_0(\psi) \otimes D = 2 \qquad \textit{or} \qquad \operatorname{ind} C_0(\psi) \otimes D = \operatorname{ind} D = 4;$$
- *in the case* $\operatorname{ind} C_0(\psi) \otimes D = 2$ *the group* $\operatorname{CH}^2(X_\psi \times X_D)$ *is trivial if and only if there exist a 3-dimensional subform $\psi_0 \subset \psi$ and a quaternion subalgebra $D_0 \subset D$ such that $C_0(\psi_0) \simeq D_0$.*

This theorem gives a complete description of the group $\operatorname{Tors}\operatorname{CH}^2(X_\psi \times X_D)$ except for the case $\operatorname{ind} C_0(\psi) \otimes D = \operatorname{ind} D = 4$ (the *exceptional case* in what follows). Unfortunately in the exceptional case we have no criterion (written in inner terms of D and ψ) of triviality of the group $\operatorname{Tors}\operatorname{CH}^2(X_\psi \times X_D)$. Nevertheless, we have examples of trivial and nontrivial groups $\operatorname{Tors}\operatorname{CH}^2(X_\psi \times X_D)$ in the exceptional case (an example of trivial torsion can be easily derived from Theorem 4.5; an example of nontrivial torsion is given in Corollary 8.5).

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1. Terminology and notation

1.1. **Quadratic forms.** Mainly, we use the notation of [25] and [35]. However there is a slight difference: we denote by $\langle\langle a_1, \dots, a_n \rangle\rangle$ the n-fold Pfister form $\langle 1, -a_1 \rangle \otimes \cdots \otimes \langle 1, -a_n \rangle$.

We write $\phi \simeq \psi$ if the form ϕ is isometric to ψ and $\phi \sim \psi$ if ϕ is similar to ψ. We do not use a special notation for the Witt class of quadratic forms; notation like $\phi + \psi$ always means the sum of the classes of ϕ and ψ in the Witt ring.

We denote by $P_n(F)$ the set of all n-fold Pfister forms; $GP_n(F)$ is the set of the forms similar to a form from $P_n(F)$. For any field extension L/F, we denote by $P_n(L/F)$ the set of forms from $P_n(F)$ hyperbolic over L; $GP_n(L/F)$ is the set of the forms similar to a form from $P_n(L/F)$.

We recall that a quadratic form ψ is called a *(Pfister) neighbor* (of a Pfister form π), if it is similar to a subform in π and $\dim \phi > \frac{1}{2} \dim \pi$. We call π the *associated* Pfister form of ϕ ([22, Def. 7.4]). Two quadratic forms ϕ and ϕ^* are *half-neighbors*, if $\dim \phi = \dim \phi^*$ and there exists $s \in F^*$ such that the sum $\phi \perp s\phi^*$ is similar to a Pfister form.

If ϕ is a quadratic F-form of dimension ≥ 3, we write X_ϕ for the projective variety given by the equation $\phi = 0$. We set $F(\phi) = F(X_\phi)$ if $\dim \phi \geq 3$; $F(\phi) = F(\sqrt{d})$ if $\dim \phi = 2$ and $d \overset{\text{def}}{=} -\det \phi \neq 1$; and $F(\phi) = F$ otherwise.

1.2. **Algebras.** We consider only finite-dimensional F-algebras.

For a central simple F-algebra D, we denote by $\deg(D)$, $[D]$, and $\exp(D)$ respectively the degree of D, the class of D in the Brauer group $\mathrm{Br}(F)$, and the exponent of D, i.e., the order of $[D]$ in the Brauer group.

For a simple F-algebra A, we write $\mathrm{ind}(A)$ for the Schur index of A. For an algebra B of the form $B = A \times \cdots \times A$ with A simple, we set $\mathrm{ind}\, B = \mathrm{ind}\, A$.

Let ϕ be a quadratic form. We write $C(\phi)$ for the Clifford algebra of ϕ and $C_0(\phi)$ for the even part of $C(\phi)$. Note that for any quadratic F-form ψ and any central simple F-algebra D, the index of $C_0(\psi) \otimes_F D$ is well-defined.

If $\phi \in I^2(F)$ then $C(\phi)$ is a central simple algebra. Its class $[C(\phi)]$ in the Brauer group $\mathrm{Br}_2(F)$ is denoted by $c(\phi)$.

Let D be a central simple algebra. We denote by X_D the Severi-Brauer variety of D and by $F(D)$ the function field $F(X_D)$. For another central simple F-algebra D' and for a quadratic F-form ψ of dimension ≥ 3, we set $F(D', D) \stackrel{\text{def}}{=} F(X_{D'} \times X_D)$ and $F(\psi, D) \stackrel{\text{def}}{=} F(X_\psi \times X_D)$.

1.3. **Cohomology groups.** We write $H^*(F)$ for the graded Galois cohomology ring $H^*(F, \mathbb{Z}/2\mathbb{Z}) \stackrel{\text{def}}{=} H^*(\mathrm{Gal}(F_{\mathrm{sep}}/F), \mathbb{Z}/2\mathbb{Z})$.

For $n = 0, 1, 2, 3$, there is a homomorphism $e^n : I^n(F) \to H^n(F)$ which is uniquely determined by the condition $e^n(\langle\langle a_1, \ldots, a_n \rangle\rangle) = (a_1, \ldots, a_n)$ (see [2]). The homomorphism e^n is surjective and $\ker e^n = I^{n+1}(F)$ for $n = 0, 1, 2, 3$ (see [27], [28], and [32]).

We also work with the cohomology groups $H^n(F, \mathbb{Q}/\mathbb{Z}(i))$, where $i = 0, 1, 2$ (for a definition see [17]). For $n = 1, 2, 3$, the group $H^n(F)$ is naturally identified with $\mathrm{Tors}_2\, H^n(F, \mathbb{Q}/\mathbb{Z}(n-1))$.

1.4. *K*-**theory and Chow groups.** For a smooth algebraic F-variety X, its Grothendieck ring is denoted by $K(X)$. This ring is supplied with the filtration by codimension of support (also called the topological filtration).

We fix an algebraic closure $\bar{F}$ of the base field F and denote by $\bar{X}$ the $\bar{F}$-variety $X_{\bar{F}}$. If the variety X is projective homogeneous, we identify $K(X)$ with a subring of $K(\bar{X})$ via the restriction homomorphism which is injective by [29].

For a ring (or a group) with filtration A, we denote by G^*A the adjoint graded ring (resp., the adjoint graded group).

There is a canonical surjective homomorphism of the graded Chow ring $\mathrm{CH}^*(X)$ onto $G^*K(X)$. Its kernel consists only of torsion elements and is trivial in the 0-th, 1-st, and 2-nd graded components ([36, §9]).

Let X_1 and X_2 be two smooth F-varieties. For any $x_1 \in K(X_1)$ and $x_2 \in K(X_2)$, we denote by $x_1 \boxtimes x_2$ the product $\mathrm{pr}_1^*(x_1) \cdot \mathrm{pr}_2^*(x_2) \in K(X_1 \times X_2)$ where pr_1^* and pr_2^* are the pull-backs with respect to the projections pr_1 and pr_2 of $X_1 \times X_2$ on X_1 and X_2 respectively. For an $\mathcal{O}_{X_1}$-module $\mathcal{F}_1$ and an $\mathcal{O}_{X_2}$-module $\mathcal{F}_2$, we denote by $\mathcal{F}_1 \boxtimes \mathcal{F}_2$ the tensor product $\mathrm{pr}_1^*(\mathcal{F}_1) \otimes_{\mathcal{O}_X} \mathrm{pr}_2^*(\mathcal{F}_2)$.

1.5. **Relative groups.** Let Φ be an arbitrary functor on the category of fields (of characteristic $\neq 2$) with values in the category of abelian groups. For a field extension L/F we use the notation $\Phi(L/F)$ for $\ker(\Phi(F) \to \Phi(L))$. Some examples of this which we will be using in this paper are: $W(L/F)$, $I^n(L/F)$, $H^n(L/F)$, and $H^n(L/F, \mathbb{Q}/\mathbb{Z}(i))$.

2. The groups $H^3(F(\psi, D)/F)$ and $I^3(F(\psi, D)/F)$

Proposition 2.1. *Let D be a central simple F-algebra of exponent 2 and ψ be a quadratic form of dimension ≥ 3. Then there exists a natural isomorphism*

$$\frac{H^3(F(\psi, D)/F)}{H^3(F(\psi)/F) + H^3(F(D)/F)} \simeq \frac{\operatorname{Tors} \operatorname{CH}^2(X_\psi \times X_D)}{\operatorname{pr}_\psi^* \operatorname{Tors} \operatorname{CH}^2(X_\psi) + \operatorname{pr}_D^* \operatorname{Tors} \operatorname{CH}^2(X_D)}$$

where pr_ψ^* and pr_D^* are the pull-backs with respect to the projection pr_ψ and pr_D of $X_\psi \times X_D$ to X_ψ and X_D.

Proof. By [13, Prop. 2.2] the factor group

$$\frac{H^3(F(\psi, D)/F, \mathbb{Q}/\mathbb{Z}(2))}{H^3(F(\psi)/F, \mathbb{Q}/\mathbb{Z}(2)) + H^3(F(D)/F), \mathbb{Q}/\mathbb{Z}(2))}$$

is isomorphic to

$$\frac{\operatorname{Tors} \operatorname{CH}^2(X_\psi \times X_D)}{\operatorname{pr}_\psi^* \operatorname{Tors} \operatorname{CH}^2(X_\psi) + \operatorname{pr}_D^* \operatorname{Tors} \operatorname{CH}^2(X_D)}.$$

Now it is sufficient to apply the isomorphisms

$$H^3(F(\psi)/F, \mathbb{Q}/\mathbb{Z}(2)) = H^3(F(\psi)/F),$$
$$H^3(F(D)/F, \mathbb{Q}/\mathbb{Z}(2)) = H^3(F(D)/F), \text{ and}$$
$$H^3(F(\psi, D)/F, \mathbb{Q}/\mathbb{Z}(2)) = H^3(F(\psi, D)/F)$$

(see, for example, [13, Lemma 2.8], [10, Lemma A.8], and [14, Cor. 6.6]). □

Corollary 2.2. *Let D be a biquaternion algebra and let ψ be a 4-dimensional quadratic form. Then there exists a natural isomorphism*

$$\frac{H^3(F(\psi, D)/F)}{H^3(F(\psi)/F) + H^3(F(D)/F)} \simeq \operatorname{Tors} \operatorname{CH}^2(X_\psi \times X_D).$$

In particular, $2 \cdot \operatorname{Tors} \operatorname{CH}^2(X_\psi \times X_D) = 0$.

Proof. Since $\dim \psi = 4$, we have $\operatorname{Tors} \operatorname{CH}^2(X_\psi) = 0$ (see, for example, [18, Th. 6.1] or [13, Lemma 2.4]). By [20, Prop. 5.3], we have $\operatorname{Tors} \operatorname{CH}^2(X_D) = 0$. To complete the proof it is sufficient to apply Proposition 2.1. □

Lemma 2.3. *Let ψ be a quadratic form of dimension ≥ 3. Then*

1. *the map $e^3 : P_3(F(\psi)/F) \to H^3(F(\psi)/F)$ is surjective;*
2. *$I^3(F(\psi)/F) + I^4(F) = P_3(F(\psi)/F) + I^4(F)$.*

Proof. 1. This is really the statement that the set $H^3(F(\psi)/F)$ consists of symbols, which is proved in [2, Satz 5.6].
2. This follows from Item 1 and from the injectivity of $e^3 : I^3(F)/I^4(F) \to H^3(F)$. □

Lemma 2.4. *Let D be a biquaternion algebra and let q be an Albert form of D. Then*

1. $H^3(F(D)/F) = [D] \cup H^1(F)$;
2. $I^3(F(D)/F) + I^4(F) = \{q \langle\langle s \rangle\rangle \mid s \in F^*\} + I^4(F)$;
3. *the map* $e^3 : I^3(F(D)/F) \to H^3(F(D)/F)$ *is surjective.*

Proof. 1. See [30, Cor. 4.5].
2. Obviously $e^3(q \langle\langle s \rangle\rangle) = [D] \cup (s)$. Now, it is sufficient to apply Item 1 and the injectivity of $e^3 : I^3(F)/I^4(F) \to H^3(F)$.
3. This follows from Items 1 and 2. □

Lemma 2.5. *Let D be a biquaternion algebra and let ψ be a quadratic form of dimension ≥ 3. Then the natural homomorphism*

$$\frac{I^3(F(\psi, D)/F) + I^4(F)}{I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F)} \to \frac{H^3(F(\psi, D)/F)}{H^3(F(\psi)/F) + H^3(F(D)/F)}$$

is injective.

In particular, the condition $H^3(F(\psi, D)/F) = H^3(F(\psi)/F) + H^3(F(D)/F)$ implies that $I^3(F(\psi, D)/F) \subset I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F)$.

Proof. This is an obvious consequence of the following facts:
a) $I^3(F)/I^4(F) \to H^3(F)$ is injective;
b) $I^3(F(\psi)/F)) \to H^3(F(\psi)/F)$ is surjective (Lemma 2.3);
c) $I^3(F(D)/F)) \to H^3(F(D)/F)$ is surjective (Lemma 2.4). □

Corollary 2.6. *Let D be a biquaternion algebra and let ψ be a quadratic form of dimension ≥ 3 such that $\operatorname{Tors} \operatorname{CH}^2(X_\psi \times X_D) = 0$. Then $I^3(F(\psi, D)/F) \subset I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F)$.*

Proof. This follows from Lemma 2.5 and Proposition 2.1. □

3. The group $K(X_\psi \times X_D)$

In this section, ψ is a quadratic F-form of dimension 4 and D is a biquaternion F-algebra.

Lemma 3.1. *Consider the tensor product $K(X_\psi) \otimes_{\mathbb{Z}} K(X_D)$ together with the filtration induced by the topological filtrations on $K(X_\psi)$ and $K(X_D)$. The adjoint graded group $G^*(K(X_\psi) \otimes K(X_D))$ is torsion-free.*

Proof. The adjoint graded group $G^*K(X_\psi)$ is torsion-free (see, for example, [18]). The adjoint graded group $G^*K(X_D)$ is torsion-free as well (see [19, Example]). We have a surjection

$$G^*K(X_\psi) \otimes G^*K(X_D) \twoheadrightarrow G^*(K(X_\psi) \otimes K(X_D)) .$$

The left-hand term is a finitely generated torsion-free abelian group, i.e. a free abelian group of finite rank. This rank coincides with the rank of the right-hand term.[1] Therefore, the map is an isomorphism. □

[1]This statement is a formal consequence of two following assertions: (i) for an abelian group with filtration A, one has $\operatorname{rk} A = \operatorname{rk} G^*A$; (ii) for any two abelian groups A and B, one has $\operatorname{rk}(A \otimes B) = \operatorname{rk} A \cdot \operatorname{rk} B$.

We also consider the subgroup $K(X_\psi) \boxtimes K(X_D)$ of $K(X_\psi \times X_D)$ together with the filtration induced by the topological filtration on $K(X_\psi \times X_D)$.

Lemma 3.2. *The homomorphism $K(X_\psi) \otimes K(X_D) \to K(X_\psi) \boxtimes K(X_D)$ is an isomorphism of groups with filtrations.*

Proof. The map is a homomorphism of groups respecting the filtrations. First of all let us check that it is an isomorphism of groups, regardless of the filtrations. It is evidently an epimorphism. So, we only have to check the injectivity. Since for any extension of the base field F, the restriction homomorphism on the product $K(X_\psi) \otimes K(X_D)$ is injective, it suffices to check the injectivity in the case where D is split. However in that case X_D is isomorphic to a projective space; therefore the map $K(X_\psi) \otimes K(X_D) \to K(X_\psi) \boxtimes K(X_D)$ is an isomorphism.

To finish the proof, it suffices to show that the homomorphism of the adjoint graded groups is injective. Consider the commutative diagram

$$\begin{array}{ccc} G^*(K(\bar{X}_\psi) \otimes K(\bar{X}_D)) & \longrightarrow & G^*(K(\bar{X}_\psi) \boxtimes K(\bar{X}_D)) \\ \big\uparrow & & \big\uparrow \\ G^*(K(X_\psi) \otimes K(X_D)) & \longrightarrow & G^*(K(X_\psi) \boxtimes K(X_D)) \end{array}$$

The arrow on the left is injective since the group $G^*(K(X_\psi) \otimes K(X_D))$ is torsion-free (Lemma 3.1). The top arrow is injective because $\bar{X}_D$ is isomorphic to a projective space. Therefore the bottom arrow is injective as well. $\square$

Corollary 3.3. *The group $G^*(K(X_\psi) \boxtimes K(X_D))$ is torsion-free.* $\square$

We write C for the even Clifford algebra $C_0(\psi)$. Let $\mathcal{U}(2)$ be Swan's sheaf $\mathcal{U}$ on the quadric X_ψ ([37, §6]), twisted twice. It has the structure of a C-module. Let $\mathcal{J}$ be the canonical sheaf on the Severi-Brauer variety X_D ([31, §8.4]). It has the structure of a D-module.

Set $\mathcal{F} \stackrel{\text{def}}{=} \mathcal{U}(2) \boxtimes \mathcal{J}$. It is a sheaf on $X_\psi \times X_D$ with the structure of a $C \otimes_F D$-module. Denote by f the homomorphism $K(C \otimes D) \to K(X_\psi \times X_D)$ given by the functor of tensor multiplication by $\mathcal{F}$ over $C \otimes D$.

Set $\mathcal{G} \stackrel{\text{def}}{=} \mathcal{U}(2) \boxtimes \mathcal{J}^{\otimes 3}$. It is a sheaf on $X_\psi \times X_D$ with the structure of a $C \otimes_F D^{\otimes 3}$-module. Consider the homomorphism $K(C \otimes D^{\otimes 3}) \to K(X_\psi \times X_D)$ given by the functor of tensor multiplication by $\mathcal{G}$ over $C \otimes D^{\otimes 3}$. Since the algebra $D^{\otimes 2}$ is split, the group $K(C \otimes D^{\otimes 3})$ is canonically isomorphic (via Morita equivalence) to $K(C \otimes D)$. Denote by g the composition $K(C \otimes D) \xrightarrow{\sim} K(C \otimes D^{\otimes 3}) \to K(X_\psi \times X_D)$.

Lemma 3.4. 1. *The homomorphism*

$$K(C \otimes D)^{\oplus 2} \to K(X_\psi \times X_D)/(K(X_\psi) \boxtimes K(X_D))$$

induced by the homomorphism $f + g$ is surjective.

2. *If $C \otimes D$ is a skew field, then $K(X_\psi \times X_D) = K(X_\psi) \boxtimes K(X_D)$.*

Proof. 1. Using Swan's computation of the K-theory for quadrics [37, Th. 9.1] (with $\mathcal{U}(2)$ instead of $\mathcal{U}$) and a generalized Peyre's version [30, Prop. 3.1] of the Quillen's computation of K-theory for Severi-Brauer schemes [31, Th. 4.1 of §8], we get an isomorphism

$$K(F)^{\oplus 4} \oplus K(C)^{\oplus 2} \oplus K(D)^{\oplus 4} \oplus K(C \otimes D)^{\oplus 2} \simeq K(X_\psi \times X_D)$$

such that the image of $K(F)^{\oplus 4} \oplus K(C)^{\oplus 2} \oplus K(D)^{\oplus 4}$ is contained in $K(X_\psi) \boxtimes K(X_D)$ and the summand $K(C \otimes D)^{\oplus 2}$ is mapped into $K(X_\psi \times X_D)$ via $f + g$. Therefore, $K(C \otimes D)^{\oplus 2} \to K(X_\psi \times X_D)/(K(X_\psi) \boxtimes K(X_D))$ is an epimorphism.
2. If the algebra $C \otimes D$ is a skew field, then its class generates the group $K(C \otimes D)$. The images of this class under f, g are $\mathcal{F}, \mathcal{G} \in K(X_\psi) \boxtimes K(X_D)$. $\square$

Corollary 3.5. *If $C \otimes D$ is a skew field, then the group $G^* K(X_\psi \times X_D)$ is torsion-free.*

Proof. By Lemma 3.4, $K(X_\psi \times X_D) = K(X_\psi) \boxtimes K(X_D)$. By Corollary 3.3, the group $G^*(K(X_\psi) \boxtimes K(X_D))$ is torsion-free. $\square$

4. The group $\operatorname{Tors} \operatorname{CH}^2(X_\psi \times X_D)$

Theorem 4.1. *Let D be a biquaternion algebra and let ψ be an anisotropic 4-dimensional quadratic form with $\det \psi \neq 1$. Then the group $\operatorname{CH}^2(X_\psi \times X_D)$ is torsion-free except (possibly) the following two cases:*

1. $\operatorname{ind} C_0(\psi) \otimes D = \operatorname{ind} D = 4$,
2. $\operatorname{ind} C_0(\psi) \otimes D = 2$.

Proof. Set $C \overset{\text{def}}{=} C_0(\psi)$ and $s \overset{\text{def}}{=} \operatorname{ind}(C \otimes D)$. The possible values of s are 1, 2, 4, and 8.

Assume that $s = 8$. Since $\det \psi \neq 1$, it follows that $C \otimes D$ is a skew field. Therefore, the group $\operatorname{CH}^2(X_\psi \times X_D) \simeq G^2 K(X_\psi \times X_D)$ is torsion-free by Corollary 3.5.

Now assume $s = 1$. Then $\operatorname{ind}(C_0(\psi_{F(D)})) = 1$. Therefore the quadratic form $\psi_{F(D)}$ is isotropic. Hence the extension $F(\psi, D)/F(D)$ is purely transcendental and $H^3(F(\psi, D)/F) = H^3(F(D)/F)$. Now, Corollary 2.2 shows that $\operatorname{Tors} \operatorname{CH}^2(X_\psi \times X_D) = 0$.

Finally, assume $s = 4$ and $\operatorname{ind} D \neq 4$. Then the biquaternion algebra D is Brauer equivalent to a quaternion F-algebra D'. Since $\operatorname{Tors} \operatorname{CH}^2(X_\psi \times X_D)$ depends only on the Brauer class of D (see, for example, [14]), we may replace D by D'. Let ψ' be a 3-dimensional quadratic F-form such that $C_0(\psi') \simeq D'$. The Severi-Brauer variety $X_{D'}$ is isomorphic to the conic $X_{\psi'}$. Since the tensor product $C_0(\psi) \otimes C_0(\psi')$ has index 4, it is a division algebra. Therefore, by [13, Cor. 4.4], the group $\operatorname{CH}^2(X_\psi \times X_{\psi'})$ is torsion-free. $\square$

Remark 4.2. The assumption that ψ is anisotropic in the theorem is not essential: if ψ is isotropic, then $H^3(F(\psi, D)/F) = H^3(F(D)/F)$ and therefore the group $\operatorname{CH}^2(X_\psi \times X_D)$ is torsion-free as well (by Corollary 2.2).

Proposition 4.3. *Let D be a biquaternion algebra and let ψ be a 4-dimensional quadratic form with* $\det\psi \neq 1$. *Then the group* $\operatorname{Tors}\operatorname{CH}^2(X_\psi \times X_D)$ *is equal to zero or is isomorphic to* $\mathbb{Z}/2$.

Proof. Since we already know that the torsion in the group $\operatorname{CH}^2(X_\psi \times X_D)$ is annihilated by 2 (Corollary 2.2), it suffices to show that the torsion is cyclic.

Once again we set $C \stackrel{\text{def}}{=} C_0(\psi)$. According to Theorem 4.1, it suffices to consider the case where $\operatorname{ind}(C \otimes D)$ equals 2 or 4.

Consider the quotient $K(X_\psi \times X_D)/(K(X_\psi) \boxtimes K(X_D))$ with the filtration induced by the topological filtration on $K(X_\psi \times X_D)$. Since in the exact sequence of the adjoint graded groups

$$0 \to G^*(K(X_\psi) \boxtimes K(X_D)) \to G^*K(X_\psi \times X_D) \to$$
$$\to G^*(K(X_\psi \times X_D)/(K(X_\psi) \boxtimes K(X_D))) \to 0$$

the first term is torsion-free (Corollary 3.3), we have an injection

$$\operatorname{Tors} G^*K(X_\psi \times X_D) \hookrightarrow G^*(K(X_\psi \times X_D)/(K(X_\psi) \boxtimes K(X_D)))\ .$$

Since $\operatorname{CH}^2(X_\psi \times X_D) \simeq G^2K(X_\psi \times X_D)$, it suffices to show that the group $G^2(K(X_\psi \times X_D)/(K(X_\psi) \boxtimes K(X_D)))$ is cyclic.

Denote by $h \in K(X_\psi)$ the class of a general hyperplane section of X_ψ. Let $\xi \in K(\bar{X}_D)$ be the class of the tautological linear bundle on the projective space $\bar{X}_D$. Note that for any $i \geq 0$, the multiple $(\operatorname{ind} D^{\otimes i}) \cdot \xi^i$ of ξ^i belongs to $K(X_D)$. Thus $\xi^i \in K(X_D)$ for i even and $4\xi^i \in K(X_D)$ for i odd.

The algebra $C \otimes D$ is simple. Therefore, its Grothendieck group is cyclic. By Lemma 3.4, it follows that the quotient $K(X_\psi \times X_D)/(K(X_\psi) \boxtimes K(X_D))$ is generated by two elements, namely by $(s/2)x$ and $(s/2)y$, where $x \stackrel{\text{def}}{=} (4 + 2h + h^2) \boxtimes \xi$, $y \stackrel{\text{def}}{=} (4 + 2h + h^2) \boxtimes \xi^3$, and $s \stackrel{\text{def}}{=} \operatorname{ind} C \otimes D$ (here we are using the equality $[\mathcal{U}(2)] = 4 + 2h + h^2 \in K(X_\psi)$, [18, Lemma 3.6]).

We have a congruence $x \equiv h^2 \boxtimes (\xi - 1) - h \boxtimes (\xi - 1)^2 \pmod{K(X_\psi) \boxtimes K(X_D)}$. The element on the right side belongs to $K(X_\psi \times X_D)^{(2)}$, because it is in $K(\bar{X}_\psi \times \bar{X}_D)^{(2)}$ and $K(X_\psi \times X_D)^{(2)} = K(\bar{X}_\psi \times \bar{X}_D)^{(2)} \cap K(X_\psi \times X_D)$ (see [34, Lemme 6.3 (i)]). Therefore, $(s/2)x \in (K(X_\psi \times X_D)/(K(X_\psi) \boxtimes K(X_D)))^{(2)}$ (take into account that the coefficient $s/2$ is an integer).

We also have another congruence modulo $K(X_\psi) \boxtimes K(X_D)$:

$$y - x \equiv \big(h^2 \boxtimes (\xi - 1) - h \boxtimes (\xi - 1)^2\big) \cdot \big(1 \boxtimes (\xi^2 - 1)\big)\ .$$

The element on the right side is in $K(X_\psi \times X_D)^{(3)}$ as a product of an element in $K(X_\psi \times X_D)^{(2)}$ and the element $1 \boxtimes (\xi^2 - 1) \in K(X_\psi \times X_D)^{(1)}$. Therefore, $(s/2)(y - x) \in (K(X_\psi \times X_D)/(K(X_\psi) \boxtimes K(X_D)))^{(3)}$ and it follows that the group $G^2(K(X_\psi \times X_D)/(K(X_\psi) \boxtimes K(X_D)))$ is generated by $(s/2)x$. So, in particular, this group is cyclic. $\square$

Remark 4.4. The assumption that $\det\psi \neq 1$ in the proposition is not essential: if $\det\psi = 1$, then $H^3(F(\psi, D)/F) = H^3(F(\psi', D)/F)$ and $H^3(F(\psi)/F) =$

$H^3(F(\psi')/F)$, where ψ' is an arbitrary 3-dimensional subform of ψ ([13, Lemma 5.2]). Therefore,

$$\mathrm{CH}^2(X_\psi \times X_D) \simeq \mathrm{CH}^2(X_{\psi'} \times X_D) \simeq \mathrm{CH}^2(X_{D'} \times X_D)$$

where D' is the even Clifford algebra of ψ'. The group $\operatorname{Tors}\mathrm{CH}^2(X_{D'} \times X_D)$ is zero or $\mathbb{Z}/2$ according to [21, Th. 6.1].

Theorem 4.5. *Let D be a biquaternion algebra and let ψ be a 4-dimensional quadratic form with* $\det\psi \neq 1$. *Suppose that* $\operatorname{ind} C_0(\psi) \otimes D = \operatorname{ind} C_0(\psi_0) \otimes D$ *for some* 3*-dimensional subform* $\psi_0 \subset \psi$. *Then* $\operatorname{Tors}\mathrm{CH}^2(X_\psi \times X_D) = 0$.

Proof. According to Theorem 4.1, it suffices to consider the case where $s \stackrel{\mathrm{def}}{=} \operatorname{ind}(C_0(\psi) \otimes D)$ equals 2 or 4. In this case it suffices to show that the element $x \stackrel{\mathrm{def}}{=} (s/2)\big(h^2 \boxtimes (\xi - 1) - h \boxtimes (\xi - 1)^2\big)$ is in $K(X_\psi \times X_D)^{(3)}$ (see the proof of Proposition 4.3).

Consider the element

$$x_0 \stackrel{\mathrm{def}}{=} (s/2)\big(h_0 \boxtimes (\xi - 1) - 1 \boxtimes (\xi - 1)^2\big) \in K(\bar{X}_{\psi_0} \times \bar{X}_D),$$

where $h_0 \in K(X_{\psi_0})$ is the class of a hyperplane section of the conic X_{ψ_0}. Since $\operatorname{ind} C_0(\psi_0) \otimes D = s$, one has $x_0 \in K(X_{\psi_0} \times X_D)$. Moreover, since $x_0 \in K(\bar{X}_{\psi_0} \times \bar{X}_D)^{(2)}$ and

$$K(X_{\psi_0} \times X_D)^{(2)} = K(\bar{X}_{\psi_0} \times \bar{X}_D)^{(2)} \cap K(X_{\psi_0} \times X_D)$$

(see [34, Lemme 6.3 (i)]), it follows that $x_0 \in K(X_{\psi_0} \times X_D)^{(2)}$. The product $X_{\psi_0} \times X_D$ is a closed subvariety of codimension 1 in $X_\psi \times X_D$ and the image of x_0 under the push-forward $K(X_{\psi_0} \times X_D) \to K(X_\psi \times X_D)$ equals x. Consequently $x \in K(X_\psi \times X_D)^{(3)}$. □

Corollary 4.6. *Suppose that* $\psi = \langle -x, -y, xy, d\rangle$ *(with* $d \notin F^{*2}$*) and* $D = (x, y) \otimes (u, v)$ *where* $x, y, d, u, v \in F^*$. *Then* $\operatorname{Tors}\mathrm{CH}^2(X_\psi \times X_D) = 0$.

Proof. The even Clifford algebra of the quadratic form ψ is isomorphic to the quaternion algebra $(x, y)_{F(\sqrt{d})}$. Therefore, the tensor product $C_0(\psi) \otimes D$ is Brauer-equivalent to the quaternion algebra $(u, v)_{F(\sqrt{d})}$ and in particular has index 2 or 1. In the case where the index is 1, we are done by Theorem 4.1. Otherwise we have $2 = \operatorname{ind} C_0(\psi) \otimes D = \operatorname{ind} C_0(\psi_0) \otimes D$, where ψ_0 is the 3-dimensional subform $\langle -x, -y, xy\rangle$ in ψ; therefore we are done by Theorem 4.5. □

Remark 4.7. The assumption $d \notin F^{*2}$ in the Corollary is not essential: if $d \in F^{*2}$, then $\operatorname{Tors}\mathrm{CH}^2(X_\psi \times X_D) \simeq \operatorname{Tors}\mathrm{CH}^2(X_{C_0(\psi')} \times X_D)$, where $\psi' \stackrel{\mathrm{def}}{=} \langle -x, -y, xy\rangle$; since $C_0(\psi') \simeq (x, y)$, we have $\operatorname{ind} C_0(\psi') \otimes D \leq 2$; therefore, by [21, Th. 6.1], the latter group is zero.

5. L-standard isotropy

In this section we discuss condition (L2) in the definition of L-standard isotropy. Since the case $\dim\psi = 2$ is rather obvious, we can assume that $\dim\psi \geq 3$. The general problem of isotropy of $\phi_{F(\psi)}$ naturally splits into two cases:

(a) $\dim\psi = 3$ or $\psi \in GP_2(F)$. In other words, ψ is a 2-fold Pfister neighbor.
(b) $\dim\psi \geq 4$ and $\psi \notin GP_2(F)$. In other words, ψ is not similar to a subform of a 2-fold Pfister form.

In case (a) there is a good description of isotropy of quadratic forms over the function fields of ψ in terms of minimal forms ([9], [8]). In fact, the following statement includes several results of these papers (see also [5, Prop. 5.1 and 6.1]).

Proposition 5.1. *Let ψ be a 3-dimensional form or 4-dimensional form with trivial determinant. Let $\langle\langle a,b\rangle\rangle$ be the associated Pfister form of ψ. Let ϕ be an anisotropic form such that $\phi_{F(\psi)}$ is isotropic. Then there exists a subform $\phi_0 \subset \phi$ and a quadratic form τ such that*

- *the form $(\phi_0)_{F(\psi)}$ is isotropic,*
- *ϕ_0 is isometric to a $(2\dim\tau + 1)$-dimensional subform of $\langle\langle a,b\rangle\rangle\,\tau$,*
- *the form $\langle\langle a,b\rangle\rangle\,\tau$ is anisotropic.*

Moreover, if $\dim\phi = 8$ then at least one of the following conditions holds:

- *ϕ_0 is similar to $\langle 1,-a,-b\rangle$,*
- *ϕ_0 is a 5-dimensional Pfister neighbor,*
- *$\dim\phi_0 = 7$, $\dim\tau = 3$ and $\phi_0 \subset \langle\langle a,b\rangle\rangle\,\tau$.*

Proof. Since the forms ψ and $\langle 1,-a,-b\rangle$ are Pfister neighbors of $\langle\langle a,b\rangle\rangle$, it follows that the field $F(\psi)$ is stably equivalent to the function field of $\langle 1,-a,-b\rangle$. Hence, we can suppose that $\psi = \langle 1,-a,-b\rangle$. Let us define ϕ_0 as an $F(\psi)$-minimal subform of ϕ. To complete the proof it suffices to apply results concerning $F(\psi)$-minimal forms given in [9, Rem. 1.7] and [8, §1]. $\square$

Corollary 5.2. *Let ϕ and ψ be anisotropic forms such that $\phi_{F(\psi)}$ is isotropic. Suppose additionally that $\dim\psi = 3$ or $\psi \in GP_2(F)$. Then condition (L2) in the definition of L-standard isotropy holds. In particular, the isotropy of $\phi_{F(\psi)}$ is L-standard.*

Proof. Let $a,b \in F^*$, ϕ_0 and τ be as in Lemma 5.1. Setting $\pi = \langle\langle a,b\rangle\rangle\,\tau$, one can see that condition (L2) holds. $\square$

Now we consider case (b), where $\dim\psi \geq 4$ and $\psi \notin GP_2(F)$. We start with the following well-known statement concerning the forms from $W(F(\psi)/F)$.

Lemma 5.3. *Let ψ be an anisotropic form such that $\dim\psi \geq 4$ and $\psi \notin GP_2(F)$. Let π be anisotropic form such that $\pi_{F(\psi)}$ is hyperbolic. Then there exist $a,b \in F^*$ and an even-dimensional form τ such that $\pi \simeq \langle\langle a,b\rangle\rangle\,\tau$. In particular, $\pi \in I^3(F)$ and $\dim\pi$ is divisible by 8.*

Proof. Let ψ_0 be a 3-dimensional subform of ψ and let $\langle\langle a, b\rangle\rangle$ be the associated Pfister form of ϕ_0. We have $\pi \in W(F(\psi)/F) \subset W(F(\psi_0)/F) = \langle\langle a, b\rangle\rangle\, W(F)$ (the last equality is a well-known assertion, cf. [35, Th. 5.4 (iv) of Chap. 4]). Hence there exists a form τ such that $\pi \simeq \langle\langle a, b\rangle\rangle\, \tau$. If $\dim \tau$ is even, we are done. If $\dim \tau$ is odd, one has $c(\tau) = (a, b)$. Since $\pi_{F(\psi)}$ is hyperbolic, we have $(a, b)_{F(\psi)} = 0$ and hence $\langle\langle a, b\rangle\rangle_{F(\psi)}$ is hyperbolic. By the Cassels-Pfister subform theorem, ψ is similar to a subform of $\langle\langle a, b\rangle\rangle$. This contradicts the assumption on ψ. □

Corollary 5.4. *Let ϕ be a quadratic form of dimension ≤ 8 and let ψ be a quadratic form of dimension ≥ 4 such that $\phi \notin GP_2(F)$. Then the following conditions are equivalent:*

(1) *there exists a subform $\phi_0 \subset \phi$ and an anisotropic form $\pi \in W(F(\psi)/F)$ such that $\phi_0 \subset \pi$ and $\dim \phi_0 > \frac{1}{2} \dim \pi$;*
(2) *there exists a 5-dimensional Pfister neighbor $\phi_0 \subset \phi$ such that $(\phi_0)_{F(\psi)}$ is isotropic;*
(3) *there exists a Pfister neighbor $\phi_0 \subset \phi$ such that $(\phi_0)_{F(\psi)}$ is isotropic.*

Proof. (1)⇒(2). By Lemma 5.3, there exist $a, b \in F^*$ and an even-dimensional form τ such that $\pi \simeq \langle\langle a, b\rangle\rangle\, \tau$. We have $\dim \tau = \frac{1}{4} \dim \pi < \frac{1}{2} \dim \phi_0 \leq 4$. Since $\dim \tau$ is even, one has $\dim \tau = 2$. Hence $\pi \in GP_3(F)$. Thus, ϕ_0 is a Pfister neighbor of $\pi \in GP_3(F)$. In particular, $\dim \phi_0 \geq 5$. Replacing ϕ_0 by a 5-dimensional subform of ϕ_0, we can assume that $\dim \phi_0 = 5$. Since $\pi \in W(F(\psi)/F)$ and ϕ_0 is a Pfister neighbor of π, it follows that $(\phi_0)_{F(\psi)}$ is isotropic.

Finally, implication (2)⇒(3) is obvious, and implication (3)⇒(1) was proved in the introduction. □

Corollary 5.5. *Let ϕ be an 8-dimensional form with trivial determinant and ψ be a quadratic form such that $\dim \psi \geq 4$ and $\psi \notin GP_2(F)$. Then condition* (L2) *is equivalent to condition* (L2$'$). □

6. Standard isotropy in the case $\operatorname{Tors} \mathrm{CH}^2(X_\psi \times X_D) = 0$

In this section we need the following

Theorem 6.1 ([24, Th. 4] and [4, Cor. 9.3], see also [14])**.** *Let ϕ be an anisotropic 8-dimensional quadratic form with $\det \phi = 1$ and let D be an algebra such that $c(\phi) = [D]$. Then $\phi_{F(D)}$ is anisotropic.*

Definition 6.2. We say that (ϕ, D, q) is a *special triple* if the following conditions hold:

1) ϕ is an 8-dimensional anisotropic form with $\det \phi = 1$,
2) D is a biquaternion algebra,
3) q is an Albert form,
4) $[D] = c(\phi) = c(q) \in \mathrm{Br}_2(F)$.

Our study of the isotropy of 8-dimensional forms over function field of quadrics is based on the following assertion.

Proposition 6.3. *Let (ϕ, D, q) be a special triple and ψ be a quadratic form. Then*

1. *The following two conditions are equivalent:*
   (i) $\phi + q \in I^3(F(\psi, D)/F)$;
   (ii) $\phi_{F(\psi)}$ *is isotropic.*
2. *The following two conditions are equivalent:*
   (i) $\phi + q \in I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F)$;
   (ii) *there exists a* 5*-dimensional Pfister neighbor* ϕ_0 *such that* $\phi_0 \subset \phi$ *and* $(\phi_0)_{F(\psi)}$ *is isotropic.*

Proof. (1i)⇒(1ii). Condition (1i) implies that the quadratic form $(\phi \perp q)_{F(\psi,D)}$ is hyperbolic. Since $q_{F(D)}$ is hyperbolic, it follows that $\phi_{F(\psi,D)}$ is hyperbolic. Let $E \stackrel{\text{def}}{=} F(\psi)$. We see that $\phi_{E(D)}$ is hyperbolic. Theorem 6.1 implies that ϕ_E is isotropic, i.e., condition (1ii) holds.
(1ii)⇒(1i). Suppose that $\phi_{F(\psi)}$ is isotropic. Since $c(\phi) = c(q)$, it follows that $\phi + q \in I^3(F)$. Hence it is sufficient to prove that $\phi_{F(\psi,D)}$ and $q_{F(\psi,D)}$ are hyperbolic. The form $q_{F(\psi,D)}$ is hyperbolic because $q_{F(D)}$ is. Since $c(\phi) = [D]$, we have $\phi_{F(\psi,D)} \in I^3(F(\psi, D))$. Since $\phi_{F(\psi)}$ is isotropic, we have $\dim(\phi_{F(\psi,D)})_{an} < 8$. The Arason-Pfister Hauptsatz shows that $\phi_{F(\psi,D)}$ is hyperbolic.
(2i)⇒(2ii). By Lemmas 2.3 and 2.4, there exist $\pi \in P_3(F(\psi)/F)$ and $s \in F^*$ such that

$$\phi + q \equiv \pi + q \langle\langle s \rangle\rangle \pmod{I^4(F)}.$$

We have $\phi + sq \equiv \pi \pmod{I^4(F)}$. Since $\pi \in P_3(F)$, $\pi_{F(\pi)}$ is hyperbolic. Hence $(\phi + sq)_{F(\pi)} \equiv \pi_{F(\pi)} \equiv 0 \pmod{I^4(F)}$. Since $\dim(\phi \perp sq) = 8 + 6 < 16$, the Arason-Pfister Hauptsatz shows that $(\phi \perp sq)_{F(\pi)}$ is hyperbolic. Hence there exists a form γ such that $(\phi \perp sq)_{an} \simeq \pi\gamma$. Clearly $0 < 8 - 6 \leq \dim(\phi \perp sq)_{an} \leq 8 + 6 < 16$. This implies that $\dim \gamma = 1$, i.e., there exists $k \in F^*$ such that $\gamma = \langle k \rangle$. Thus $\phi + sq = k\pi$. Therefore $(\phi \perp -k\pi)_{an} \simeq -sq_{an}$. Hence ϕ and $k\pi$ contain a common subform of dimension

$$\frac{\dim \phi + \dim \pi - \dim q}{2} = \frac{8 + 8 - 6}{2} = 5.$$

Let us denote such a form by ϕ_0. Since $\dim \phi_0 = 5$ and $\phi_0 \subset k\pi$, it follows that ϕ_0 is a Pfister neighbor of π. Since $\pi \in P_3(F(\psi)/F)$, it follows that $(\phi_0)_{F(\psi)}$ is isotropic.
(2ii)⇒(2i). Let ϕ_0 be a 5-dimensional quadratic form as in (2ii). By the assumption, there exists $\pi \in GP_3(F)$ such that $\phi_0 \subset \pi$. Since $(\phi_0)_{F(\psi)}$ is isotropic, it follows that $\pi \in GP_3(F(\psi)/F)$.

Since $\phi_0 \subset \phi$ and $\phi_0 \subset \pi$, there exist 3-dimensional quadratic forms ρ', ρ'' such that $\phi = \phi_0 \perp \rho'$ and $\pi = \phi_0 \perp \rho''$. We set $\rho = \rho'' \perp -\rho'$. Clearly $\dim \rho = 6$. In the Witt ring $W(F)$ we have $\rho = \rho'' - \rho' = \pi - \phi$. In particular, $\rho \in I^2(F)$. Hence ρ is an Albert form.

We have $c(\rho) = c(\pi) + c(\phi) = 0 + c(\phi) = c(q)$. Hence ρ is similar to q ([16, Th. 3.12]). Let $s \in F^*$ be such that $\rho \simeq sq$. We have $\pi - \phi = \rho = sq$. Hence

$\phi = \pi - sq$. Therefore

$$\phi + q = \pi + q\,\langle\!\langle s\rangle\!\rangle \in GP_3(F(\psi)/F) + [q]\cdot I(F) \subset \\ \subset I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F).$$

□

Corollary 6.4. *Let (ϕ, D, q) be a special triple and let ψ be a quadratic form of dimension ≥ 3. Suppose that* $\operatorname{Tors}\operatorname{CH}^2(X_\psi \times X_D) = 0$. *Then the following conditions are equivalent:*

(1) *$\phi_{F(\psi)}$ is isotropic,*
(2) *there exists a 5-dimensional Pfister neighbor ϕ_0 such that $\phi_0 \subset \phi$ and $(\phi_0)_{F(\psi)}$ is isotropic.*

Proof. This is obvious in view of Proposition 6.3 and Corollary 2.6. □

7. The group $H^3(F(\psi, D)/F)$ in the case $\operatorname{ind}(C_0(\psi)\otimes_F D) = 2$

In this section we study the group $H^3(F(\psi, D)/F)$ in the case where ψ is a 4-dimensional quadratic form with a nontrivial determinant, D is a biquaternion division F-algebra, and $\operatorname{ind}(C_0(\psi)\otimes_F D) = 2$.

Let $d \stackrel{\text{def}}{=} \det\psi$. By our assumption, $d \notin F^{*2}$. Replacing ψ by a similar form, we can suppose $\psi = \langle -a, -b, ab, d\rangle$ with $a, b \in F^*$. Let $L \stackrel{\text{def}}{=} F(\sqrt{d})$. By our assumption, we have $\operatorname{ind}((a,b)\otimes_F D)_L = 2$. Hence there exists a quaternion F-algebra Q such that $(a,b)_L + [D_L] = [Q_L]$ in $\operatorname{Br}_2(L)$ (see [1, Th. 10.21]). Write Q in the form $Q = (r,s)$ with $r, s \in F^*$ and set $\psi' = \langle -r, -s, rs, d\rangle$.

Let q be an Albert form corresponding to D.

Lemma 7.1. *There exist $k, k' \in F^*$ such that $k\psi + k'\psi' + q \in I^3(F)$.*

Proof. Since $(a,b)_L + [D_L] = [Q_L] = (r,s)_L$, it follows that $(a,b) + (r,s) + [D] \in \operatorname{Br}_2(L/F)$. Hence there exists $k \in F^*$ such that $(a,b) + (r,s) + [D] = (d,k)$. Let $k' \stackrel{\text{def}}{=} -1$ and $\phi \stackrel{\text{def}}{=} k\psi \perp k'\psi' \perp q$. We claim that $\phi \in I^3(F)$. To prove this, it is sufficient to verify that $\phi \in I^2(F)$ and $c(\phi) = 0$. We have

$$\phi = k\psi + k'\psi' + q = k\,\langle -a, -b, ab, d\rangle - \langle -r, -s, rs, d\rangle + q = \\ = k(\langle\!\langle a,b\rangle\!\rangle - \langle\!\langle d\rangle\!\rangle) - (\langle\!\langle r,s\rangle\!\rangle - \langle\!\langle d\rangle\!\rangle) + q = \\ = k\,\langle\!\langle a,b\rangle\!\rangle + \langle\!\langle d,k\rangle\!\rangle - \langle\!\langle r,s\rangle\!\rangle + q.$$

Hence $\phi \in I^2(F)$ and $c(\phi) = (a,b) + (d,k) + (r,s) + c(q) = (a,b) + (d,k) + (r,s) + [D] = 0$. □

Definition 7.2. Let D be a biquaternion algebra and let ψ be a 4-dimensional quadratic form such that $\det\psi \neq 1$ and $\operatorname{ind}(C_0(\psi)\otimes_F D) = 2$. We denote by $\Gamma(\psi, D)$ the set defined as follows

$$\{\gamma \in I^3(F) \mid \text{there exist } k, k', l \in F^* \text{ such that } \gamma = k\psi + k'\psi' + lq\},$$

where q is an Albert form corresponding to D and ψ' is a 4-dimensional quadratic form satisfying the following two properties: $\det\psi' = \det\psi$ and $C_0(\psi')$ is Brauer-equivalent to $C_0(\psi)\otimes_F D$.

Remark 7.3. 1. The set $\Gamma(\psi, D)$ does not depend on the choice of q and ψ': indeed, the condition on q and ψ' determines them uniquely up to similarity.

2. Lemma 7.1 shows the set $\Gamma(\psi, D)$ is not empty.

Lemma 7.4. $\Gamma(\psi, D) \subset I^3(F(\psi, D)/F)$.

Proof. Let $\gamma = k\psi + k'\psi' + lq \in \Gamma(\psi, D)$. By the definition of $\Gamma(\psi, D)$, we have $\gamma \in I^3(F)$. Thus it is sufficient to prove that $\gamma_{F(\psi,D)}$ is hyperbolic. We have $\dim(\psi_{F(\psi)})_{an} \le 2$ and $\dim(q_{F(D)})_{an} = 0$. Therefore $\dim(\gamma_{F(\psi,D)})_{an} = \dim((k\psi \perp k'\psi' \perp lq)_{F(\psi,D)})_{an} \le 2+4+0 = 6 < 8$. Since $\gamma \in I^3(F)$, the Arason-Pfister Hauptsatz shows that $\gamma_{F(\psi,D)}$ is hyperbolic. $\square$

Lemma 7.5. *Let $\gamma \in \Gamma(\psi, D)$, $\pi \in P_3(F(\psi)/F)$, and $s \in F^*$. Then there exists $\gamma' \in \Gamma(\psi, D)$ such that $\gamma' \equiv \gamma + \pi + q\langle\langle s\rangle\rangle \pmod{I^4(F)}$.*

Proof. Let us write γ in the form $\gamma = k\psi + k'\psi' + lq$. Since $\pi \in P_3(F(\psi)/F)$, there exists $r \in F^*$ such that $\pi = \psi\langle\langle r\rangle\rangle$ ([13, Lemmas 6.1 and 6.2]).

We have $k\psi + \pi \equiv k\psi - k\pi = k\psi - k\psi\langle\langle r\rangle\rangle = rk\psi \pmod{I^4(F)}$ and $lq + q\langle\langle s\rangle\rangle \equiv lq - lq\langle\langle s\rangle\rangle = lsq \pmod{I^4(F)}$. Therefore

$$\gamma + \pi + q\langle\langle s\rangle\rangle = k\psi + k'\psi' + lq + \pi + q\langle\langle s\rangle\rangle \equiv rk\psi + k'\psi' + lsq \pmod{I^4(F)}.$$

Now it is sufficient to set $\gamma' = rk\psi + k'\psi' + lsq$. $\square$

Corollary 7.6. $\Gamma(\psi, D) + I^4(F) = \Gamma(\psi, D) + I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F)$.

Proof. Obvious in view of Lemmas 7.5, 2.3, and 2.4. $\square$

Lemma 7.7. *The following conditions are equivalent:*

(1) $I^3(F(\psi, D)/F) \subset I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F)$;
(2) $\Gamma(\psi, D) \subset I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F)$;
(3) *there exists* $\gamma \in \Gamma(\psi, D)$ *such that* $\gamma \in I^3(F(\psi)/F)+I^3(F(D)/F)+I^4(F)$;
(4) $\Gamma(\psi, D)$ *contains a hyperbolic form, i.e.* $0 \in \Gamma(\psi, D)$;
(5) *there exist* $x, y, u, v, d \in F^*$ *such that* $\psi \sim \langle -x, -y, xy, d\rangle$ *and* $D \simeq (x, y) \otimes_F (u, v)$;
(6) $\operatorname{Tors}\operatorname{CH}^2(X_\psi \times X_D) = 0$;
(7) $H^3(F(\psi, D)/F) = H^3(F(\psi)/F) + H^3(F(D)/F)$.

Proof. (1)⇒(2). Obvious in view of Lemma 7.4.
(2)⇒(3). Obvious, because $\Gamma(\psi, D)$ is not empty.
(3)⇒(4). Condition (3) implies that $0 \in \Gamma(\psi, D)+I^3(F(\psi)/F)+I^3(F(D)/F)+I^4(F)$. It follows from Corollary 7.6 that $0 \in \Gamma(\psi, D) + I^4(F)$. Hence there exists $\gamma = k\psi + k'\psi' + lq \in \Gamma(\psi, D)$ such that $\gamma \in I^4(F)$. Since $\dim\gamma = 4+4+6 = 14 < 16$, the Arason-Pfister Hauptsatz shows that $\gamma = 0$.

(4)⇒(5). Let $\gamma = k\psi + k'\psi' + lq$ be a hyperbolic form. We have $(k\psi \perp lq)_{an} = -k'\psi'_{an}$. Therefore $k\psi$ and $-lq$ contain a common subform of dimension

$$\frac{1}{2}(\dim\psi + \dim q - \dim\psi') = \frac{1}{2}(4+6-4) = 3.$$

Let us denote such a 3-dimensional form by τ. Let $x, y \in F^*$ be such that $\tau \sim \langle -x, -y, xy\rangle$. Thus $\langle -x, -y, xy\rangle$ is similar to a subform of ψ and similar to a subform of q. Let $d = \det\psi$. Since $\langle -x, -y, xy\rangle$ is similar to a subform of ψ it follows that $\langle -x, -y, xy, d\rangle$ is similar to ψ. Since $\langle -x, -y, xy\rangle$ is similar to a subform of the Albert form q, it follows that there exist $u, v \in F^*$ such that q is similar to $\langle -x, -y, xy, u, v, -uv\rangle$. Then $[D] = c(q) = (x, y) + (u, v)$. Therefore $D \simeq (x, y) \otimes_F (u, v)$.
(5)⇒(6). See Corollary 4.6.
(6)⇒(7). See Proposition 2.1.
(7)⇒(1). See Lemma 2.5. □

Proposition 7.8. *Let D be a biquaternion algebra and let ψ be a 4-dimensional quadratic form such that $\det\psi \neq 1$ and $\operatorname{ind}(D \otimes_F C_0(\psi)) = 2$. Then for any $\gamma \in \Gamma(\psi, D)$ one has*

$$H^3(F(\psi, D)/F) = H^3(F(\psi)/F) + H^3(F(D)/F) + e^3(\gamma)H^0(F).$$

Proof. By Lemma 7.4, the element $e^3(\gamma)$ belongs to $H^3(F(\psi, D)/F)$. If the group $\mathrm{CH}^2(X_\psi \times X_D)$ is torsion-free, then by Proposition 2.1 we have

$$H^3(F(\psi, D)/F) = H^3(F(\psi)/F) + H^3(F(D)/F)$$

and the proof is complete. If $\operatorname{Tors}\mathrm{CH}^2(X_\psi \times X_D) \neq 0$, Lemma 7.7 shows that $\gamma \notin I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F)$. It follows from Lemma 2.5 that $e^3(\gamma) \notin H^3(F(\psi)/F) + H^3(F(D)/F)$. To complete the proof it is sufficient to apply Proposition 4.3 and Proposition 2.1. □

Corollary 7.9. $I^3(F(\psi, D)/F) \subset I^3(F(\psi)/F) + I^3(F(D)/F) + \Gamma'(\psi, D) + I^4(F)$, *where* $\Gamma'(\psi, D) = \Gamma(\psi, D) \cup \{0\}$.

Proof. Let $\tau \in I^3(F(\psi, D)/F)$. Choose an element $\gamma \in \Gamma(\psi, D)$. By Proposition 7.8, either the element $e^3(\tau)$ or the element $e^3(\tau - \gamma)$ is in $H^3(F(\psi)/F) + H^3(F(D)/F)$. It remains to apply Lemma 2.5. □

Proposition 7.10. *Let $\lambda \in I^3(F(\psi, D)/F)$. Then at least one of the following conditions holds*

1) $\lambda \in I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F)$;
2) $\lambda \in \Gamma(\psi, D) + I^4(F)$.

Proof. This is obvious in view of Corollaries 7.9 and 7.6. □

8. Main theorem

Theorem 8.1. *Let ϕ be an anisotropic 8-dimensional quadratic form with $\det\phi = 1$ and let ψ be a quadratic form such that $\phi_{F(\psi)}$ is isotropic. Let the case where $\dim\psi = 4$, $\det\psi \neq 1$, and $\operatorname{ind} C(\phi) = \operatorname{ind}(C(\phi) \otimes_F C_0(\psi)) = 4$ be excluded. Then the isotropy of $\phi_{F(\psi)}$ is L-standard.*

Proof. Results described in the introduction and in §5 reduce the proof to the case where $\dim\psi = 4$, $\det\psi \neq 1$, and $\operatorname{ind} C(\phi) = 4$. Then there exists a biquaternion algebra D such that $c(\phi) = [D]$. Let q be an Albert form corresponding to D. Clearly (ϕ, D, q) is a special triple.

By Corollary 6.4, we can suppose that $\operatorname{Tors}\mathrm{CH}^2(X_\psi \times X_D) \neq 0$. Theorem 4.1 asserts that at least one of the following conditions holds:

1) $\operatorname{ind}(D \otimes C_0(\psi)) = \operatorname{ind} D = 4$,
2) $\operatorname{ind}(D \otimes C_0(\psi)) = 2$.

If $\operatorname{ind}(D \otimes C_0(\psi)) = 2$, then all the conditions of Definition 7.2 hold. Thus we have a well-defined set $\Gamma(\psi, D)$.

By Proposition 6.3, we have $\phi + q \in I^3(F(\psi, D)/F)$. By Proposition 7.10, we see that at least one of the following condition holds:

1) $\phi + q \in I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F)$;
2) $\phi + q \in \Gamma(\psi, D) + I^4(F)$.

In the first case, Proposition 6.3 shows that there exists a 5-dimensional Pfister neighbor ϕ_0 such that $\phi_0 \subset \phi$ and $(\phi_0)_{F(\psi)}$ is isotropic. This implies that the isotropy of $\phi_{F(\psi)}$ is L-standard.

Therefore, we may assume that $\phi + q \in \Gamma(\psi, D) + I^4(F)$. Thus there exist $k, k', l \in F^*$ (and a 4-dimensional quadratic form ψ') such that

$$\phi + q \equiv k\psi + k'\psi' + lq \pmod{I^4(F)}.$$

Since $k\psi + k'\psi' + lq \in I^3(F)$ it follows that

$$k\psi + k'\psi' + lq \equiv l(k\psi + k'\psi' + lq) \pmod{I^4(F)}.$$

Therefore

$$\phi + q \equiv lk\psi + lk'\psi' + q \pmod{I^4(F)}.$$

Thus $\phi \equiv lk\psi + lk'\psi' \pmod{I^4(F)}$. Let $\phi^* \overset{\text{def}}{=} \psi \perp kk'\psi'$. Since $\phi \equiv lk\phi^*$ $\pmod{I^4(F)}$ and $\dim\phi^* = 8$, it follows that ϕ and ϕ^* are half-neighbors. Since $\psi \subset \phi^*$, the isotropy is L-standard and the proof is complete. □

Corollary 8.2. *Let ϕ be an anisotropic 8-dimensional quadratic form with $\det\phi = 1$ and let ψ be a quadratic form such that $\phi_{F(\psi)}$ is isotropic but the isotropy is not L-standard. Then ϕ can be written in the form $\phi = \pi_1 \perp \pi_2$ with $\pi_1, \pi_2 \in GP_2(F)$.*

Proof. By Theorem 8.1, we have $\dim\psi = 4$ and $\det\psi \neq 1$. Since $\det\phi = 1$, it is sufficient to verify that ϕ contains a 4-dimensional quadratic form with trivial determinant. Suppose for the moment that ϕ contains no 4-dimensional quadratic subform with trivial determinant. Then [11, Th. 6.1] implies that there exists a homomorphism of F-algebras $C_0(\psi) \to C_0(\phi)$. Since $\det\phi = 1$ and $\dim\phi = 8$, there exists a 3-quaternion algebra A such that $C_0(\phi)$ has the form $A \times A$ and $C(\phi) \simeq M_2(A)$. Thus we get a homomorphism $C_0(\psi) \to A$ which is injective because $C_0(\psi)$ is a simple algebra. Then $\operatorname{ind}(C_0(\psi) \otimes_F A) = 2$. Since $M_2(A) \simeq C(\phi)$, we have $\operatorname{ind}(C_0(\psi) \otimes_F C(\phi)) = 2$. Theorem 8.1 implies that the isotropy of $\phi_{F(\psi)}$ is L-standard. This contradiction completes the proof. $\square$

It is a natural question if there exists an example of non-L-standard isotropy. One possible way to construct such an example is based on the following

Lemma 8.3. *Let q be a 6-dimensional quadratic form and let ψ be a 4-dimensional quadratic form over a field k. Suppose that q is a $k(\psi)$-minimal form (for a definition, see [8]). Let $F \overset{\text{def}}{=} k((t))$ and $\phi \overset{\text{def}}{=} q \perp t\,\langle 1, \det(q)\rangle$. Then the form $\phi_{F(\psi)}$ is isotropic, but the isotropy is not L-standard.*

Proof. The proof of this lemma is absolutely analogous to the proof of [12, Th. 5.1] and we omit it. $\square$

Corollary 8.4. *There exist a field F, an 8-dimensional quadratic form $\phi \in I^2(F)$, and a 4-dimensional F-form ψ with nontrivial determinant such that $\phi_{F(\psi)}$ is isotropic, but the isotropy is not L-standard.*

Proof. By [15] there exists a field k, a 6-dimensional quadratic form q and a 4-dimensional quadratic form ψ over a field k such that q is a $k(\psi)$-minimal form. Now the required result follows from Lemma 8.3. $\square$

Corollary 8.5. *There exist a field F, a biquaternion F-algebra D, and a 4-dimensional F-form ψ with nontrivial determinant such that*

$$\operatorname{ind} D = \operatorname{ind}(C_0(\psi) \otimes_F D) = 4 \quad \textit{and} \quad \operatorname{Tors} \operatorname{CH}^2(X_\psi \times X_D) \simeq \mathbb{Z}/2\mathbb{Z}\,.$$

Proof. This is an obvious consequence of Theorem 8.1, Corollaries 8.4 and 6.4, and Proposition 4.3. $\square$

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