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CANONICAL DIMENSION OF ORTHOGONAL GROUPS

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Abstract. We prove Berhuy–Reichstein's conjecture on the canonical dimension of orthogonal groups showing that for any integer $n \geqslant 1$, the canonical dimension of SO_{2n+1} and of SO_{2n+2} is equal to $n(n+1)/2$. More precisely, for a given $(2n+1)$-dimensional quadratic form ϕ defined over an arbitrary field F of characteristic $\neq 2$, we establish a certain property of the correspondences on the orthogonal grassmannian X of n-dimensional totally isotropic subspaces of ϕ, provided that the degree over F of any finite splitting field of ϕ is divisible by 2^n; this property allows us to prove that the function field of X has the minimal transcendence degree among all generic splitting fields of ϕ.

1. Results

Let F be an arbitrary field of characteristic $\neq 2$, ϕ a nondegenerate $(2n+1)$-dimensional quadratic form over F (with $n \geqslant 1$), X the orthogonal grassmannian of n-dimensional totally isotropic subspaces of ϕ. The variety X is projective, smooth, and geometrically connected; $\dim X = n(n+1)/2$. We write $d(X)$ for the greatest common divisor of the degrees of all closed points on X.

In this paper, a field extension E/F is called a *splitting field* of ϕ if the Witt index (see [10] for the definition of the Witt index of a quadratic form) of the form ϕ_E is maximal (i.e., equal to n). Note that a field extension E/F is a splitting field of ϕ if and only if the set $X(E)$ is nonempty. We write $d(\phi)$ for the greatest common divisor of the degrees of all finite splitting fields of ϕ.

Clearly, $d(\phi) = d(X)$. Moreover, this integer is a power of 2 not exceeding 2^n. The equality $d(\phi) = 2^n$ holds if, for example, the even Clifford algebra $C_0(\phi)$ of the quadratic form ϕ is a division algebra. Of course, this is so for the $(2n+1)$-dimensional generic quadratic form $\langle t_1, \ldots, t_{2n+1}\rangle$ (defined over the field $F(t_1, \ldots, t_{2n+1})$ of rational functions in variables $t_1, \ldots, t_{2n+1}$).

A splitting field L/F of ϕ is called *generic* if it is finitely generated and for any splitting field E/F and any nonzero element $a \in L$ there exists an F-place $f\colon L \to E$ such that $f(a)$ is neither 0 nor ∞. The function field $F(X)$ is a generic splitting field

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of ϕ. In fact, it is even *very generic* in the sense of [1] (where it is also explained how the "very generic" property implies the "generic" one): indeed, if E/F is a splitting field, the variety X_E is rational (as any projective homogeneous variety with a rational point is), and therefore $F(X)$ is contained in a purely transcendental extension of E (namely, in $E(X)$).

Following [1], we define the *canonical dimension* $\operatorname{cd}(\phi)$ of ϕ as the minimum of the transcendence degrees of all generic splitting fields of ϕ (the canonical dimension of SO_{2n+1} is then the maximum of $\operatorname{cd}(\phi)$ when ϕ runs over all $(2n+1)$-dimensional quadratic forms over (finitely generated) extensions of F; the canonical dimension of SO_{2n+2} coincides with the canonical dimension of SO_{2n+1}, see [1]).

Our main result here reads as follows.

Theorem 1.1. *If $d(\phi) = 2^n$, then $\operatorname{cd}(\phi) = n(n+1)/2$. In particular,*

$$\operatorname{cd}(\mathrm{SO}_{2n+1}) = \operatorname{cd}(\mathrm{SO}_{2n+2}) = n(n+1)/2\ .$$

The proof is given in Section 2. It immediately follows from Theorem 1.2 (also proved in Section 2), dealing with correspondences on X. A similar situation occurs in the proof of [1, Theorem 11.3] based on [4, Theorem 2.1] dealing with correspondences on a Severi–Brauer variety. An alternative proof of [4, Theorem 2.1], making use of a degree formula, is given in [8, §7.2]. However, for the similar statement [4, Theorem 6.4] concerning correspondences on quadrics (producing a result [5, Theorem 4.3], similar to Theorem 1.1, on the minimum of transcendence degree of generic *isotropy* fields of a quadratic form), there is no proof making use of a degree formula. In the present article as well, we use neither degree formulas nor Steenrod operations.

Theorem 1.2. *If $d(X) = 2^n$, then the multiplicity of any correspondence $\alpha\colon X \rightsquigarrow X$ is congruent modulo 2 to the multiplicity of the transpose of α. In particular, any rational map $X \to X$ is necessarily dominant.*

Here, by a correspondence $X \rightsquigarrow X$ we mean an algebraic cycle on $X \times X$ of dimension $\dim X$. The multiplicity $\operatorname{mult}(\alpha)$ of such a correspondence α is defined by the formula $(\mathrm{pr}_1)_*(\alpha) = \operatorname{mult}(\alpha)\cdot[X]$, where $\mathrm{pr}_1\colon X\times X \to X$ is the projection onto the first factor, while $(\mathrm{pr}_1)_*$ is the push-forward homomorphism of the group of algebraic cycles, see [3] (we do not use any equivalence relation on algebraic cycles yet). For the transpose α^t of α, we clearly have $\operatorname{mult}(\alpha^t)\cdot[X] = (\mathrm{pr}_2)_*(\alpha)$. The statement on rational maps is obtained by consideration of the correspondence given by the closure in $X \times X$ of the graph of a given rational map $X \to X$.

Remark 1.3. Assume that $d(X) = 2^n$. Although we have Theorem 1.2, we do not know whether the variety X is 2-*incompressible* in the sense of [8, §7]. Note that the only known proof of p-incompressibility of Severi–Brauer varieties of p-primary division algebras (p is an arbitrary prime), given in [8, §7.2], makes use of a degree formula, while the incompressibility of quadrics with first Witt index 1 [5, Corollary 3.4] cannot be proved by a degree formula.

In its turn, Theorem 1.2 follows (very similar to the way [4, Theorem 2.1] follows from [4, Corollary 2.3]) from the following computation of the reduced modulo 2 Chow group $\overline{\mathrm{Ch}}(X)$. Recall that $\overline{\mathrm{Ch}}(X)$ is defined as the image of the restriction homomorphism $\mathrm{Ch}(X) \to \mathrm{Ch}(\overline{X})$ of the usual modulo 2 Chow groups, where $\overline{X}$ is X over an algebraic closure $\overline{F}$ of F (a general reference for Chow groups is [3]).

Proposition 1.4. *If* $d(X) = 2^n$*, then* $\overline{\mathrm{Ch}}^{>0}(X) = 0$.

The next section starts with the proof of Proposition 1.4.

2. Proofs

In the proof of Proposition 1.4, we are going to use the description of the integral Chow ring $\mathrm{CH}(\overline{X})$ given in [9] (we borrowed this reference from Totaro's beautiful paper [11]). The graded ring $\mathrm{CH}^*(\overline{X})$ is isomorphic to the quotient of the polynomial ring $\mathbb{Z}[e_1, \dots, e_n]$ by the ideal generated by the polynomials

$$e_i^2 - 2e_{i-1}e_{i+1} + 2e_{i-2}e_{i+2} - \cdots + (-1)^{i-1}2e_1e_{2i-1} + (-1)^i e_{2i}$$

with $i = 1, \dots, n$ (e_i should be understood as 0 for $i > n$ in this formula), where the degree of e_i is i. The element of $\mathrm{CH}(\overline{X})$ corresponding to the class of e_i is a special Schubert class; we still write e_i for it. For any i, the element $2e_i$ is, up to a sign, the i-th Chern class of the tautological vector bundle on the grassmannian, therefore is rational, that is, lies in the integral reduced Chow group $\overline{\mathrm{CH}}(X) \subset \mathrm{CH}(\overline{X})$.

For any subset I of the set $\{1, 2, \dots, n\}$, let us define an element $e_I \in \mathrm{CH}(\overline{X})$ as the product $\prod_{i \in I} e_i$. Defining $|I|$ as $\sum_{i \in I} i$, we have $\operatorname{codim} e_I = |I|$. The element $e_{\{1,2,\dots,n\}}$ of the maximal codimension $1 + 2 + \cdots + n = \dim X$ is equal to the class of a rational point.

A basis of the modulo 2 Chow group $\mathrm{Ch}(\overline{X})$ is given by the classes of the elements e_I, where I runs over all subsets of the set $\{1, 2, \dots, n\}$ (in particular, the dimension of $\mathrm{Ch}(\overline{X})$ (as a vector space over $\mathbb{Z}/2\mathbb{Z}$) is equal to 2^n).

Proof of Proposition 1.4. Assume the contrary: there exists a homogeneous element $\alpha \in \overline{\mathrm{CH}}(X)$ of a positive codimension such that $\alpha \pmod 2$ is a nonzero element of $\overline{\mathrm{Ch}}(X)$. Decompose α in a sum of some e_I (without repetitions) plus 2β with $\beta \in \mathrm{CH}(\overline{X})$; let us fix a set I such that the element e_I occurs in the decomposition. Let J be the complement of I. Let m be the number of elements in J (note that $m < n$). Then the product $2^m e_J$ is rational. We claim that the degree of the rational 0-cycle $\alpha \cdot (2^m e_J)$ is an odd multiple of 2^m: indeed, the product $e_I \cdot (2^m e_J) = 2^m e_{\{1,2,\dots,n\}}$ has the degree 2^m, while the product $e_{I'} \cdot (2^m e_J)$, for any $I' \neq I$ with $|I'| = |I|$ as well as the product $(2\beta) \cdot (2^m e_J)$, are 0 modulo 2^{m+1}. We now have a contradiction with the assumption on $d(X)$. $\square$

In the proof of Theorem 1.2, which follows, we use a motivic decomposition of $X \times X$ (in the category of the integral Chow motives), produced in [2]. This motivic decomposition arises from the relative cellular structure on $X \times X$, where the cells are the orbits of the diagonal G-action for $G = \mathrm{SO}(\phi)$. Every summand of this decomposition is a Tate twist of the motive of X. More precisely, there is one copy of the motive of X (without twist, that is, with zero twist), while the remaining summands have some positive twists (although we do not need the precise description of the positive twists, here it is: for any i, the number of summands twisted i times is equal to the rank of the group $\mathrm{CH}_i(\overline{X})$).

To be absolutely precise, we have to say that the motivic decomposition of X given in [2] is not yet the decomposition described above: it also contains motives of certain flag varieties of the tautological vector bundle on X. However, the motive of each such flag variety decomposes in the sum of some twists of the motive of X by [7].

Proof of Theorem 1.2. First, since X is projective, the multiplicity homomorphism factors through the Chow group so that we have mult : $\mathrm{CH}_N(X \times X) \to \mathbb{Z}$, where $N = \dim X = n(n+1)/2$. Since the multiplicity of a cycle is not changed under extensions of the base field, the multiplicity homomorphism factors even through the reduced Chow group, so that we may replace $\mathrm{CH}(X \times X)$ by $\overline{\mathrm{CH}}(X \times X)$. Since we are interested in multiplicities modulo 2, we consider the induced homomorphism of the modulo 2 Chow group (still denoted by mult): mult: $\overline{\mathrm{Ch}}_N(X \times X) \to \mathbb{Z}/2\mathbb{Z}$.

The theorem we are trying to prove claims that the image of the homomorphism

$$f\colon \overline{\mathrm{Ch}}(X \times X) \to \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}\ , \quad f\colon \alpha \mapsto \big(\mathrm{mult}(\alpha), \mathrm{mult}(\alpha^t)\big)$$

is contained in the diagonal subgroup of $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Using the above description of motivic decomposition of $X \times X$, we get a decomposition of $\overline{\mathrm{Ch}}_N(X \times X)$ in the direct sum, where the summands are: one copy of $\overline{\mathrm{Ch}}_N(X)$ and several copies of $\overline{\mathrm{Ch}}_i(X)$ with various $i < N$. Since $\overline{\mathrm{Ch}}_i(X) = 0$ for any $i < N$ by Proposition 1.4, the image of the homomorphism f is cyclic. Since, on the other hand, $f([\Delta_X]) = (1, 1)$, the image of f is generated by $(1, 1)$, that is, coincides with the diagonal subgroup of $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. □

Proof of Theorem 1.1. We repeat the proof of [1, Theorem 11.3] using Theorem 1.2 instead of [8, §7.2] (and meaning by X our orthogonal grassmannian instead of a Severi–Brauer variety).

Since the field $F(X)$ is a generic splitting field of ϕ and has the transcendence degree $n(n+1)/2$, the inequality $\mathrm{cd}(\phi) \leqslant n(n+1)/2$ holds (the assumption on $d(\phi)$ is not needed for this bound).

If L is now another generic splitting field of ϕ, then we show that $\mathrm{tr.deg}(L/F) \geqslant n(n+1)/2$ as follows. Let Y be a projective model of L/F. Since both $F(X)$ and $F(Y)$ are generic splitting fields of ϕ, there exist rational morphisms $f\colon X \to Y$ and $g\colon Y \to X$. Moreover, for any nonempty open subset $U \subset Y$, there exists a rational morphism $X \to Y$ with an image meeting U, so that we may assume that f and g are composable. Since the rational map $X \to X$ given by the composition $g \circ f$ is dominant by Theorem 1.2, the dimension of Y is at least equal to $\dim X = n(n+1)/2$. □

Remark 2.1. Let us say that a splitting field L/F of ϕ is *quasigeneric* if it simply has a place to any splitting field of ϕ (see [6]). Then we can define a quasicanonical dimension $\mathrm{qcd}(\phi)$ as the minimal transcendence degree of a quasigeneric splitting field. Clearly, $\mathrm{qcd}(\phi) \leqslant \mathrm{cd}(\phi)$. But in the situation of Theorem 1.1, we can show that the equality holds. Namely (see [5, proof of Theorem 4.3]), if Y is a projective model of a quasigeneric splitting field of ϕ, the places $F(X) \to F(Y)$ and $F(Y) \to F(X)$ produce rational morphisms $Y \to X$ and $X \to Y$; replacing Y by the closure in $Y \times X$ of the graph of the rational morphism $Y \to X$, we get a model of the same field possessing a regular morphism to X, so that we can compose it with the rational morphism $X \to Y$ and conclude that $\dim Y \geqslant \dim X$ by Theorem 1.2. Similarly, $\mathrm{qcd}(A) = \mathrm{cd}(A)$ for any p-primary central division algebra A.

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