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Isotropy of Virtual Albert Forms over Function Fields of Quadrics

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Abstract. Let F be a field of characteristic different from 2 and let ϕ be a virtual Albert form over F, i.e., an anisotropic 6-dimensional quadratic form over F which is still anisotropic over the field $F\left(\sqrt{d_\pm\,\phi}\right)$. We give a complete description of the quadratic forms ψ such that ϕ becomes isotropic over the function field $F(\psi)$. This completes the series of works ([H6], [Lag6], [Lag], [Lee], [M2]) where the question was considered previously.

0. Introduction

Let F be a field of characteristic different from 2 and let ϕ and ψ be two anisotropic quadratic forms over F. An important problem in the algebraic theory of quadratic forms is to find conditions on ϕ and ψ so that $\phi_{F(\psi)}$ is isotropic. In the case where $\dim\phi \leq 5$ the problem was completely solved in [H5] and [Schap]. For 6-dimensional quadratic forms, the problem was studied by D. W. Hoffmann ([H6]), A. Laghribi ([Lag6], [Lag]), D. Leep ([Lee]), and A. S. Merkurjev ([M2]) and was solved fully except for the following two cases (see [Lag6] and [Lag]):

1) $\dim\psi = 4$, $d_\pm\,\psi \neq 1$, and $\operatorname{ind}(C_0(\phi)) = 2$;

2) $\dim\psi = 4$, $d_\pm\,\psi \neq 1$, $\operatorname{ind}(C_0(\phi)) = 4$, and $d_\pm\,\phi = d_\pm\,\psi$.

In this paper the second case is studied completely. Our result (Theorem 5.1) and results of Laghribi, Leep and Merkurjev give rise to the following

Theorem. *Let ϕ be a 6-dimensional quadratic form such that* $\operatorname{ind}(C_0(\phi)) = 4$. *In the case where $\psi \notin GP_2(F)$, the quadratic form $\phi_{F(\psi)}$ is isotropic if and only if ψ is*

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similar to a subform of ϕ. In the case where $\psi \in GP_2(F)$, the form $\phi_{F(\psi)}$ is isotropic if and only if a 3-dimensional subform of ψ is similar to a subform of ϕ.

We deduce Theorem 5.1 from a result on 8-dimensional forms (Proposition 4.1), which also has an independent value: together with [Lag8], it gives rise to Theorem 4.3 answering the question about isotropy of an 8-dimensional quadratic form ϕ with $\det\phi = 1$ and $\operatorname{ind}(C(\phi)) = 8$ over the function fields of quadrics.

1. Terminology, notation, and backgrounds

1.1. Quadratic forms

We write $\phi \perp \psi$ for the orthogonal sum of the quadratic forms. The class of ϕ in the Witt ring $W(F)$ of the field F is also denoted by ϕ. For a quadratic form ϕ of dimension n, we set $d_\pm\,\phi = (-1)^{n(n-1)/2}\det\phi$. We consider $d_\pm\,\phi$ as an element of F^*/F^{*2}. The maximal ideal of $W(F)$ consisting of the classes of the even-dimensional forms is denoted by $I(F)$. The anisotropic part of ϕ is denoted by ϕ_{an}. We denote by $\langle\langle a_1,\ldots,a_n\rangle\rangle$ the n-fold Pfister form $\langle 1,-a_1\rangle\otimes\cdots\otimes\langle 1,-a_n\rangle$ and by $P_n(F)$ the set of all n-fold Pfister forms. The set of all forms similar to n-fold Pfister forms is denoted by $GP_n(F)$. For any field extension L/F, we put $\phi_L = \phi\otimes L$, $W(L/F) = \operatorname{Ker}(W(F)\to W(L))$, and $I^n(L/F) = \operatorname{Ker}(I^n(F)\to I^n(L))$.

For a quadratic form ϕ of dimension ≥ 3, we denote by X_ϕ the projective variety given by the equation $\phi = 0$. We set $F(\phi) = F(X_\phi)$ if $\dim\phi \geq 3$; $F(\phi) = F\big(\sqrt{d}\big)$ if $\dim\phi = 2$ and $d = d_\pm\,\phi \neq 1$; and $F(\phi) = F$ otherwise.

Let $\psi \in GP_2(F)$ and let ψ_0 be a 3-dimensional subform of ψ. Then the quadratic forms $\psi_{F(\psi_0)}$ and $(\psi_0)_{F(\psi)}$ are isotropic. Hence for any quadratic form ϕ, isotropy of $\phi_{F(\psi)}$ is equivalent to isotropy of $\phi_{F(\psi_0)}$. Thus, to give a complete description of the quadratic forms ψ such that ϕ becomes isotropic over the function field $F(\psi)$, it is sufficient to consider the case where $\psi \notin GP_2(F)$.

We say that a quadratic form ϕ is a Pfister neighbor if for some n there exists $\pi \in P_n(F)$ such that ϕ is similar to a subform of π and $\dim\phi > 2^{n-1}$.

Let ϕ be a quadratic form of dimension 2^n. We say that ϕ^* is a half-neighbor of ϕ, if $\dim\phi^* = 2^n$ and there exists $k \in F^*$ such that $\phi^* \equiv k\phi \pmod{I^{n+1}(F)}$.

1.2. Algebras

Let A be a central simple algebra over F. We write $\deg(A)$, $\operatorname{ind}(A)$, $[A]$, and $\exp(A)$ for the degree of A, the Schur index of A, the class of A in the Brauer group $\operatorname{Br}(F)$, and the order of $[A]$ in the Brauer group respectively. The Severi-Brauer variety of A is denoted by $\operatorname{SB}(A)$. If an algebra B has the form $B = A\times A$, we set $\operatorname{ind} B = \operatorname{ind} A$.

Let ϕ be a quadratic form. We write $C(\phi)$ for the Clifford algebra of ϕ and $C_0(\phi)$ for the even part of $C(\phi)$. If $\phi \in I^2(F)$ then $C(\phi)$ is a central simple algebra. Hence we get a well defined element $[C(\phi)]$ of $\operatorname{Br}_2(F)$ which we denote by $c(\phi)$.

1.3. Quadratic forms of dimension 6

Let ϕ be an anisotropic quadratic form of dimension 6 and let $d = d_\pm\, \phi$. If $d = 1$, then ϕ is an Albert form. In this case the problem of isotropy of ϕ over the function field of a quadratic form ψ is completely solved ([Lee], [M2]): in the case where $\psi \notin GP_2(\psi)$, the form $\phi_{F(\psi)}$ is isotropic if and only if ψ is similar to a subform in ϕ.

Suppose now that $d \neq 1$. Then $C_0(\phi)$ is a central simple algebra over the field $L = F(\sqrt{d})$. In this case we have the following classification of anisotropic 6-dimension forms:

Type 1 is defined by one of the following equivalent conditions:
1) $\mathrm{ind}(C_0(\phi)) = 1$;
2) ϕ_L is hyperbolic;
3) ϕ has the form $\langle\langle d\rangle\rangle \otimes \mu$ where μ is a quadratic form of dimension 3;
4) ϕ is a Pfister neighbor.

Type 2 is defined by one of the following equivalent conditions:
1) $\mathrm{ind}(C_0(\phi)) = 2$;
2) ϕ_L is isotropic but not hyperbolic;
3) ϕ is similar to a form of the kind $\langle\langle a, b\rangle\rangle \perp c\,\langle\langle d\rangle\rangle$, where $\langle\langle a, b\rangle\rangle_L$ is not isotropic.

Type 3 is defined by one of the following equivalent conditions:
1) $\mathrm{ind}(C_0(\phi)) = 4$;
2) ϕ_L is anisotropic.

The quadratic form of the type 3 is called a virtual Albert form.

For the quadratic forms ϕ of type 1 (i. e., for the Pfister neighbors), the problem of isotropy $\phi_{F(\psi)}$ is completely solved by the Cassels-Pfister subform theorem [Schar, Th. 5.4 (ii) of Ch. 4]. The case of quadratic forms of type 2 was studied by D. HOFFMANN in [H6]: he found the conditions on ϕ and ψ so that $\phi_{F(\psi)}$ is isotropic excepting the case $\dim\psi = 4$. The case where ϕ is of type 2 and $\dim\psi = 4$ is recently studied in [IK6].

The case of the quadratic forms ϕ of type 3 (virtual Albert forms) was studied completely by A. LAGHRIBI in [Lag6, Lag] except for the case where $\dim\psi = 4$ and $d_\pm\, \psi = d_\pm\, \phi$. In this paper we complete the investigation of isotropy of virtual Albert forms over the function field of a quadric.

1.4. Cohomology groups

By $H^*(F)$ we denote the graded ring of Galois cohomology

$$H^*(F, \mathbb{Z}/2\mathbb{Z}) \;=\; H^*(\mathrm{Gal}(F_{\mathrm{sep}}/F), \mathbb{Z}/2\mathbb{Z})\,.$$

For any field extension L/F, we set $H^*(L/F) = \mathrm{Ker}(H^*(F) \to H^*(L))$.

We use the standard canonical isomorphisms $H^0(F) = \mathbb{Z}/2\mathbb{Z}$, $H^1(F) = F^*/F^{*2}$, and $H^2(F) = \mathrm{Br}_2(F)$. Thus any element $a \in F^*$ gives rise to an element of $H^1(F)$; it is denoted by (a). The cup product $(a_1) \cup \ldots \cup (a_n)$ is denoted by $(a_1, \ldots, a_n)$.

For $n = 0, 1, 2$ there is a homomorphism $e^n : I^n(F) \to H^n(F)$ defined as follows: $e^0(\phi) = \dim\phi \pmod 2$, $e^1(\phi) = d_\pm\, \phi$, and $e^2(\phi) = c(\phi)$. Moreover, there exists a homomorphism $e^3 : I^3(F) \to H^3(F)$ which is uniquely determined by the condition $e^3(\langle\langle a_1, a_2, a_3\rangle\rangle) = (a_1, a_2, a_3)$ (see [Ara]). The homomorphism e^n is surjective and $\mathrm{Ker}\, e^n = I^{n+1}(F)$ for $n = 0, 1, 2, 3$ (see [M1], [MS], and [R]).

1.5. K – theory and Chow groups

Let X be a smooth algebraic F – variety. The Grothendieck ring of X is denoted by $K(X)$. This ring is supplied with the filtration "by codimension of support" (which respects the multiplication); the adjoint graded ring is denoted by $G^*K(X)$. There is a canonical surjective homomorphism of the graded Chow ring $\mathrm{CH}^*(X)$ onto $G^*K(X)$; its kernel consists only of torsion elements and is trivial in the 0th, 1st, and 2nd graded components ([Su, §9]).

We fix a separable closure $\overline{F}$ of the ground field F and denote by $\overline{X}$ the variety $X_{\overline{F}}$. The image of the restriction homomorphism $G^*K(X) \to G^*K(\overline{X})$ is denoted by $\overline{G}^*K(X)$.

We denote by $|S|$ the order of a finite set S.

2. Computation of $H^3(F(\mathrm{SB}(A) \times \mathrm{SB}(B))/F)$

Theorem 2.1. *Let A and B be biquaternion division F – algebras with* $\mathrm{ind}(A \otimes B) = 8$. *Suppose that there exists a quadratic extension L/F such that both A_L and B_L are not division algebras. Then*

$$H^3(F(\mathrm{SB}(A) \times \mathrm{SB}(B))/F) = [A] \cup H^1(F) + [B] \cup H^1(F).$$

Proof. We put $X = \mathrm{SB}(A) \times \mathrm{SB}(B)$.
The following formula is proved in [K, Prop. 2]:

Lemma 2.2.

$$|\operatorname{Tors} G^*K(X)| = \frac{|G^*K(\overline{X})/\overline{G}^*K(X)|}{|K(\overline{X})/K(X)|}.$$

Lemma 2.3. $|K(\overline{X})/K(X)| = 2^{28}$.

Proof. Applying [Q, Th. 4.1 of §8], one gets an isomorphism

$$K(X) \simeq K(F)^{\oplus 4} \oplus K(A)^{\oplus 4} \oplus K(B)^{\oplus 4} \oplus K(A \otimes B)^{\oplus 4}.$$

Thus $|K(\overline{X})/K(X)| = (\mathrm{ind}\, A)^4 \cdot (\mathrm{ind}\, B)^4 \cdot (\mathrm{ind}\, A \otimes B)^4 = 2^{28}$. □

The variety $\overline{\mathrm{SB}(A)}$ is isomorphic to a projective space; denote by f the class of a hyperplane in $G^1K(\overline{\mathrm{SB}(A)})$.

Lemma 2.4. *For any $i \geq 0$, the group $\overline{G}^iK(\mathrm{SB}(A_L))$ contains $2f^i$; for any even $i \geq 0$ it contains f^i.*

Proof. By [K, Lemma 3], for any i, one has an inclusion

$$\overline{G}^iK(\mathrm{SB}(A_L)) \ni \frac{\mathrm{ind}\, A_L}{(i, \mathrm{ind}\, A_L)} f^i$$

where $(\cdot,\cdot)$ denotes the greatest common divisor. Since $\operatorname{ind} A_L = 2$, the statement follows. □

Lemma 2.5. $\overline{G}^1 K(\mathrm{SB}(A)) \ni 2f$.

Proof. By the computation [Art, §2] of the Picard group of a Severi–Brauer variety, one knows that

$$\overline{G}^1 K(\mathrm{SB}(A)) \ni (\exp A) f .$$

Since $\exp A = 2$, the statement follows. □

The variety $\overline{\mathrm{SB}(B)}$ is isomorphic to a projective space; denote by g the class of a hyperplane in $G^1 K(\overline{\mathrm{SB}(B)})$.

Corollary 2.6. *For any i, $j \geq 0$, the group $\overline{G}^{i+j} K(X)$ contains*

1) $f^i \times g^j$, *if* $i = j = 0$;

2) $2(f^i \times g^j)$, $\begin{cases} \text{if } i \text{ and } j \text{ are even or} \\ \text{if } i = 0,\ j = 1 \text{ or} \\ \text{if } i = 1,\ j = 0; \end{cases}$

3) $4(f^i \times g^j)$, $\begin{cases} \text{if } i + j \text{ is odd or} \\ \text{if } i = j = 1; \end{cases}$

4) $8(f^i \times g^j)$ *for any* i, j.

Proof. The case $i = j = 0$ is evident.

If i and j are even, then $f^i \in \overline{G}^i K(\mathrm{SB}(A_L))$ and $g^j \in \overline{G}^j K(\mathrm{SB}(B_L))$ by Lemma 2.4. Thus $f^i \times g^j \in \overline{G}^{i+j} K(X_L)$ and the transfer argument shows that $2(f^i \times g^j) \in \overline{G}^{i+j} K(X)$.

By Lemma 2.5, $\overline{G}^1 K(\mathrm{SB}(A)) \ni 2f$ and $\overline{G}^1 K(\mathrm{SB}(B)) \ni 2g$. Therefore $\overline{G}^1 K(X)$ contains $2(f \times 1)$ and $2(1 \times g)$; moreover, $\overline{G}^2 K(X) \ni 4(f \times g)$.

If $i+j$ is odd, then $2(f^i \times g^j) \in \overline{G}^{i+j} K(X_L)$ by Lemma 2.4 and the transfer argument shows that $4(f^i \times g^j) \in \overline{G}^{i+j} K(X)$.

Since there exists a field extension of degree 8 splitting the algebras A and B simultaneously, the inclusion $8(f^i \times g^j) \in \overline{G}^{i+j} K(X)$ holds for any i, j. □

Corollary 2.7. $\left|G^* K(\overline{X}) / \overline{G}^* K(X)\right| \leq 2^{28}$.

Proof. Since $\overline{\mathrm{SB}(A)}$ and $\overline{\mathrm{SB}(B)}$ are projective spaces, $G^* K(\overline{X})$ is an abelian group freely generated by $f^i \times g^j$ with $i, j = 0, 1, 2, 3$. By Lemma 2.6, we know that the following multiples of these generators are in $\overline{G}^*K(X)$:

$$\begin{array}{llll}
2^0\cdot\left(f^0\times g^0\right), & 2^1\cdot\left(f^0\times g^1\right), & 2^1\cdot\left(f^0\times g^2\right), & 2^2\cdot\left(f^0\times g^3\right),\\
2^1\cdot\left(f^1\times g^0\right), & 2^2\cdot\left(f^1\times g^1\right), & 2^2\cdot\left(f^1\times g^2\right), & 2^3\cdot\left(f^1\times g^3\right),\\
2^1\cdot\left(f^2\times g^0\right), & 2^2\cdot\left(f^2\times g^1\right), & 2^1\cdot\left(f^2\times g^2\right), & 2^2\cdot\left(f^2\times g^3\right),\\
2^2\cdot\left(f^3\times g^0\right), & 2^3\cdot\left(f^3\times g^1\right), & 2^2\cdot\left(f^3\times g^2\right), & 2^3\cdot\left(f^3\times g^3\right).
\end{array}$$

Taking the product of the coefficients, we get 2^{28}. □

Corollary 2.8. $\operatorname{Tors} G^*K(X)=0$.

Proof. Follows from Lemma 2.2, Lemma 2.3, and Corollary 2.7. □

Since the Chow group $\mathrm{CH}^2(X)$ is isomorphic to $G^2K(X)$ (see §1.5.), we also get

Corollary 2.9. $\operatorname{Tors}\mathrm{CH}^2(X)=0$.

To complete the proof of Theorem 2.1, we apply [Pe, Th. 4.1 with Rem. 4.1]. By that result, there is a monomorphism

$$\frac{H^3(F(X)/F)}{[A]\cup H^1(F)+[B]\cup H^1(F)}\hookrightarrow \operatorname{Tors}\mathrm{CH}^2(X)$$

and so, by Corollary 2.9, we are done. □

Remark 2.10. In the hypotheses of Theorem 2.1 there are obvious inclusions:

$$\begin{array}{lclcl}
[A]\cup H^1(F) & \subset & H^3(F(\mathrm{SB}(A))/F) & \subset & H^3(F(\mathrm{SB}(A)\times\mathrm{SB}(B))/F);\\
[B]\cup H^1(F) & \subset & H^3(F(\mathrm{SB}(B))/F) & \subset & H^3(F(\mathrm{SB}(A)\times\mathrm{SB}(B))/F).
\end{array}$$

Therefore $H^3(F(\mathrm{SB}(A)\times\mathrm{SB}(B))/F)=H^3(F(\mathrm{SB}(A))/F)+H^3(F(\mathrm{SB}(B))/F)$.

3. Computation of $H^3(F(X_\psi\times\mathrm{SB}(D))/F)$

Theorem 3.1. *Let ψ be an anisotropic 4 - dimensional quadratic form over F with $d_\pm\psi\neq 1$. Let D be a 3 - quaternion division algebra over F such that $D_{F(\psi)}$ is not a division algebra. Then the group $H^3(F(X_\psi\times\mathrm{SB}(D))/F)$ is equal to*

$$\left\{e^3(\psi\langle\langle k\rangle\rangle)\mid k\in F^* \textit{ is such that } \psi\langle\langle k\rangle\rangle\in GP_3(F)\right\}+[D]\cup H^1(F).$$

Proof. We start with the following observation:

Lemma 3.2. *In the hypotheses of Theorem 3.1, assume that $\psi=\langle -a,-b,ab,d\rangle$ with some a, b, $d\in F^*$. Then there exist u, v, $s\in F^*$ such that $D\simeq(a,b)\otimes(u,v)\otimes(d,s)$.*

Proof. Clearly, $C_0(\psi) \simeq (a,b)_{F(\sqrt{d})}$. Since $D_{F(\psi)}$ is not a division algebra, the index reduction formula [M3] shows that $\operatorname{ind}(C_0(\psi) \otimes_F D) = 2$. Therefore

$$\operatorname{ind}((a,b) \otimes_F D)_{F(\sqrt{d})} = 2.$$

Hence there are $u, v, s \in F^*$ such that $[(a,b) \otimes_F D] = [(u,v) \otimes_F (d,s)]$ in $\mathrm{Br}_2(F)$. Consequently $[D] = [(a,b) \otimes_F (u,v) \otimes_F (d,s)]$. Since $\deg D = 8$, it follows that $D \simeq (a,b) \otimes (u,v) \otimes (d,s)$. $\square$

Replacing ψ by a similar form, we may assume that $\psi = \langle -a, -b, ab, d \rangle$ with some $a, b, d \in F^*$. We choose $u, v, s \in F^*$ as in Lemma 3.2. Put $\widehat{F} = F((t))$ and consider two biquaternion algebras $A = (a,b) \otimes (d,t)$ and $B = (d,st) \otimes (u,v)$ over $\widehat{F}$.

Since D is a division algebra, it follows that $\operatorname{ind}(D) = 8$. Therefore

$$\operatorname{ind}((a,b) \otimes (u,v))_{F(\sqrt{d})} = \operatorname{ind} D_{F(\sqrt{d})} = 4.$$

Hence $(a,b)_{F(\sqrt{d})}$ and $(u,v)_{F(\sqrt{d})}$ are division $F(\sqrt{d})$-algebras. Consequently, by Tignol's theorem [T, Prop. 2.4], A and B are division $\widehat{F}$-algebras as well.

Since $\operatorname{ind}(D) = \operatorname{ind}\left(D_{\widehat{F}}\right)$ (see [T, Prop. 2.4]) and $[A \otimes B] = \left[D_{\widehat{F}}\right]$ in $\mathrm{Br}\left(\widehat{F}\right)$, we have $\operatorname{ind}\left(A \otimes_{\widehat{F}} B\right) = \operatorname{ind}(D) = 8$. Since $A_{\widehat{F}(\sqrt{d})}$ and $B_{\widehat{F}(\sqrt{d})}$ are not division algebras, the conditions of Theorem 2.1 hold for the field $\widehat{F}$ and the algebras A and B over $\widehat{F}$. Therefore

$$H^3\left(\widehat{F}(\mathrm{SB}(A) \times \mathrm{SB}(B))/\widehat{F}\right) = [A] \cup H^1(\widehat{F}) + [B] \cup H^1(\widehat{F}).$$

Let $E = \widehat{F}(\mathrm{SB}(A) \times \mathrm{SB}(B))$. Clearly $[A_E] = [B_E] = 0$. Hence $[D_E] = [A_E] + [B_E] = 0$. Thus $\mathrm{SB}(D)_E$ is a rational variety.

Since $[A_E] = 0$, the Albert form $\langle -a, -b, ab, d, t, -dt \rangle_E$ of the biquaternion algebra $A_E = ((a,b) \otimes (d,t))_E$ is hyperbolic. Hence $\langle -a, -b, ab, d \rangle_E = \langle dt, -t \rangle_E$ in the Witt ring $W(E)$. Therefore ψ_E is isotropic. Hence $(X_\psi)_E$ is a rational variety.

Let $Y = X_\psi \times \mathrm{SB}(D)$. Since $(X_\psi)_E$ and $\mathrm{SB}(D)_E$ are rational, it follows that Y_E is rational. Hence $E(Y)/E$ is a purely transcendental extension. Therefore $H^3(E(Y)/F) = H^3(E/F)$. We have $H^3(F(Y)/F) \subset H^3(E(Y)/F) = H^3(E/F)$.

Let $u \in H^3(F(X_\psi \times \mathrm{SB}(D))/F) = H^3(F(Y)/F)$. To prove the theorem, it is enough to show that u can be written in the form

$$u = e^3(\psi \langle\langle k \rangle\rangle) + [D] \cup (r)$$

with some $k, r \in F^*$.

Since $H^3(F(Y)/F) \subset H^3(E/F)$, it follows that

$$u_{\widehat{F}} \in H^3(E/\widehat{F}) = H^3\left(\widehat{F}(\mathrm{SB}(A) \times \mathrm{SB}(B))/\widehat{F}\right) = [A] \cup H^1(\widehat{F}) + [B] \cup H^1(\widehat{F}).$$

Since $[A] + [B] = \left[D_{\widehat{F}}\right]$, we have $u_{\widehat{F}} \in [A] \cup H^1(\widehat{F}) + \left[D_{\widehat{F}}\right] \cup H^1(\widehat{F})$. Hence there are $\alpha, \beta \in \widehat{F}$ such that $u_{\widehat{F}} = [A] \cup (\alpha) + \left[D_{\widehat{F}}\right] \cup (\beta)$. Since $\widehat{F}^*/\widehat{F}^{*2} \simeq F^*/F^{*2} \times \{1, t\}$, we may assume that $\alpha = kt^i$ and $\beta = rt^j$, where $k,\ r \in F^*$ and $i,\ j \in \{0,1\}$. We have

$$\begin{aligned} u_{\widehat{F}} &= [A] \cup (\alpha) + \left[D_{\widehat{F}}\right] \cup (\beta) \\ &= ((a,b) + (d,t)) \cup \left(kt^i\right) + [D] \cup \left(rt^j\right) \\ &= ((a,b,k) + [D] \cup (r)) + (t) \cup \left(i(a,b) + \left(d, k(-1)^i\right) + j[D]\right). \end{aligned}$$

Using the well-known isomorphism $H^i(F((t))) = H^i(F) \oplus H^{i-1}(F)$, we have

$$u = (a,b,k) + [D] \cup (r)$$

and

$$i(a,b) + \left(d, k(-1)^i\right) + j[D] = 0.$$

We claim that $j = 0$. Indeed, if $j \neq 0$ then $j = 1$ and hence $[D] = i(a,b) + \left(d, k(-1)^i\right)$. Therefore $[D] = \left[(a, b^i) \otimes \left(d, k(-1)^i\right)\right]$. Thus $\operatorname{ind}(D) \leq 4$, a contradiction.

So $j = 0$, and we have $i(a,b) + \left(d, k(-1)^i\right) = 0$. Thereby $i(a,b)_{F(\sqrt{d})} = 0$. Since $(a,b)_{F(\sqrt{d})} \neq 0$, it follows that $i = 0$.

Since $i(a,b) + \left(d, k(-1)^i\right) = 0$ and $i = 0$, we have $(d,k) = 0$. Hence $\langle\langle d,k\rangle\rangle = 0$ in $W(F)$. Since $\psi = \langle -a, -b, ab, d\rangle = \langle\langle a,b\rangle\rangle - \langle\langle d\rangle\rangle$, we have

$$\psi\,\langle\langle k\rangle\rangle = \left(\langle\langle a,b\rangle\rangle - \langle\langle d\rangle\rangle\right)\langle\langle k\rangle\rangle = \langle\langle a,b,k\rangle\rangle - \langle\langle d,k\rangle\rangle = \langle\langle a,b,k\rangle\rangle.$$

Therefore $\psi\,\langle\langle k\rangle\rangle \in GP_3(F)$ and $e^3(\psi\,\langle\langle k\rangle\rangle) \doteq (a,b,k)$.

Hence the element $u = (a,b,k) + [D] \cup (r)$ belongs to the set

$$\left\{e^3(\psi\,\langle\langle k\rangle\rangle) \mid k \text{ is such that } \psi\,\langle\langle k\rangle\rangle \in GP_3(F)\right\} + [D] \cup H^1(F).$$

The proof is complete. □

Remark 3.3. In the hypotheses of Theorem 3.1 there are obvious inclusions:

$$\begin{aligned} &\left\{e^3(\psi\,\langle\langle k\rangle\rangle) \mid k \in F^* \text{ is such that } \psi\,\langle\langle k\rangle\rangle \in GP_3(F)\right\} \subset H^3(F(\psi)/F) \\ &\qquad\qquad\qquad\qquad \subset H^3(F(X_\psi \times \mathrm{SB}(D))/F); \\ &[D] \cup H^1(F) \subset H^3(F(\mathrm{SB}(D))/F) \subset H^3(F(X_\psi \times \mathrm{SB}(D))/F). \end{aligned}$$

Therefore $H^3(F(X_\psi \times \mathrm{SB}(D))/F) = H^3(F(\psi)/F) + H^3(F(\mathrm{SB}(D))/F)$.

Proposition 3.4. *In the hypotheses of Theorem* 3.1, *let* $\xi \in I^2(F)$ *be a quadratic form such that* $c(\xi) = [D]$. *Then for an arbitrary element* $\pi \in I^3(F(X_\psi \times \mathrm{SB}(D))/F)$ *there are* $k_1,\ k_2 \in F^*$ *such that*

$$\pi \equiv \psi\,\langle\langle k_1\rangle\rangle + \xi\,\langle\langle k_2\rangle\rangle \pmod{I^4(F)}.$$

Proof. Obviously $e^3(\pi) \in H^3(F(X_\psi \times \mathrm{SB}(D))/F)$. It follows from Theorem 3.1 that there are $k_1,\ k_2 \in F^*$ such that $e^3(\pi) = e^3\left(\psi\,\langle\langle k_1\rangle\rangle\right) + [D] \cup (k_2)$. Clearly $[D] \cup (k_2) = e^2(\xi) \cup e^1\left(\langle\langle k_2\rangle\rangle\right) = e^3\left(\xi\,\langle\langle k_2\rangle\rangle\right)$. Hence $e^3(\pi) = e^3\left(\psi\,\langle\langle k_1\rangle\rangle\right) + e^3\left(\xi\,\langle\langle k_2\rangle\rangle\right)$. Since $\operatorname{Ker}\left(e^3 : I^3(F) \to H^3(F)\right) = I^4(F)$, we have $\pi \equiv \psi\,\langle\langle k_1\rangle\rangle + \xi\,\langle\langle k_2\rangle\rangle \pmod{I^4(F)}$. □

4. 8 – dimensional quadratic forms

Proposition 4.1. *Let ϕ be an 8 – dimensional quadratic form with $d_{\pm}\,\phi = 1$ and* $\operatorname{ind} C(\phi) = 8$. *Let ψ be a 4 – dimensional quadratic form with $d_{\pm}\,\psi \neq 1$. Suppose that* $\phi_{F(\psi)}$ *is isotropic. Then there exists a half – neighbor ϕ^* of ϕ such that $\psi \subset \phi^*$.*

Proof. Replacing ψ by a similar form, we may assume that $\psi = \langle -a, -b, ab, d\rangle$ with some a, b, $d \in F^*$. Then $C_0(\psi) = (a,b)_{F(\sqrt{d})}$. Besides, since $\operatorname{ind} C(\phi) = 8$, there exists a 3 – quaternion division algebra D such that $c(\phi) = [D]$.

Since $\phi_{F(\psi)}$ is isotropic, it follows that $D_{F(\psi)}$ is not a division algebra. Therefore, by Lemma 3.2, there exist u, v, $s \in F^*$ such that $D \simeq (a,b) \otimes (u,v) \otimes (d,s)$.

Consider the quadratic form

$$\gamma \;=\; \langle -a, -b, ab, d\rangle \perp -s\,\langle -u, -v, uv, d\rangle\,.$$

One can verify that $d_{\pm}\,\gamma = 1$ and $c(\gamma) = [D]$. Hence $c(\phi) = [D] = c(\gamma)$. Therefore $\phi + \gamma \in I^3(F)$.

Lemma 4.2. $\phi + \gamma \in I^3(F(X_\psi \times \mathrm{SB}(D))/F)$.

Proof. Let $E = F(X_\psi \times \mathrm{SB}(D))$. Since $\phi + \gamma \in I^3(F)$, it is sufficient to verify that ϕ_E and γ_E are hyperbolic. Obviously $[D_E] = 0$, and the form ψ_E is isotropic. Since $c(\phi_E) = c(\gamma_E) = [D_E] = 0$ and $\dim\phi = \dim\gamma = 8$, we have ϕ_E, $\gamma_E \in GP_3(E)$. Hence it is sufficient to prove that ϕ_E and γ_E are isotropic. Since $\phi_{F(\psi)}$ and ψ_E are isotropic, ϕ_E is isotropic as well. Since $\psi \subset \gamma$ and ψ_E is isotropic, we see that γ_E is isotropic. □

Now we can complete the proof of Proposition 4.1. By Proposition 3.4 and Lemma 4.2, there exist k_1, $k_2 \in F^*$ such that

$$\phi + \gamma \;\equiv\; \psi\,\langle\!\langle k_1\rangle\!\rangle + \phi\,\langle\!\langle k_2\rangle\!\rangle \pmod{I^4(F)}\,.$$

Let $\rho = -s\,\langle -u, -v, uv, d\rangle$. We have $\gamma = \psi + \rho$. Hence

$$\phi + \psi + \rho \equiv \psi - k_1\psi + \phi - k_2\phi \pmod{I^4(F)}.$$

Thus $k_1\psi + \rho \equiv -k_2\phi \pmod{I^4(F)}$. Hence $\psi + k_1\rho \equiv -k_1k_2\phi \pmod{I^4(F)}$. We finish the proof by setting $\phi^* = \psi \perp k_1\rho$. □

Theorem 4.3. *Let ϕ be an 8 – dimensional quadratic form with $d_{\pm}\,\phi = 1$ and* $\operatorname{ind} C(\phi) = 8$. *Let ψ be a quadratic form of dimension ≥ 4 such that $\psi \notin GP_2(F)$. The following conditions are equivalent:*

1) *$\phi_{F(\psi)}$ is isotropic;*
2) *there exists a half – neighbor ϕ^* of ϕ such that $\psi \subset \phi^*$.*

Proof. The case $\dim\psi = 4$ is Proposition 4.1. In the case $\dim\psi \neq 4$ the statement was proved by LAGHRIBI in [Lag8] and [Lag] (see also [IK, Cor. 0.2]). □

5. Main theorem

Theorem 5.1. *Let ϕ be a virtual Albert form (i. e., a 6-dimensional quadratic form with $d_\pm \phi \notin F^{*2}$ and $\mathrm{ind}(C_0(\phi)) = 4$). Let ψ be a 4-dimensional quadratic form such that $d_\pm \psi \neq 1$. The following conditions are equivalent:*
(1) $\phi_{F(\psi)}$ is isotropic;
(2) ψ is similar to a subform in ϕ.

Proof. (1) $\Rightarrow$ (2). Let $d = d_\pm \phi$. We have $\mathrm{ind}\left(C(\phi \perp \langle\langle d\rangle\rangle)_{F(\sqrt{d})}\right) = \mathrm{ind}(C_0(\phi)) = 4$. Consider the 8-dimensional quadratic form $\xi = \phi_{\widehat{F}} \perp t\langle\langle d\rangle\rangle$ over the field $\widehat{F} = F((t))$. Clearly, $c(\xi) = c(\phi \perp t\langle\langle d\rangle\rangle) = c(\phi \perp \langle\langle d\rangle\rangle) + [(d,t)]$. Applying [T, Prop. 2.4], we have $\mathrm{ind}(C(\xi)) = 2\,\mathrm{ind}\left(C(\phi \perp \langle\langle d\rangle\rangle)_{F(\sqrt{d})}\right) = 8$.

Clearly $\xi_{\widehat{F}(\psi)}$ is isotropic. It follows from Proposition 4.1 that there exists a quadratic form ξ^* over $\widehat{F}$ such that ξ and ξ^* are half-neighbors and $\psi_{\widehat{F}} \subset \xi^*$.

Lemma 5.2. *ξ^* is similar to ξ.*

Proof. Since ξ and ξ^* are half-neighbors, there exists $k \in \widehat{F}$ such that

$$\xi \equiv k\xi^* \pmod{I^4(\widehat{F})}.$$

By Springer's theorem one can write $k\xi^*$ in the form $k\xi^* = \mu_0 \perp t\mu_1$, where quadratic forms μ_0 and μ_1 are defined over F. We have

$$\phi \perp t\langle\langle d\rangle\rangle = \xi \equiv k\xi^* = \mu_0 \perp t\mu_1 \pmod{I^4(\widehat{F})}.$$

Hence $\phi \equiv \mu_0 \pmod{I^3(F)}$, $\langle\langle d\rangle\rangle \equiv \mu_1 \pmod{I^3(F)}$, and

$$\phi + \langle\langle d\rangle\rangle \equiv \mu_0 + \mu_1 \pmod{I^4(F)}.$$

Therefore $\mathrm{ind}\, C_0(\mu_0) = \mathrm{ind}\, C_0(\phi) \geq 4$. Hence $\dim \mu_0 \geq 6$. Therefore $\dim \mu_1 \leq 2$. By the Arason-Pfister Hauptsatz the condition $\langle\langle d\rangle\rangle \equiv \mu_1 \pmod{I^3(F)}$ implies that $\mu_1 = \langle\langle d\rangle\rangle$. Hence $\phi \equiv \mu_0 \pmod{I^4(F)}$. Applying Arason-Pfister Hauptsatz once again, we have $\phi = \mu_0$. Therefore $\xi = k\xi^*$. □

Now we return to the proof of Theorem 5.1. Since ψ is similar to a subform in ξ^*, and ξ^* is similar to ξ, it follows that ψ is similar to a subform in $\xi = \phi \perp t\langle\langle d\rangle\rangle$. Thus ψ is similar to a subform of ϕ by the following obvious observation.

Lemma 5.3. *Let ψ, γ_0 and γ_1 be anisotropic quadratic forms over F. The following conditions are equivalent:*
a) $\psi_{F((t))}$ is similar to a subform in $\gamma_0 \perp t\gamma_1$,
b) ψ is similar either to a subform in γ_0 or to a subform in γ_1.

Thus we have proved that condition (1) of Theorem 5.1 implies condition (2). On the other hand, condition (2) obviously implies condition (1). The proof of Theorem

5.1 is complete. □

Theorem 5.4. *Let ϕ be a virtual Albert form and let $\psi \notin GP_2(F)$. The quadratic form $\phi_{F(\psi)}$ is isotropic if and only if ψ is similar to a subform in ϕ.*

Proof. This theorem was proved by A. LAGHRIBI in the following cases ([Lag6], [Lag]):
(a) $\dim \psi \neq 4$;
(b) $\dim \psi = 4$, $d_\pm \psi \neq d_\pm \phi$.
Thus we may assume that $\dim \psi = 4$. To complete the proof it is sufficient to apply Theorem 5.1. □

In the special case which was not covered by the results of A. LAGHRIBI, we get the following

Corollary 5.5. *Let ϕ be a virtual Albert form and ψ be a 4-dimensional form such that $d_\pm \psi = d_\pm \phi$. Then $\phi_{F(\psi)}$ is anisotropic.*

Proof. If ψ is similar to a subform in ϕ, then ϕ is isotropic, a contradiction. Therefore ψ is not similar to a subform in ϕ. By Theorem 5.1, it means that $\phi_{F(\psi)}$ is anisotropic. □

Together with results described in §1, Theorem 5.4 gives rise to the following

Corollary 5.6. *Let ϕ be a 6-dimensional quadratic form with* $\operatorname{ind}(C_0(\phi)) = 4$. *In the case where $\psi \notin GP_2(F)$, the quadratic form $\phi_{F(\psi)}$ is isotropic if and only if ψ is similar to a subform of ϕ. In the case where $\psi \in GP_2(F)$, the form $\phi_{F(\psi)}$ is isotropic if and only if a 3-dimensional subform of ψ is similar to a subform of ϕ.*

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