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Codimension 2 Cycles on Severi–Brauer Varieties

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Abstract. For a given sequence of integers $(n_i)_{i=1}^{\infty}$ we consider all the central simple algebras A (over all fields) satisfying the condition ind $A^{\otimes i} = n_i$ and find among them an algebra having the biggest torsion in the second Chow group CH^2 of the corresponding Severi–Brauer variety ('biggest' means that it can be mapped epimorphically onto each other). We give a description of this biggest torsion in the general case (via the gamma filtration) and find out when (i.e. for which sequences $(n_i)_{i=1}^{\infty}$) it is nontrivial. We also make an explicit computation in some special situations, e.g. in the situation of algebras of a square-free exponent e the biggest torsion turns out to be (cyclic) of order e. As an application we prove indecomposability for certain algebras of a prime exponent.

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0. Introduction

We consider finite-dimensional central simple algebras over fields. Let A be such an algebra and $X = \mathrm{SB}(A)$, the corresponding Severi–Brauer variety ([3, §1]). We are interested to describe the torsion in the second Chow group $\mathrm{CH}^2(X)$ of 2-codimensional cycles on X modulo rational equivalence (the question seems more natural if one takes in account that the groups $\mathrm{CH}^0(X)$ and $\mathrm{CH}^1(X)$ never have a torsion). Here are some preliminary observations. The group Tors $\mathrm{CH}^2(X)$ is finite and annihilated by ind A. Further, if A' is another algebra Brauer equivalent to A and $X' = \mathrm{SB}(A')$ then by [21, lemma (1.12)] or [15, cor. 1.3.2]

$$\mathrm{Tors}\ \mathrm{CH}^2(X) \simeq \mathrm{Tors}\ \mathrm{CH}^2(X')\ .$$

Finally, if $A = \otimes_p A_p$ is the decomposition of an algebra A into the tensor product of its primary components and $X_p = \mathrm{SB}(A_p)$ for each prime p, then

$$\mathrm{Tors}\ \mathrm{CH}^2(X) \simeq \bigoplus_p \mathrm{Tors}\ \mathrm{CH}^2(X_p)$$

or, in other words, the p-primary part of the group Tors $\mathrm{CH}^2(X)$ is isomorphic to Tors $\mathrm{CH}^2(X_p)$ (Proposition 1.3).

Summarizing, we see that the problem to compute Tors $\mathrm{CH}^2(X)$ for all algebras reduces itself to the case of primary division algebras.

Now consider the Grothendieck group $K(X) = K_0(X)$ together with the gamma filtration (Definition 2.6):

$$K(X) = \Gamma^0 K(X) \supset \Gamma^1 K(X) \supset \dots .$$

One has a canonical epimorphism (see the proof of Corollary 2.15)

$$\Gamma^{2/3} K(X) \twoheadrightarrow \mathrm{CH}^2(X)$$

of the quotient

$$\Gamma^{2/3} K(X) = \Gamma^2 K(X) / \Gamma^3 K(X) .$$

We consider the group $\Gamma^{2/3} K(X)$ as an upper bound for $\mathrm{CH}^2(X)$ and will show that in the primary case this upper bound is in certain sense the least one.

To formulate it precisely, let us call the sequence $(\mathrm{ind}\, A^{\otimes i})_{i=1}^{\infty}$ the *behaviour of* A. A sequence of integers $(n_i)_{i=1}^{\infty}$ will be called a *(p-primary) behaviour* if it is the behaviour of a (p-primary) algebra.

Suppose that A is a division algebra. The Grothendieck group $K(X)$ depends only on the behaviour of A (Theorem 3.1). Moreover, $K(X)$ together with the gamma filtration (and the group $\Gamma^{2/3} K(X)$ in particular) depend only on the behaviour in Corollary 3.2 and our main observation is Theorem 3.13:

For any primary behaviour (and any given field) there exists a division algebra $\tilde{A}$ *(over an extension of the field) of the given behaviour for which the canonical epimorphism* $\Gamma^{2/3} K(\tilde{X}) \twoheadrightarrow \mathrm{CH}^2(\tilde{X})$ *with* $\tilde{X} = \mathrm{SB}(\tilde{A})$ *is bijective.*

The construction of the algebra $\tilde{A}$ is rather simple (Definition 3.12). We take a division algebra (over a suitable extension of the field) of the index as in the given behaviour and of the exponent coinciding with the index. After that we pass to the function field of a product of certain generalized Severi–Brauer varieties in order to change the behaviour in the way prescribed.

Since the groups $\Gamma^{2/3} K(X)$ and $\mathrm{CH}^2(X)$ have the same rank (Proposition 2.14) (rank 1 if X is a Severi–Brauer variety of dimension at least 2), we also have an epimorphism of the torsion subgroups

$$\mathrm{Tors}\, \Gamma^{2/3} K(X) \twoheadrightarrow \mathrm{Tors}\, \mathrm{CH}^2(X)$$

which is moreover bijective iff $\Gamma^{2/3} K(X) \twoheadrightarrow \mathrm{CH}^2(X)$ is (Corollary 2.15). So, formulating the main observation we may replace (and we do replace) both the groups $\Gamma^{2/3} K(X)$ and $\mathrm{CH}^2(X)$ by their torsion subgroups.

The gamma filtration for a Severi–Brauer variety X and the group $\mathrm{Tors}\, \Gamma^{2/3} K(X)$ in particular are from the so to say 'algebro-geometrical' point of view very easy to compute (Propositions 4.1, 4.6 and 4.10): $K(X)$ is a subring of $K(\mathbb{P})$ where $\mathbb{P}$ is a $\dim X$-dimensional projective space and the Chern classes on X with values in K (Definition 2.1 and Remark 2.2) needed to determine the gamma filtration are simply the restrictions of the Chern classes on $K(\mathbb{P})$. However to get the answer in a final form (say, to find the canonical decomposition of the finite Abelian group $\operatorname{Tors} \Gamma^{2/3} K(X)$ for any primary behaviour) further calculations are required which can be done, e.g. by computer in every particular situation (i.e. for every particular behaviour) but seem to be not easy in the general case. Our main efforts in this direction are made in Propositions 4.7, 4.9, 4.13 and 4.14 where we firstly find out when this group is nontrivial (Propositions 4.7 and 4.9) and after that describe a wide class of situations when this group is cyclic and compute its order (Propositions 4.13, 4.14, see also Example 4.15).

To the structure of the article.

In Section 1 we reduce the problem of computation of $\operatorname{Tors} \operatorname{CH}^2(\operatorname{SB}(A))$ for an arbitrary central simple algebra A to the case when $\operatorname{ind} A$ is a power of a prime. In Section 2 we recall and partially prove certain general facts on the Chern classes (with various values) and on the gamma-filtration. In Section 3 we make the main observation. In Section 4 we investigate the group $\Gamma^{2/3} K$ for various primary behaviours.

In Section 5 we consider algebras of prime exponent. We show that the group $\operatorname{Tors} \operatorname{CH}^2(X)$, where $X = \operatorname{SB}(A)$ for an algebra A of a prime exponent p, is (cyclic) of order p or trivial (Proposition 5.1). Moreover, if A *decomposes* (into a tensor product of two smaller algebras), then $\operatorname{Tors} \operatorname{CH}^2(X) = 0$ (Proposition 5.3). However, the torsion group is non-trivial if A is a 'generic' division algebra of index p^n and exponent p (see Example 4.12 for the definition) with $n \geqslant 2$ for an odd p and $n \geqslant 3$ for $p = 2$ (Proposition 5.1). Thus we obtain a wide family of indecomposable algebras (Corollary 5.4) which can be constructed over an extension of any given field (without any restriction on the characteristic in particular). Here is a list of some articles where the question of indecomposability for central simple algebras was considered previously [2, 26, 29, 9, 13]. The method of [13] is close to but different from the one presented here; it does not cover the case $p = 2$.

Some additional notations concerning filtrations on $K(X)$ are introduced in Section 2.

The 57 items of suggestions and remarks made by the referee essentially improved the final version of the article.

1. Reduction to the Primary Case

In this Section, A is a central simple algebra over a field F, A_p (for every prime number p) stays for the p-primary component of A, finally

$$X = \operatorname{SB}(A) \quad \text{and} \quad X_p = \operatorname{SB}(A_p).$$

For an Abelian group C, we denote by C_p its p-primary part.

Let E/F be a finite field extension. Consider the homomorphisms

$$\operatorname{res}_{E/F}\colon \operatorname{CH}^2(X) \to \operatorname{CH}^2(X_E) \quad \text{and} \quad N_{E/F}\colon \operatorname{CH}^2(X_E) \to \operatorname{CH}^2(X).$$

The projection formula shows that the composition $N_{E/F} \circ \mathrm{res}_{E/F}$ coincides with the multiplication by $[E:F]$.

LEMMA 1.1. *The composition* $\mathrm{res}_{E/F} \circ N_{E/F}$ *coincides with the multiplication by* $[E{:}\,F]$ *as well.*

Proof. Consider the homomorphisms $\mathrm{res}_{E/F}$ and $N_{E/F}$ on the Grothendieck groups $K(X)$ and $K(X_E)$. Since these groups are torsion-free and have the same rank (Theorem 3.1), and since the composition $N_{E/F} \circ \mathrm{res}_{E/F}$ is the multiplication by $[E:F]$, the composition taken in the other order is the multiplication by $[E:F]$ as well. Since the second Chow group coincides with the second successive quotient of the topological filtration on the Grothendieck group (see, e.g., [10, (3.1)]), we are done.

COROLLARY 1.2. *If* $[E:F]$ *is not divisible by a given prime number* p*, then*

$$\mathrm{CH}^2(X)_p \simeq \mathrm{CH}^2(X_E)_p \ . \qquad \square$$

PROPOSITION 1.3. *For every prime* p*, the* p*-primary part of the group* $\mathrm{CH}^2(X)$ *coincides with the torsion of* $\mathrm{CH}^2(X_p)$*.*

Proof. Fix a prime p and a finite field extension E/F of degree prime to p such that the algebra A_E is Brauer equivalent to $(A_p)_E$. We have

$$\mathrm{CH}^2(X)_p \simeq \mathrm{CH}^2(X_E)_p \simeq \mathrm{CH}^2((X_p)_E)_p \simeq \mathrm{CH}^2(X_p)_p$$

(for the first and the third steps, we use the corollary). Since $\mathrm{Tors}\,\mathrm{CH}^2(X_p)$ is annihilated by $\mathrm{ind}\,A_p$, we finally get

$$\mathrm{CH}^2(X_p)_p = \mathrm{Tors}\,\mathrm{CH}^2(X_p) \ . \qquad \square$$

2. Chern Classes and Gamma Filtration

In this Section, we are working with the category of smooth projective irreducible algebraic varieties over a fixed field. The Grothendieck ring K is considered as a contravariant functor on this category.

DEFINITION 2.1 (Chern classes with values in K). The total Chern class c_t is a homomorphism of functors

$$c_t : K^+ \longrightarrow K[[t]]^\times$$

(where the left-hand side is the additive group of the ring K while the right-hand side is the multiplicative group of series in one variable t over K) satisfying the following property: if $\xi \in K(X)$ is a class of an invertible sheaf on a variety X then $c_t(\xi) = 1 + (\xi - 1)t$.

One defines the Chern classes $c^i\colon K \to K$ by putting

$$c_t = \sum_{i=0}^{\infty} c^i \cdot t^i \ .$$

Remark 2.2. Usually, one does not use the name 'Chern classes' for the maps c^i defined above, e.g., since unlike the Chern classes in Definitions 2.7, 2.8 and 2.11 they do not satisfy the rule $c^1(\xi \cdot \eta) = c^1(\xi) + c^1(\eta)$ for classes of invertible sheaves ξ and η.

PROPOSITION 2.3. *Chern classes with values in K are unique.*

Proof. Follows from the

LEMMA 2.4 (Splitting principle, [20, prop. 5.6]). *For any variety X and any $x \in K(X)$ there exists a morphism $f: Y \to X$ such that:*

(1) f is a composition of some projective bundle morphisms;
(2) $f^*(x) \in K(Y)$ is a linear combination (with integral coefficients) of classes of some invertible sheaves.

To obtain uniqueness of the Chern classes just note that the homomorphism $f^*: K(X) \to K(Y)$ in the lemma is injective.

PROPOSITION 2.5. *Chern classes with values in K exist.*

Proof. Here is the way of constructing due to Grothendieck with the original notations ([20, Theorem 3.10 and Section 8]).

Take a variety X. First one constructs a homomorphism $\lambda_t: K^+ \longrightarrow K[[t]]^\times$ by sending the class of a locally free sheaf $\mathcal{E}$ to $\lambda_t([\mathcal{E}]) = \sum_{i=0}^{\infty} [\Lambda^i \mathcal{E}] \cdot t^i$ where $\Lambda^i \mathcal{E}$ is the ith exterior power of $\mathcal{E}$.

After that one considers another homomorphism $\gamma_t: K^+ \longrightarrow K[[t]]^\times$ namely, $\gamma_t = \lambda_{\frac{t}{1-t}}$ (this γ_t gave the name of the gamma filtration).

Finally, one puts $c_t = \gamma_t \circ (\mathrm{id} - \mathrm{rk})$ where $\mathrm{rk}: K(X) \to \mathbb{Z}$ is the rank homomorphism (followed by the inclusion $\mathbb{Z} \hookrightarrow K(X)$ more precisely). □

DEFINITION 2.6 (Gamma filtration). The gamma-filtration

$$K(X) \supset \Gamma^0 K(X) \supset \Gamma^1 K(X) \supset \dots$$

is the smallest ring filtration on $K(X)$ such that $\Gamma^0 K(X) = K(X)$ and

$$c^i\big(K(X)\big) \subset \Gamma^i K(X) \quad \text{for all} \quad i \geqslant 1 .$$

In other words, for every $l \geqslant 0$, $\Gamma^l K(X)$ is the subgroup of $K(X)$ generated by all the products

$$c^{i_1}(x_1) \dots c^{i_r}(x_r) \quad \text{with} \quad x_j \in K(X) \quad \text{and} \quad \sum_{i=1}^{r} i_j \geqslant l$$

(it might be not immediately clear but it is nevertheless easy to see that the group $K(X) = \Gamma^0 K(X)$ is really also generated by these products).

In particular, $\Gamma^1 K(X) = \mathrm{Ker}(\mathrm{rk}\colon K(X) \to \mathbb{Z})$.

We denote by $\mathrm{G}^*\Gamma K(X)$ the adjoint graded ring.

DEFINITION 2.7 (Chern classes with values in $G^*\Gamma K$). For any variety X, we call the induced maps $c^i: K(X) \to G^i\Gamma K(X)$ the Chern classes with values in $G^*\Gamma K$. The total Chern class c_t is the homomorphism

$$c_t: K(X)^+ \longrightarrow \left(\sum_{i=0}^{\infty} G^i\Gamma K(X) \cdot t^i\right)^\times$$

It is a morphism of functors and $c_t(\xi) = 1 + (\xi - 1)t$ for a class $\xi \in K(X)$ of an invertible sheaf on X (($\xi - 1$) is considered as an element of $G^1\Gamma K(X)$ in the last formula).

Side by side with the gamma filtration we consider the topological filtration on $K(X)$ (in fact defined on $K_0'(X)$) ([25, Section 7]):

$$K(X) = T^0K(X) \supset T^1K(X) \supset \dots .$$

Note that

$$T^1K(X) = \mathrm{Ker}(\mathrm{rk}: K(X) \to \mathbb{Z}) = \Gamma^1K(X) .$$

We will denote by $G^*TK(X)$ the adjoint graded ring.

DEFINITION 2.8 (Chern classes with values in G^*TK). The total Chern class c_t is a homomorphism of functors

$$c_t: K^+ \longrightarrow \left(\sum_{i=0}^{\infty} G^iTK \cdot t^i\right)^\times$$

satisfying the property $c_t(\xi) = 1 + (\xi - 1)t$.

One defines the Chern classes $c^i: K \to G^iTK$ by putting $c_t = \sum_{i=0}^{\infty} c^i \cdot t^i$.

PROPOSITION 2.9. *Chern classes with values in* G^*TK *are unique.*

Proof. Follows from the splitting principle (Lemma 2.4) since the homomorphism

$$f^*: G^*TK(X) \to G^*TK(Y)$$

is injective ([7, lemma 3.8 of chapter V]). □

PROPOSITION 2.10. *Chern classes with values in* G^*TK *exist.*

Proof. Simply compose the Chern classes with values in CH^* (Definition 2.11) with the canonical epimorphism $CH^* \twoheadrightarrow G^*TK$ mapping a class $[Z] \in CH^*(X)$ of a simple cycle $Z \subset X$ to the class of the structure sheaf $\mathcal{O}_Z$ of Z prolonged to X by 0. □

DEFINITION 2.11 (Chern classes with values in CH^*). We repeat Definition 2.8, replacing G^*TK by CH^*. In the formula $c_t(\xi) = 1 + (\xi - 1)t$, we consider $(\xi - 1)$ as an element of $CH^1(X)$ via the canonical isomorphism $CH^1(X) \simeq G^1TK(X)$ ([25, Section 7.5]) described in the proof of Proposition 2.10.

PROPOSITION 2.12. *Chern classes with values in* CH^* *are unique.*

Proof. Follows from the splitting principle (Lemma 2.4) since the homomorphism $f^*\colon \mathrm{CH}^*(X) \to \mathrm{CH}^*(Y)$ is injective.

PROPOSITION 2.13 ([6, Section 3.2]). *Chern classes with values in* CH^* *exist.*

Now we establish certain connections between the gamma filtration and the topological one.

PROPOSITION 2.14. *For any variety* X,

(1) $\Gamma^i K(X) \subset \mathrm{T}^i K(X)$ *for all* i*;*
(2) $\Gamma^i K(X) = \mathrm{T}^i K(X)$ *for* $i \leqslant 2$*;*
(3) $\Gamma^i K(X) \otimes \mathbb{Q} = \mathrm{T}^i K(X) \otimes \mathbb{Q}$ *for all* i.

Proof. (1) [7, Theorem 3.9 of chapter V]. (2) We only need to manage the case $i = 2$.

There are canonical isomorphisms

$$\mathrm{G}^1\Gamma K(X) \simeq \mathrm{Pic}(X) \text{ ([7, remark 1 in Section 3, chapter IV]);}$$

$$\mathrm{CH}^1(X) \simeq \mathrm{G}^1\mathrm{T}K(X) \text{ ([25, Section 7.5])}$$

(the definition of the second map is given in the proof of Proposition 2.10).

Since $\Gamma^2 K(X) \subset \mathrm{T}^2 K(X)$ we have a surjection $\mathrm{G}^1\Gamma K(X) \twoheadrightarrow \mathrm{G}^1\mathrm{T}K(X)$ which gives an epimorphism $\mathrm{Pic}(X) \to \mathrm{CH}^1(X)$. But the latter map is an isomorphism ([8, cor. 6.16]). Thus $\Gamma^2 K(X) = \mathrm{T}^2 K(X)$.

(3) [7, proposition 5.5 of chapter VI]. □

COROLLARY 2.15. *One has an exact sequence*

$$0 \to \mathrm{T}^3 K(X)/\Gamma^3 K(X) \to \mathrm{Tors}\, \mathrm{G}^2\Gamma K(X) \to \mathrm{Tors}\, \mathrm{CH}^2(X) \to 0\,.$$

Proof. The equality $\Gamma^2 K(X) = \mathrm{T}^2 K(X)$ and the inclusion $\Gamma^3 K(X) \subset \mathrm{T}^3 K(X)$ stated in the proposition give an exact sequence

$$0 \to \mathrm{T}^3 K(X)/\Gamma^3 K(X) \to \mathrm{G}^2\Gamma K(X) \to \mathrm{G}^2\mathrm{T}K(X) \to 0\,.$$

Consider the commutative diagram with exact columns

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathrm{Tors}\, \mathrm{G}^2\Gamma K(X) & \xrightarrow{(1)} & \mathrm{Tors}\, \mathrm{G}^2\mathrm{T}K(X) \\
\downarrow & & \downarrow \\
\mathrm{G}^2\Gamma K(X) & \xrightarrow{(2)} & \mathrm{G}^2\mathrm{T}K(X) \\
\downarrow & & \downarrow \\
\mathrm{G}^2\Gamma K(X)/\mathrm{Tors} & \xrightarrow{(3)} & \mathrm{G}^2\mathrm{T}K(X)/\mathrm{Tors} \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

The map (2) is surjective. Hence, the map (3) is surjective as well. Since by the proposition the map (3)$\otimes\mathbb{Q}$ is bijective, the map (3) itself is bijective as well. Thus (1) is surjective and the kernels of (1) and (2) coincide. So, we get the exact sequence

$$0 \to \mathrm{T}^3K(X)/\Gamma^3K(X) \to \text{Tors } \mathrm{G}^2\Gamma K(X) \to \text{Tors } \mathrm{G}^2\mathrm{T}K(X) \to 0\ .$$

Finally, the canonical map $\mathrm{CH}^2(X) \to \mathrm{G}^2\mathrm{T}K(X)$ is an isomorphism (see, e.g., [10, (3.1)], the definition of the map is given in the proof of Proposition 2.10). □

As a corollary of the uniqueness assertion 2.9, we get a connection between Chern classes with different values:

LEMMA 2.16. *The following diagram of maps commutes:*

$$\begin{array}{ccc} K(X) & \xrightarrow{c^i} & \mathrm{CH}^i(X) \\ \downarrow{\scriptstyle c^i} & & \downarrow{\scriptstyle can.} \\ \mathrm{G}^i\Gamma K(X) & \longrightarrow & \mathrm{G}^i\mathrm{T}K(X) \end{array}$$

Proof. Both the compositions are Chern classes with values in $\mathrm{G}^*\mathrm{T}K$ (Definition 2.8) which are unique (Proposition 2.9).

Remark 2.17. One can formulate a criterion (which will be used later) for when the gamma filtration coincides with the topological one. It is clear from the very Definition 2.6 that for any variety X the ring $\mathrm{G}^*\Gamma K(X)$ is generated by the Chern classes (with values in $\mathrm{G}^*\Gamma K$). So, if the gamma and the topological filtrations are the same, the ring $\mathrm{G}^*\mathrm{T}K$ is generated by the Chern classes (with values in $\mathrm{G}^*\mathrm{T}K$ this time) as well.

The other way round, if $\mathrm{G}^*\mathrm{T}K$ is generated by the Chern classes, then the homomorphism $\mathrm{G}^*\Gamma K(X) \to \mathrm{G}^*\mathrm{T}K(X)$ is surjective whence the filtrations coincide.

3. Main Observation

From now on, X denotes the Severi–Brauer variety corresponding to a central simple algebra A over a field F.

Denote by $\mathbb{P}$ the projective space $X_{\bar{F}}$ where $\bar{F}$ is an algebraic closure of F and let $\xi \in K(\mathbb{P})$ be the class of $\mathcal{O}_{\mathbb{P}}(-1)$. The ring $K(\mathbb{P})$ is generated by ξ subject to only one relation $(\xi - 1)^n = 0$, where $n = \dim X + 1 = \deg A$. We consider the restriction map $K(X) \to K(\mathbb{P})$ which is a ring homomorphism commuting with the Chern classes of Definition 2.1.

THEOREM 3.1 ([25, Section 8, Theorem 4.1]). *The map $K(X) \to K(\mathbb{P})$ is injective; its image is additively generated by* $(\text{ind } A^{\otimes i}) \cdot \xi^i$ *($i \geqslant 0$).* □

COROLLARY 3.2. *For a division algebra A, the group $K(X)$ together with the gamma filtration depends only on the behaviour of A.* □

PROPOSITION 3.3 ([11, Theorem 1]). *If* $\operatorname{ind} A = \exp A$ *for an algebra* A *then (for any* $l \geqslant 0$*) the* l*th term of the topological filtration* $\mathrm{T}^l K(X)$ *is generated by all*

$$\frac{\operatorname{ind} A}{(i, \operatorname{ind} A)}(\xi - 1)^i \quad \textit{with} \quad l \leqslant i < \deg A$$

where $(\cdot, \cdot)$ *denotes the greatest common divisor. In particular, the group* $\mathrm{G}^*\mathrm{T}K(X)$ *is torsion-free.*

PROPOSITION 3.4. *If* A *is a primary algebra then*

$$\frac{\operatorname{ind} A}{(i, \operatorname{ind} A)}(\xi - 1)^i \in \Gamma^i K(X) \quad \textit{for any} \quad i \geqslant 0 \,.$$

Proof. Put $n = \operatorname{ind} A$. For $n\xi \in K(X)$ we have

$$c_t(n\xi) = c_t(\xi)^n = \big(1 + (\xi - 1)t\big)^n$$

where c_t is the total Chern class with values in K (the last equality holds by Definition 2.1). Whence

$$c^i(n\xi) = \binom{n}{i}(\xi - 1)^i \in \Gamma^i K(X) \,.$$

In particular, $(\xi - 1)^n \in \Gamma^n K(X)$, thereby for the rest of the proof we may assume that $i \leqslant n$. Moreover,

$$n^i(\xi - 1)^i = c^1(n\xi)^i \in \Gamma^i K(X) \,.$$

The last observation is

LEMMA 3.5. *If* n *is a power of a prime* p *and* $n \geqslant i \geqslant 0$ *then*

$$\left(n^i, \binom{n}{i}\right) = \frac{n}{(i, n)} \,.$$

Moreover, if $i \neq 0$ *then* $v_p\binom{n}{i} = v_p(n) - v_p(i)$, *where* $v_p(i)$ *is the multiplicity of* p *in* i.

Proof. The case $i = 0$ is evident. Suppose that $i \neq 0$. If $1 \leqslant j < n$ then $v_p(j) < v_p(n)$ and so $v_p(n - j) = v_p(j)$. Hence

$$v_p\left(\frac{(n-1)(n-2)\cdots(n-(i-1))}{1 \cdot 2 \cdots (i-1)}\right) = 0$$

and

$$v_p\binom{n}{i} = v_p\left(\frac{n}{i}\right) = v_p\left(\frac{n}{(i, n)}\right) \,.$$

□

COROLLARY 3.6. *If A is a primary algebra and* ind $A = \exp A$ *then the gamma filtration on $K(X)$ coincides with the topological one.* □

THEOREM 3.7. *Let A be as in Corollary 3.6, $X = \mathrm{SB}(A)$. Let $Y_1, \ldots, Y_m$ be some generalized Severi–Brauer varieties ([4, Section 4]) of some algebras which are (Brauer equivalent to) some tensor powers of A.*

The gamma filtration on the Grothendieck group of the variety X over the function field $F(Y_1 \times \cdots \times Y_m)$ coincides with the topological one. In particular, the epimorphism (Corollary 2.15) $\mathrm{Tors}\, G^2\Gamma K \twoheadrightarrow \mathrm{Tors}\, \mathrm{CH}^2$ *for this variety is bijective.*

Proof. For every Y_i the product $X \times Y_i$ is a Grassmann bundle over X (with respect to the first projection) (Corollary 6.4). Hence $\mathrm{CH}^*(X \times Y_i)$ is generated as a $\mathrm{CH}^*(X)$-algebra by the Chern classes of a locally free sheaf (see, e.g., [6, Proposition 14.6.5] or [19, (3.2)]). Taking the product of all $X \times Y_i$ over X we obtain that

$$\mathrm{CH}^*(X \times Y_1 \times \cdots \times Y_m)$$

is generated as a $\mathrm{CH}^*(X)$-algebra by the Chern classes (of some locally free sheaves).

The homomorphism of $\mathrm{CH}^*(X)$-algebras

$$\mathrm{CH}^*(X \times Y_1 \times \cdots \times Y_m) \to \mathrm{CH}^*(X_{F(Y_1 \times \cdots \times Y_m)})$$

(given by the pull-back) is surjective (see, e.g., [17, Theorem 3.1]). Whence the right-hand side is generated as a $\mathrm{CH}^*(X)$-algebra by the Chern classes as well.

Using the epimorphism $\mathrm{CH}^* \twoheadrightarrow G^*TK$ of the Chow ring onto the adjoint graded Grothendieck ring we obtain the same statement as in the previous paragraph but for G^*TK instead of CH^* by meaning the Chern classes with values in G^*TK this time.

The gamma filtration on $K(X)$ coincides with the topological one (Corollary 3.6) and therefore the ring $G^*TK(X)$ is generated by the Chern classes (Remark 2.17). Consequently, $G^*TK(X_{F(Y_1 \times \cdots \times Y_m)})$ is generated by the Chern classes as a ring, not only as a $G^*TK(X)$-algebra. It means that the gamma filtration on $K(X_{F(Y_1 \times \cdots \times Y_m)})$ coincides with the topological one (Remark 2.17). □

DEFINITION 3.8. Let A be a p-primary algebra. The sequence of integers

$$(\log_p \ \mathrm{ind} \ A^{\otimes p^i})_{i=0}^{\log_p \exp A}$$

is called the reduced behaviour of A.

EXAMPLE 3.9. The reduced behaviour of a p-primary algebra A with $\mathrm{ind}\, A = \exp A = p^n$ is $n, \ n-1, \ n-2, \ \ldots, \ 1, \ 0$.

Proof. Suppose that $n > 0$. By [1, lemma 7 on page 76], $\mathrm{ind}\, A^{\otimes p} < \mathrm{ind}\, A$. Moreover, $\mathrm{ind}\, A^{\otimes p} \geqslant \exp A^{\otimes p} = p^{n-1}$. Thus $\mathrm{ind}\, A^{\otimes p} = p^{n-1}$.

LEMMA 3.10. *The behaviour of a primary algebra is completely determined by its reduced behaviour. The reduced behaviour of an algebra is a finite strong decreasing* *sequence of integers with 0 in the end. Any finite strong decreasing sequence of integers with 0 in the end is for any prime p the reduced behaviour of a p-primary division algebra.*

Proof. Let A be a p-primary algebra. If i is an integer prime to p then the splitting fields of the algebra A are the same as the splitting fields of the algebra $A^{\otimes i}$. Therefore $\operatorname{ind} A^{\otimes i} = \operatorname{ind} A$, what proves the first sentence of the lemma.

If in addition $\operatorname{ind} A \neq 1$ then $\operatorname{ind} A^{\otimes p} < \operatorname{ind} A$ ([1, lemma 7 on page 76]). It proves the second sentence.

Finally, fix a sequence $n_0 > n_1 > \cdots > n_m = 0$ and a prime p. A construction of a p-primary algebra having the reduced behaviour $(n_i)_{i=0}^m$ is given in [27, construction 2.8]. This construction involves function fields of usual Severi–Brauer varieties only. We describe another known construction which involves function fields of generalized Severi–Brauer varieties as well and is more suitable for our purposes.

We start with a division algebra A (over a suitable field) for which $\operatorname{ind} A = \exp A = p^{n_0}$.

For each $i = 1, 2, \ldots, m$ consider the generalized Severi–Brauer variety $Y_i = \mathrm{SB}(p^{n_i}, A^{\otimes p^i})$ (we define here $\mathrm{SB}(p^{n_i}, A^{\otimes p^i})$ to be the variety of rank p^{n_i} *left* ideals in $A^{\otimes p^i}$; its function field is a generic extension making the index of $A^{\otimes p^i}$ to be equal to p^{n_i}).

Finally, we denote the function field $F(Y_1 \times \cdots \times Y_m)$ by $\tilde{F}$ and put $\tilde{A} = A_{\tilde{F}}$. Using the index reduction formula [4, theorem 5] or an improved version of this formula [22, formula I] one can easily show that the algebra $\tilde{A}$ has the reduced behaviour $(n_i)_{i=0}^m$. □

Remark 3.11. In the construction described in the proof above, it is not necessary to use all of the varieties Y_i: if $n_i = n_{i-1} - 1$ for some i then the variety Y_i can be omitted.

DEFINITION 3.12. We refer to an algebra $\tilde{A}$ constructed like as in the above proof (with taking the remark into account) as to a *'generic' p-primary division algebra of the reduced behaviour* $(n_i)_{i=0}^m$. Note that it can be constructed over an extension of any given field.

THEOREM 3.13. *Fix a prime p and a reduced behaviour. If $\tilde{A}$ is a 'generic' p-primary division algebra of the given reduced behaviour (Definition 3.12) then the epimorphism (Corollary 2.15)*

$$\operatorname{Tors} \mathrm{G}^2\Gamma K(\tilde{X}) \twoheadrightarrow \operatorname{Tors} \mathrm{CH}^2(\tilde{X}) \quad \textit{(where } \tilde{X} = \mathrm{SB}(\tilde{A}))$$

is bijective. If A is an arbitrary p-primary algebra of the same reduced behaviour as $\tilde{A}$ then there exists an epimorphism

$$\operatorname{Tors} \mathrm{CH}^2(\tilde{X}) \twoheadrightarrow \operatorname{Tors} \mathrm{CH}^2(X)\ .$$

Proof. The first part follows from Theorem 3.7 and from the definition of 'generic' algebras (Definition 3.12). The second part follows from Corollaries 2.15 and 3.2 and from the first one. □

4. Computation of Gamma Filtration

Recall that we have put $X = \mathrm{SB}(A)$.

PROPOSITION 4.1. *Let A be a p-primary algebra and $(n_i)_{i=0}^m$ its reduced behaviour. For any $l \geqslant 0$, the group $\Gamma^l K(X)$ is generated by all the products*

$$\prod_{i=0}^{m} \frac{p^{n_i}}{(j_i,\, p^{n_i})} (\xi^{p^i} - 1)^{j_i} \quad \textit{with} \quad j_i \geqslant 0 \quad \textit{and} \quad \sum_{i=0}^{m} j_i \geqslant l \tag{$*$}$$

where $\xi = [\mathcal{O}(-1)] \in K(\mathbb{P})$.

Proof. The formula

$$c^j(p^{n_i}\xi^{p^i}) = \left(p^{n_i} atop j\right)\ (\xi^{p^i} - 1)^j \quad \text{where} \quad 0 \leqslant j \leqslant p^{n_i}$$

and Lemma 3.5 show that for any $j \geqslant 0$ (even for $j > p^{n_i}$) the element

$$\frac{p^{n_i}}{(j,\ p^{n_i})} (\xi^{p^i} - 1)^j$$

lies in $\Gamma^j K(X)$ (compare with the proof of Proposition 3.4). Therefore, each product $(*)$ lies in $\Gamma^l K(X)$.

For the opposite inclusion we need

LEMMA 4.2. *Consider the polynomials over $\mathbb{Z}$ in one variable ζ. For any integers $j,\ r \geqslant 0$ the polynomial $(\zeta^r - 1)^j$ is equal to a sum*

$$\sum_{s \geqslant j} a_s (\zeta - 1)^s$$

with integers a_s such that $s \cdot a_s$ is a multiple of $j \cdot r$.

Proof. It is clear that

$$(\zeta^r - 1)^j = \sum_{s=j}^{j\cdot r} a_s (\zeta - 1)^s$$

for some (uniquely determined) $a_s \in \mathbb{Z}$. Taking the derivative we obtain the statement on the coefficients. □

As follows from Theorem 3.1, the additive group $K(X)$ is generated by all $p^{n_i}\xi^{rp^i}$ with $0 \leqslant i \leqslant m$ and $r \geqslant 0$. We have,

$$c^j(p^{n_i}\xi^{rp^i}) = \binom{p^{n_i}}{j}\ (\xi^{rp^i} - 1)^j = \sum_{s \geqslant j} \binom{p^{n_i}}{j} \cdot a_s \cdot (\xi^{p^i} - 1)^s\ .$$

Since by the lemma $j \mid s \cdot a_s$ and $v_p \binom{p^{n_i}}{j}$ equals $n_i - v_p(j)$ or ∞ (3.5), the coefficient

$$\binom{p^{n_i}}{j} \cdot a_s \quad \text{is divisible by} \quad \frac{p^{n_i}}{(s,\, p^{n_i})} \,.$$

Thus the Chern class $c^j(p^{n_i}\xi^{rp^i})$ is a linear combination (with integral coefficients) of

$$\frac{p^{n_i}}{(s,\, p^{n_i})}(\xi^{p^i} - 1)^s \quad \text{with} \quad s \geqslant j \,.$$

DEFINITION 4.3. Fix a prime number p, a reduced behaviour $(n_i)_{i=0}^m$, and consider a polynomial ring $\mathbb{Z}[\zeta]$. Let $\mathbb{K} \subset \mathbb{Z}[\zeta]$ be the additive subgroup generated by all $p^{n_i} \cdot \zeta^{r \cdot p^i}$ where $0 \leqslant i \leqslant m$ and $r \geqslant 0$. Consider a filtration Γ on $\mathbb{K}$ defined by the formula of Proposition 4.1: for any $l \geqslant 0$, the group $\Gamma^l\mathbb{K}$ is generated by all the products

$$\prod_{i=0}^{m} \frac{p^{n_i}}{(j_i,\, p^{n_i})}(\zeta^{p^i} - 1)^{j_i} \quad \text{with } j_i \geqslant 0 \text{ and } \sum_{i=0}^{m} j_i \geqslant l \,. \qquad (*)$$

Note that $\mathbb{K}$ is a ring and that for any $l_1, l_2 \geqslant 0$ one has $\Gamma^{l_1}\mathbb{K} \cdot \Gamma^{l_2}\mathbb{K} \subset \Gamma^{l_1+l_2}\mathbb{K}$.

LEMMA 4.4. *In the notation of the definition, one has:*

(1) $\mathbb{K} = \Gamma^0\mathbb{K}$;
(2) *for any* $l \geqslant 0$, $\mathbb{K} \supset \Gamma^l\mathbb{K} \supset \Gamma^{l+1}\mathbb{K}$;
(3) *if* n *is a multiple of* p^{n_0}, *then* $\Gamma^n\mathbb{K} = (\zeta - 1)^n\mathbb{K}$.

Proof. (1) The inclusion $\mathbb{K} \subset \Gamma^0\mathbb{K}$ is evident. The inverse inclusion is a particular case of the second statement of the lemma.

(2) The inclusion $\Gamma^l\mathbb{K} \supset \Gamma^{l+1}\mathbb{K}$ is evident. Fix an arbitrary $l \geqslant 0$. We prove the inclusion $\mathbb{K} \supset \Gamma^l\mathbb{K}$ using the third statement of the lemma. Choose a multiple n of p^{n_0} such that $n > l$. Since $\Gamma^l\mathbb{K} \supset \Gamma^n\mathbb{K} = (\zeta - 1)^n\mathbb{K} \subset \mathbb{K}$, it suffices to show that $\mathbb{K}/(\zeta - 1)^n\mathbb{K} \supset \Gamma^l\mathbb{K}/(\zeta - 1)^n\mathbb{K}$. Find a p-primary algebra A (over an appropriate field F) having the reduced behaviour $(n_i)_{i=0}^m$ (see Lemma 3.10 for the existence of A). The latter inclusion follows now from Proposition 4.1.

(3) Since $(\zeta - 1)^n \in \Gamma^n\mathbb{K}$, the inclusion $\supset$ holds. One also sees immediately from the definition that any polynomial $f \in \Gamma^n\mathbb{K}$ is of the kind $f = (\zeta - 1)^n \cdot h$, where $h \in \mathbb{Z}[\zeta]$. We have to prove that $h \in \mathbb{K}$. It is a consequence of the following lemma:

LEMMA 4.5. *Let* f, g *and* h *are polynomials from* $\mathbb{Z}[\zeta]$ *such that* $f = g \cdot h$ *and suppose that the free coefficient of* g *equals* ± 1. *If* f *and* g *lie in* $\mathbb{K}$ *then* h *lies in* $\mathbb{K}$ *as well.*

Proof. Let

$$f = \sum_{i \geqslant 0} f_i \zeta^i \;, \qquad g = \sum_{i \geqslant 0} g_i \zeta^i \;, \qquad h = \sum_{i \geqslant 0} h_i \zeta^i \;.$$

We prove that $h_i \zeta^i \in \mathbb{K}$ using an induction on i. There is no problem with the base of the induction because $h_0 \in \mathbb{K}$ for any integral h_0. Suppose that $h_1\zeta, \dots, h_{i-1}\zeta^{i-1} \in \mathbb{K}$. Polynomial f is equal to the product of g and h; because of that we have:

$$f_i \zeta^i = \pm h_i \zeta^i + g_1 \zeta \cdot h_{i-1}\zeta^{i-1} + \cdots + g_{i-1}\zeta^{i-1} \cdot h_1 \zeta + g_i \zeta^i \cdot h_0 \;.$$

Since $g \in \mathbb{K}$, every its monomial $g_j \zeta^j$ is in $\mathbb{K}$ as well (see the definition of $\mathbb{K}$). By the same reason, $f_i \zeta^i \in \mathbb{K}$. Hence, $h_i \zeta^i \in \mathbb{K}$. □

If now A is a p-primary algebra of the reduced behaviour $(n_i)_{i=0}^m$, the ring homomorphism $\mathbb{K} \to K(X)$ mapping ζ to ξ respects the filtrations and thereby induces a homomorphism of graded groups $\mathrm{G}^*\Gamma\mathbb{K} \to \mathrm{G}^*\Gamma K(X)$.

PROPOSITION 4.6. *For every* $0 \leqslant l < \deg A$ *the group homomorphism* $\mathrm{G}^l\Gamma\mathbb{K} \to \mathrm{G}^l\Gamma K(X)$ *is bijective.*

Proof. It is evidently surjective by Proposition 4.1. To see the rest, put $n = \deg A$. By Lemma 4.4 and Theorem 3.1, the ring homomorphism ϕ: $\Gamma^{0/n}\mathbb{K} \to K(X)$ is bijective. Consider the induced filtration on $\Gamma^{0/n}\mathbb{K}$. We know that the bijective ring homomorphism ϕ respects the filtrations and is surjective on the successive quotients. Thus it is bijective on the successive quotients. □

The proposition gives in particular a description of the group $\mathrm{G}^2\Gamma K(X)$ for any p-primary algebra A of the reduced behaviour $(n_i)_{i=0}^m$. We want to find out when this group has a nontrivial torsion. We start with the case of an odd prime.

PROPOSITION 4.7. *Let* A *be a* p*-primary algebra with an odd* p*. The group* $\mathrm{G}^2\Gamma K(X)$ *has a torsion iff* ind $A >$ exp A.

Proof. See Proposition 3.3 with Corollary 3.6 for the ‘only if’ part.

Suppose that ind $A >$ exp A. Then in the reduced behaviour $(n_i)_{i=0}^m$ of A one has $n_s \leqslant n_{s-1} - 2$ for some s.

Using Proposition 4.6, we shall work with $\mathrm{G}^2\Gamma\mathbb{K}$ instead of $\mathrm{G}^2\Gamma K(X)$.

Consider the element

$$x = p^{n_{s-1}-2}(\zeta^{p^s} - 1)^2 - p^{n_{s-1}}(\zeta^{p^{s-1}} - 1)^2 \in \Gamma^2\mathbb{K}.$$

Since x is divisible by $(\zeta - 1)^3$ in the polynomial ring $\mathbb{Z}[\zeta]$, since, moreover, $p^{n_0}(\zeta - 1)^3 \in \Gamma^3\mathbb{K}$ and $p^{n_0} f(\zeta) \in \mathbb{K}$ for any polynomial $f(\zeta)$ by Definition 4.3, one sees that a multiple of x lies in $\Gamma^3\mathbb{K}$. So, for our purposes it suffices to show that x itself is not in $\Gamma^3\mathbb{K}$.

Let us act in the polynomial ring $\mathbb{Z}[\zeta]$ modulo $p^{n_{s-1}-1}$. We have $x \equiv p^{n_{s-1}-2}(\zeta^{p^s}-1)^2$.

Consider a generator of $\Gamma^3\mathbb{K}$ (Definition 4.3):

$$\prod_{i=0}^{m} \frac{p^{n_i}}{(j_i, p^{n_i})}(\zeta^{p^i}-1)^{j_i} \quad \text{where} \quad j_i \geqslant 0 \quad \text{and} \quad \sum_{i=0}^{m} j_i \geqslant 3\,. \tag{$*$}$$

We state that $(*) \equiv (\zeta^{p^s}-1)^3 \cdot f(\zeta^{p^s})$, where f is a polynomial. If we would manage to show it, we could proceed as follows. Suppose that $x \in \Gamma^3\mathbb{K}$. Then

$$p^{n_{s-1}-2}(\zeta^{p^s}-1)^2 = (\zeta^{p^s}-1)^3 \cdot f(\zeta^{p^s}) + p^{n_{s-1}-1} \cdot g(\zeta^{p^s})$$

for some polynomials f and g. Cancelling by $p^{n_{s-1}-2}$ and $(\zeta^{p^s}-1)^2$ and substituting $t = \zeta^{p^s}-1$ we get:

$$1 = tf_0(t) + pg_0(t) \in \mathbb{Z}[t],$$

which is a contradiction because t and p do not generate the unit ideal in the polynomial ring $\mathbb{Z}[t]$.

It remains to show that $(*) \equiv (\zeta^{p^s}-1)^3 \cdot f(\zeta^{p^s})$. If for all $i < s$ the number j_i in the product $(*)$ equals 0 then even the exact equality (not only the congruence) holds. Suppose that $j_i \neq 0$ for some $i < s$. Write down this j_i as $j_i = p^r \cdot j$ with j prime to p. If $n_i - r \geqslant n_{s-1} - 1$ then

$$\frac{p^{n_i}}{(j_i, p^{n_i})} \equiv 0$$

and, hence, $(*) \equiv 0$. So, assume that $n_i - r < n_{s-1} - 1$. We have

$$r > n_i - n_{s-1} + 1 \geqslant (s-1) - i + 1 = s - i\ .$$

In order to proceed we need

LEMMA 4.8. *In a polynomial ring* $\mathbb{Z}[t]$, *there is a congruence*

$$(t-1)^{p^k} \equiv (t^p-1)^{p^{k-1}} \mod p^k$$

for any prime p *and any integer* $k > 0$.

Proof. Induction on k starting from $k = 1$:

$$\begin{aligned}(t-1)^{p^{k+1}} &= ((t-1)^{p^k})^p \\ &= ((t^p-1)^{p^{k-1}} + p^k \cdot f(t))^p \\ &\equiv (t^p-1)^{p^k} \bmod p^{k+1}\end{aligned}$$

($f(t)$ is a polynomial, it exists by the induction hypothesis). □

According to the lemma we have

$$(\zeta^{p^i}-1)^{p^r} \equiv (\zeta^{p^s}-1)^{p^{r-s+i}} \bmod p^{r-s+i+1} .$$

Hence

$$\frac{p^{n_i}}{(j_i, p^{n_i})}(\zeta^{p^i}-1)^{j_i} \equiv \frac{p^{n_i}}{(j_i, p^{n_i})}(\zeta^{p^s}-1)^{p^{r-s+i}\cdot j} \bmod p^{n_i-s+i+1} .$$

Since $p^{r-s+i}\cdot j \geqslant p \geqslant 3$ and $n_i - s + i + 1 \geqslant n_{s-1} - 1$ we are done. □

The analogous statement in the case $p = 2$ looks out a little bit more complicated:

PROPOSITION 4.9. *Let A be a 2-primary algebra. The group* $\mathrm{G}^2\Gamma K(X)$ *has a torsion iff* ind $A >$ exp A *and the reduced behaviour of A is not of the kind*

$$n,\ n-1,\ \ldots,\ 3,\ 2,\ 0 .$$

Proof. We start with the ‘only if’ Part. The case ind $A =$ exp A is covered by Proposition 3.3 with Corollary 3.6. Suppose that A has the reduced behaviour

$$n,\ n-1,\ldots,3,\ 2,\ 0 .$$

Using the same method as in [11] one can show that the whole adjoint graded group is torsion-free. Namely, a formula like one of [11, proposition] states:

$$|\mathrm{Tors}\ \mathrm{G}^*\Gamma K(X)| = \frac{|\mathrm{G}^*\Gamma K(\mathbb{P})/\mathrm{ImG}^*\Gamma K(X)|}{|K(\mathbb{P})/K(X)|},$$

where $|\ .\ |$ denotes the order of a group. Since we know the behaviour of A we can compute that

$$|K(\mathbb{P})/K(X)| = \frac{1}{2}\prod_{i=0}^{2^n-1}\frac{2^n}{(i, 2^n)}$$

(to avoid unnecessary complications we assume here that A is a division algebra). On the other hand, Proposition 3.4 shows that

$$|\mathrm{G}^i\Gamma K(\mathbb{P})/\mathrm{ImG}^i\Gamma K(X)| \leqslant \frac{2^n}{(i, 2^n)} \quad \text{for any } i.$$

Moreover,

$$|\mathrm{G}^1\Gamma K(\mathbb{P})/\mathrm{ImG}^1\Gamma K(X)| \leqslant 2^{n-1}$$

because $\xi^{2^{n-1}} - 1 \in \Gamma^1 K(X)$ (see also the computation of $\mathrm{CH}^1(X)$ in [3, Section 2]) and therefore

$$|\mathrm{G}^*\Gamma K(\mathbb{P})/\mathrm{ImG}^*\Gamma K(X)| \leqslant \frac{1}{2}\prod_{i=0}^{2^n-1}\frac{2^n}{(i, 2^n)} .$$

Thus, $|\mathrm{Tors}\ \mathrm{G}^*\Gamma K(X)| = 1$.

Now we ‘correct’ the ‘if’ proof of the previous proposition in order to match the current 2-primary situation. Suppose that we have an algebra A for which existence of the torsion is stated. Then in the reduced behaviour $(n_i)_{i=0}^m$ of A we have

$$n_s \leqslant n_{s-1} - 2 \quad \text{and} \quad n_{s-1} \geqslant 3 \quad \text{for some} \quad s.$$

Consider the element

$$x = 2^{n_{s-1}-3}(\zeta^{2^s} - 1)^2 - 2^{n_{s-1}-1}(\zeta^{2^{s-1}} - 1)^2 \in \Gamma^2\mathbb{K}$$

where $\mathbb{K}$ is as in Definition 4.3. Since the polynomial x in $\mathbb{Z}[\zeta]$ is divisible by $(\zeta - 1)^3$, it is clear that a multiple of x lies in $\Gamma^3\mathbb{K}$. So, for our purposes it suffices to show that x itself is not in $\Gamma^3\mathbb{K}$.

Let us act in the polynomial ring $\mathbb{Z}[\zeta]$ modulo $2^{n_{s-1}-2}$. We have $x \equiv 2^{n_{s-1}-3}(\zeta^{2^s} - 1)^2$. Consider a generator of $\Gamma^3\mathbb{K}$ given in Proposition 4.1:

$$\prod_{i=0}^{m} \frac{2^{n_i}}{(j_i, 2^{n_i})}(\zeta^{2^i} - 1)^{j_i} \quad \text{where} \quad j_i \geqslant 0 \quad \text{and} \quad \sum_{i=0}^{m} j_i \geqslant 3\,. \tag{$*$}$$

We state that

$$(*) \equiv (\zeta^{2^s} - 1)^3 \cdot f(\zeta^{2^s})$$

where f is a polynomial. If we would manage to show it we could proceed in the same manner as in the proof of the previous proposition.

If for all $i < s$ the number j_i in the product $(*)$ equals 0 then even the exact equality (not only the congruence) holds. Suppose that $j_i \neq 0$ for some $i < s$. Write down this j_i as $j_i = 2^r \cdot j$ with j prime to 2. If $n_i - r \geqslant n_{s-1} - 2$ then

$$\frac{2^{n_i}}{(j_i, 2^{n_i})} \equiv 0$$

and hence $(*) \equiv 0$. So, assume that $n_i - r < n_{s-1} - 2$. We have

$$r > n_i - n_{s-1} + 2 \geqslant (s-1) - i + 2 = s - i + 1\,.$$

According to Lemma 4.8, we have

$$(\zeta^{2^i} - 1)^{2^r} \equiv (\zeta^{2^s} - 1)^{2^{r-s+i}} \bmod 2^{r-s+i+1}\,.$$

Hence

$$\frac{2^{n_i}}{(j_i, 2^{n_i})}(\zeta^{2^i} - 1)^{j_i} \equiv \frac{2^{n_i}}{(j_i, 2^{n_i})}(\zeta^{2^s} - 1)^{2^{r-s+i}\cdot j} \quad \bmod 2^{n_i-s+i+1}\,.$$

Since $2^{r-s+i} \cdot j \geqslant 2^2 \geqslant 3$ and $n_i - s + i + 1 \geqslant n_{s-1} - 1$ we are done. □

Now we want to reduce the number of generators of the filtration of Definition 4.3.

PROPOSITION 4.10. *In the notation of Definition 4.3, for every $l \geqslant 0$, the group $\Gamma^l\mathbb{K}$ is in fact also generated by a reduced number of the products $(*)$, namely by the products satisfying the additional condition: $j_i = 0$ for every i such that $n_i = n_{i-1} - 1$.*

Proof. Fix some i such that $n_i = n_{i-1} - 1$. One has

$$(\zeta^{p^i} - 1)^j = \sum_{s \geqslant j} a_s \cdot (\zeta^{p^{i-1}} - 1)^s$$

for some integers a_s with $j \cdot p \mid s \cdot a_s$ (Lemma 4.2). Consequently,

$$\begin{aligned} \frac{p^{n_i}}{(j, p^{n_i})}(\zeta^{p^i} - 1)^j &= \sum_{s \geqslant j} a_s \cdot \frac{p^{n_i}}{(j, p^{n_i})}(\zeta^{p^{i-1}} - 1)^s \\ &= \sum_{s \geqslant j} b_s \cdot \frac{p^{n_{i-1}}}{(s, p^{n_{i-1}})}(\zeta^{p^{i-1}} - 1)^s \end{aligned}$$

for some integers b_s. □

Using the proposition, we compute the group Tors $\mathrm{G}^2\Gamma K(X)$ explicitly in a special situation. The situation we mean is described in the following

DEFINITION 4.11. We say that a reduced behaviour $(n_i)_{i=0}^m$ 'makes (exactly) one jump' iff there exists exactly one s such that $n_s \leqslant n_{s-1} - 2$.

EXAMPLE 4.12. Fix a prime p and integers $n > m \geqslant 1$. One can define a 'generic' division algebra $\tilde{A}$ of index p^n and exponent p^m in spirit of Definition 3.12: take a division algebra A of index and exponent p^n, put $Y = \mathrm{SB}(A^{\otimes p^m})$ and $\tilde{A} = A_{F(Y)}$.

The resulting algebra $\tilde{A}$ can be also obtained as a 'generic' p-primary division algebra of the reduced behaviour

$$n,\ n-1,\ \dots,\ n-m+2,\ n-m+1,\ 0\ .$$

In particular, it is an example of an algebra with reduced behaviour 'making one jump'.

PROPOSITION 4.13. *Let A be a p-primary algebra with an odd p and suppose that the reduced behaviour $(n_i)_{i=0}^m$ of A 'makes one jump'. Then the torsion in $\mathrm{G}^2\Gamma K(X)$ is a cyclic group of order p to the power $\min\{s,\ n_0 - n_s - s\}$ where s is the subscript for which $n_s \leqslant n_{s-1} - 2$.*

Proof. We work with $\mathbb{K}$ instead of $K(X)$ (see Proposition 4.6). According to Proposition 4.10, for any $l \geqslant 0$, the group $\Gamma^l\mathbb{K}$ is generated by the products:

$$\frac{p^{n_0}}{(j_0, p^{n_0})}(\zeta - 1)^{j_0} \cdot \frac{p^{n_s}}{(j_s, p^{n_s})}(\zeta^{p^s} - 1)^{j_s} \quad \text{with} \quad j_i \geqslant 0 \quad \text{and} \quad j_0 + j_s \geqslant l\ .$$

In particular, residue classes in the quotient $\mathrm{G}^2\Gamma\mathbb{K}$ of the following three elements

$$u = p^{n_0}(\zeta - 1)^2 \ ;$$
$$v = p^{n_0}(\zeta - 1) \cdot p^{n_s}(\zeta^{p^s} - 1) \ ;$$
$$w = p^{n_s}(\zeta^{p^s} - 1)^2$$

of $\Gamma^2\mathbb{K}$ generate the quotient. The second one can be excluded: the difference $v - p^{n_s+s}u$ is in $\Gamma^3\mathbb{K}$ since it is divisible by $p^{n_0}(\zeta - 1)^3$ in $\mathbb{Z}[\zeta]$ and thereby can be written as a linear combination (with integral coefficients) of the polynomials

$$p^{n_0}(\zeta - 1)^3, \quad p^{n_0}(\zeta - 1)^4, \quad \dots \quad \in \Gamma^3\mathbb{K} \ .$$

Since the classes of u and w in the quotient have infinite order, any torsion element $x \in \mathrm{G}^2\Gamma\mathbb{K}$ of the kind $x = u - kw$ or $x = ku - w$ with an integer k (if exists) generates the torsion subgroup. Consider two cases: if $n_0 \geqslant n_s + 2s$ then we put $x = u - p^{n_0-n_s-2s}w$; otherwise we put $x = p^{n_s+2s-n_0}u - w$.

The element $x \in \mathrm{G}^2\Gamma\mathbb{K}$ is evidently a torsion element. We finish the proof when we show that x has order p^s in the first case and order $p^{n_0-n_s-s}$ in the second. In both cases it means the same:

(1) $p^{n_0+s}(\zeta - 1)^2 - p^{n_0-s}(\zeta^{p^s} - 1)^2 \in \Gamma^3\mathbb{K}$

and

(2) $p^{n_0+s-1}(\zeta - 1)^2 - p^{n_0-s-1}(\zeta^{p^s} - 1)^2 \notin \Gamma^3\mathbb{K}$.

In order to avoid repetition of some boring computations, we prove the inclusion (1) in the following 'tricky' way. Consider a ring $\mathbb{K}'$ with a filtration Γ constructed as in Definition 4.3 for the reduced behaviour $(n_0, n_0 - 1, \dots, 1, 0)$. The ring $\mathbb{K}'$ is contained in $\mathbb{K}$ and this inclusion respects the filtrations. The element of (1) is in $\Gamma^2\mathbb{K}'$ and there is a multiple of it lying in $\Gamma^3\mathbb{K}'$. Since $\mathrm{G}^*\Gamma\mathbb{K}'$ is torsion-free (Propositions 4.6 and 3.3) this element lies even in $\Gamma^3\mathbb{K}'$. Hence (1).

The proof of (2) goes parallel to the proof of Proposition 4.7 and does not contain any new idea. Let us act in the polynomial ring $\mathbb{Z}[\zeta]$ modulo p^{n_0-s}. The element we are interested in is congruent to $p^{n_0-s-1}(\zeta^{p^s} - 1)^2$.

Consider a generator of $\Gamma^3\mathbb{K}$:

$$\frac{p^{n_0}}{(j_0, p^{n_0})}(\zeta - 1)^{j_0} \cdot \frac{p^{n_s}}{(j_s, p^{n_s})}(\zeta^{p^s} - 1)^{j_s} \quad \text{with} \quad j_i \geqslant 0 \quad \text{and} \quad j_0 + j_s \geqslant 3 \ . \quad (*)$$

The proof is completed when we show that $(*) \equiv (\zeta^{p^s} - 1)^3 \cdot f(\zeta^{p^s})$ where f is a polynomial (compare with the proof of Proposition 4.7).

If $j_0 = 0$ then even the exact equality (not only the congruence) holds. Suppose that $j_0 \neq 0$. Write down j_0 as $j_0 = p^r \cdot j$ with j prime to p. If $n_0 - r \geqslant n_0 - s$ then

$$\frac{p^{n_0}}{(j_0, p^{n_0})} \equiv 0$$

and, hence, $(*) \equiv 0$. So, assume that $n_0 - r < n_0 - s$, i.e. that $r > s$. According to Lemma 4.8 we have $(\zeta - 1)^{p^r} \equiv (\zeta^{p^s} - 1)^{p^{r-s}} \mod p^{r-s+1}$.

Hence

$$\frac{p^{n_0}}{(j_0, p^{n_0})}(\zeta - 1)^{j_0} \equiv \frac{p^{n_0}}{(j_0, p^{n_0})}(\zeta^{p^s} - 1)^{p^{r-s} \cdot j} \mod p^{n_0-s+1} .$$

Since $p^{r-s} \cdot j \geqslant p \geqslant 3$ and $n_0 - s + 1 \geqslant n_0 - s$, we are done. □

PROPOSITION 4.14. *Let A be a 2-primary algebra. Suppose that the reduced behaviour $(n_i)_{i=0}^m$ of A 'makes one jump' and let s be the subscript for which $n_s \leqslant n_{s-1} - 2$. The group* Tors $\mathrm{G}^2\Gamma K(X)$ *is cyclic; its order equals p to the power*

$$\begin{array}{ll} \min\{s,\ n_0 - n_s - s\} & \text{if } n_s > 0, \\ \min\{s,\ n_0 - s - 1\} & \text{if } n_s = 0. \end{array}$$

Proof. We describe here only the changes which should be made in order to adopt the previous proof to the 2-primary case.

First suppose that $n_s > 0$.

The quotient $\mathrm{G}^2\Gamma\mathbb{K}$ has three generators:

$$\begin{aligned} u &= 2^{n_0-1}(\zeta - 1)^2 ; \\ v &= 2^{n_0}(\zeta - 1) \cdot 2^{n_s}(\zeta^{2^s} - 1) ; \\ w &= 2^{n_s-1}(\zeta^{2^s} - 1)^2 . \end{aligned}$$

The second one can be evidently excluded.

If $n_0 \geqslant n_s + 2s$ then we put $x = u - 2^{n_0-n_s-2s}w$; otherwise we put $x = 2^{n_s+2s-n_0}u - w$.

The element $x \in \mathrm{G}^2\Gamma\mathbb{K}$ generates the torsion subgroup. To verify the statement on its order we have to check that

$$(1) \quad 2^{n_0+s-1}(\zeta - 1)^2 - 2^{n_0-s-1}(\zeta^{2^s} - 1)^2 \in \Gamma^3\mathbb{K}$$

and

$$(2) \quad 2^{n_0+s-2}(\zeta - 1)^2 - 2^{n_0-s-2}(\zeta^{2^s} - 1)^2 \notin \Gamma^3\mathbb{K}$$

The inclusion (1) can be done in the same way as previously.

Let us do (2). We act in the polynomial ring $\mathbb{Z}[\zeta]$ modulo 2^{n_0-s-1}. The element we are interested in is congruent to $2^{n_0-s-2}(\zeta^{2^s} - 1)^2$.

Consider a generator of $\Gamma^3\mathbb{K}$:

$$\frac{2^{n_0}}{(j_0, 2^{n_0})}(\zeta - 1)^{j_0} \cdot \frac{2^{n_s}}{(j_s, 2^{n_s})}(\zeta^{2^s} - 1)^{j_s} \quad \text{with} \quad j_i \geqslant 0 \quad \text{and} \quad j_0 + j_s \geqslant 3 . \qquad (*)$$

The proof is complete when we show that $(*) \equiv (\zeta^{2^s} - 1)^3 \cdot f(\zeta^{2^s})$ where f is a polynomial.

If $j_0 = 0$ then even the exact equality (not only the congruence) holds. Suppose that $j_0 \neq 0$. Write down j_0 as $j_0 = 2^r \cdot j$ with odd j. If $n_0 - r \geqslant n_0 - s - 1$, then

$$\frac{2^{n_0}}{(j_0, 2^{n_0})} \equiv 0$$

and, hence, $(*) \equiv 0$. So, assume that $n_0 - r < n_0 - s - 1$, i.e., that $r > s + 1$. According to Lemma 4.8 we have

$$(\zeta - 1)^{2^r} \equiv (\zeta^{2^s} - 1)^{2^{r-s}} \mod 2^{r-s+1} .$$

Hence

$$\frac{2^{n_0}}{(j_0, 2^{n_0})}(\zeta - 1)^{j_0} \equiv \frac{2^{n_0}}{(j_0, 2^{n_0})}(\zeta^{2^s} - 1)^{2^{r-s}\cdot j} \mod 2^{n_0-s+1} .$$

Since $2^{r-s} \cdot j \geqslant 2^2 \geqslant 3$ and $n_0 - s + 1 \geqslant n_0 - s - 1$ we are done.

Now suppose that $n_s = 0$.

The generators of $\mathrm{G}^2\Gamma\mathbb{K}$ are

$$\begin{aligned} u &= 2^{n_0-1}(\zeta - 1)^2 ; \\ v &= 2^{n_0}(\zeta - 1) \cdot (\zeta^{2^s} - 1) ; \\ w &= (\zeta^{2^s} - 1)^2 . \end{aligned}$$

The second one can be evidently excluded.

If $n_0 \geqslant 2s+1$ then we put $x = u - 2^{n_0-2s-1}w$; otherwise we put $x = 2^{2s+1-n_0}u - w$.

The element $x \in \mathrm{G}^2\Gamma\mathbb{K}$ generates the torsion subgroup. To verify the statement on its order we have to check that

(1) $\quad 2^{n_0+s-1}(\zeta - 1)^2 - 2^{n_0-s-1}(\zeta^{2^s} - 1)^2 \in \Gamma^3\mathbb{K}$

and

(2) $\quad 2^{n_0+s-2}(\zeta - 1)^2 - 2^{n_0-s-2}(\zeta^{2^s} - 1)^2 \notin \Gamma^3\mathbb{K}$

But it was done already (the assumption $n_s > 0$ was not in use). □

EXAMPLE 4.15. Let $\tilde{A}$ be a 'generic' division algebra of index p^n and exponent p^m (Example 4.12). Put $\tilde{X} = \mathrm{SB}(\tilde{A})$. From Theorem 3.13, Propositions 4.13 and 4.14 it follows that Tors $\mathrm{CH}^2(\tilde{X})$ is a cyclic group of order p to the power

$\min\{m,\ n-m\}$ for an odd p

$\min\{m,\ n-m-1\}$ for $p = 2$.

5. Algebras of Prime Exponent

Applying Theorem 3.13, Propositions 4.13 and 4.14 to the case of a prime exponent, we can state

PROPOSITION 5.1. *Let A be an algebra of a prime exponent p. Then the group* Tors $\mathrm{CH}^2(X)$ *is trivial or (cyclic) of order p. It is trivial if* ind $A = p$ *or* ind $A \mid 4$. *It is not if A is a 'generic' division algebra of index p^n and exponent p (see Definition 3.12 or Example 4.12) where $n \geqslant 2$ in the case of odd p and $n \geqslant 3$ in the case when $p = 2$.* □

COROLLARY 5.2. *Let A be an algebra of a square-free exponent e. The group* Tors $\mathrm{CH}^2(X)$ *is (cyclic) of order dividing e; moreover, there exists an algebra $\tilde{A}$ of the exponent e with* Tors $\mathrm{CH}^2(\tilde{X})$ *of order e.*

Proof. Follows from Proposition 1.3 and the proposition. □

It would be interesting to list all algebras A of prime exponent with trivial Tors $\mathrm{CH}^2(X)$. We can only describe a class of such algebras. In [12] it was shown that any *decomposable* (into a tensor product of two smaller algebras) division algebra of index p^2 and exponent p has no torsion in $\mathrm{CH}^2(X)$ (in fact, there is no torsion in the whole graded group $\mathrm{G}^*\mathrm{T}K(X)$ ([12, theorem 1])). The 2-analogy of this fact was obtained in [14, cor. 3.1]: any decomposable division algebra of index 2^3 and exponent 2 has no torsion in $\mathrm{CH}^2(X)$ (although nontrivial torsion may exist in $\mathrm{G}^*\mathrm{T}K(X)$). These facts can be generalized as follows:

PROPOSITION 5.3. *Let A be a division algebra of prime exponent. If A decomposes then the group* $\mathrm{CH}^2(X)$ *is torsion-free.*

Proof. First consider the case when $p \neq 2$.

We have a surjection

$$\mathrm{Tors}\ \mathrm{G}^2\Gamma K(X) \twoheadrightarrow \mathrm{Tors}\ \mathrm{G}^2\mathrm{T}K(X) \simeq \mathrm{Tors}\ \mathrm{CH}^2(X)\ .$$

The group from the left-hand side is cyclic (Proposition 4.13), its generator is represented by the element

$$x = p^n(\xi - 1)^2 - p^{n-2}(\xi^p - 1)^2 \in \Gamma^2 K(X) = \mathrm{T}^2 K(X),$$

where $p^n = \mathrm{ind}\ A$.

Let $A = A_1 \otimes A_2$ be the decomposition of A into a product of two smaller algebras. Assume that the base field F has no extensions of degree prime to p (otherwise we can replace F by a maximal extension of prime to p degree; such a change has no effect on $\mathrm{CH}^2(X)$, compare with Corollary 1.2). Take an extension E/F of degree $[E : F] = p^{n-2}$ such that $\mathrm{ind}\ (A_1)_E = \mathrm{ind}\ (A_2)_E = p$ (one can obtain E/F by taking first an extension E_1/F of degree $[E_1 : F] = (\mathrm{ind}\ A_1)/p$ for which $\mathrm{ind}\ (A_1)_{E_1} = p$ and extending E_1 to E in such a way that $[E : E_1] = (\mathrm{ind}\ A_2)/p$ and $\mathrm{ind}\ (A_2)_E = p$). Consider an element

$$y = p^2(\xi - 1)^2 - (\xi^p - 1)^2 \in \mathrm{T}^2 K(X_E)\ .$$

Since the algebra A_E is Brauer equivalent to a decomposable division algebra of index p^2 the group $\mathrm{CH}^2(X_E)$ is torsion-free ([12, theorem 1]). Hence, $y \in \mathrm{T}^3 K(X)$. Taking the transfer of y we get

$$N_{E/F}(y) = p^n(\xi - 1)^2 - p^{n-2}(\xi^p - 1)^2 = x \in \mathrm{T}^3 K(X)\ .$$

Consequently Tors $\mathrm{CH}^2(X) = 0$.

Now consider the case $p = 2$.

If ind $A = 4$ then Tors $\mathrm{CH}^2(X) = 0$ (see, e.g., Proposition 4.9 or use the Albert theorem and [12, Theorem 1]). Suppose that ind $A \geqslant 8$.

The group Tors $\mathrm{G}^2\Gamma K(X)$ is cyclic (Proposition 4.14), its generator is represented by the element

$$x = 2^{n-1}(\xi - 1)^2 - 2^{n-3}(\xi^2 - 1)^2 \in \Gamma^2 K(X) = \mathrm{T}^2 K(X),$$

where $2^n =$ ind A.

Let $A = A_1 \otimes A_2$ be the decomposition of A into a product of two smaller algebras and ind $A_1 \geqslant$ ind A_2. Assume that the base field F has no extensions of odd degree. Take an extension E/F of degree $[E : F] = 2^{n-3}$ such that ind $(A_1)_E = 4$ and ind $(A_2)_E = 2$ (one can obtain E/F by taking first an extension E_1/F of degree $[E_1 : F] = (\text{ind } A_1)/4$ for which ind $(A_1)_{E_1} = 4$ and extending E_1 to E in such a way that $[E : E_1] = (\text{ind } A_2)/2$ and ind $(A_2)_E = 2$). Consider an element

$$y = 2^2(\xi - 1)^2 - (\xi^2 - 1)^2 \in \mathrm{T}^2 K(X_E)\ .$$

Since the algebra A_E is Brauer equivalent to a decomposable division algebra of index 2^3 the group $\mathrm{CH}^2(X_E)$ is torsion-free ([14, cor. 3.1]). Hence, $y \in \mathrm{T}^3 K(X)$. Taking the transfer of y, we get

$$N_{E/F}(y) = 2^{n-1}(\xi - 1)^2 - 2^{n-3}(\xi^2 - 1)^2 = x \in \mathrm{T}^3 K(X)\ .$$

Consequently, Tors $\mathrm{CH}^2(X) = 0$. □

COROLLARY 5.4. *A 'generic' algebra of prime exponent p and index p^n (Example 4.12) is always indecomposable excluding the Albert case: $p = 2 = n$.*

Proof. Follows from Propositions 5.1 and 5.3. □

6. Appendix

This section is included because we do not have an appropriate reference for Corollary 6.4. A particular case of Corollary 6.4 is proved in [24, Proposition 4.7].

We start with certain preliminary observations concerning functors of points of algebraic varieties (schemes).

Let F be a field. Denote by F-$\mathfrak{alg}$ the category of commutative associative unital F-algebras. One refers to a covariant functor from F-$\mathfrak{alg}$ to the category of sets as to an *F-functor*.

Let X be a scheme over F. For any $R \in F$-$\mathfrak{alg}$ the *set of R-points* $X(R)$ of X is by definition the set $\mathrm{Mor}_F(\mathrm{Spec}\ R, X)$ of morphisms of schemes over F. This set is evidently functorial in R, so we obtain an F-functor X called the *functor of points* of the scheme X. A morphism of F-schemes $f\colon\ X \to Y$ gives a natural transformation of their functors of points.

PROPOSITION 6.1 ([23, Proposition 2, in Section 6 of chapter 2]). *Let X and Y be F-schemes and let $\phi\colon X \to Y$ be a natural transformation of their functors of points. There exists a unique morphism of F-schemes $f\colon X \to Y$ inducing ϕ.*

COROLLARY 6.2. *Two F-schemes X and Y are isomorphic iff there exists a natural transformation of the F-functors $\phi\colon X \to Y$ such that for every $R \in F\text{-}\mathfrak{alg}$ the map of sets $\phi(R)\colon X(R) \to Y(R)$ is bijective.*

If additionally we are given morphisms $X \to Z$ and $Y \to Z$ to one more F-scheme Z, then the schemes X and Y are isomorphic over Z iff there exists a natural transformation ϕ as above commuting with the natural transformations to the F-functor Z. □

We need a couple more of natural definitions and trivial remarks.

Let $\mathcal{F}$ be an F-functor supplied with a natural transformation $\mathcal{F} \to \mathcal{G}$ to another F-functor $\mathcal{G}$. For any $R \in F\text{-}\mathfrak{alg}$, the fibre of $\mathcal{F}$ over an R-point x of $\mathcal{G}$ is by definition the inverse image of x with respect to the map $\mathcal{F}(R) \to \mathcal{G}(R)$; let us denote it by $\mathcal{F}_x$.

Let $\mathcal{F}'$ be one more F-functor supplied with a natural transformation to $\mathcal{G}$. Giving a natural transformation $\mathcal{F} \to \mathcal{F}'$ over $\mathcal{G}$ is equivalent to giving a collection of maps of sets $\mathcal{F}_x \to \mathcal{F}'_x$ for every $R \in F\text{-}\mathfrak{alg}$ and every $x \in \mathcal{G}(R)$ satisfying the evident functorial property: if $R \to S$ is a homomorphism in $F\text{-}\mathfrak{alg}$, $x \in \mathcal{G}(R)$, and if $y \in \mathcal{G}(S)$ is the image of x with respect to the map $\mathcal{G}(R) \to \mathcal{G}(S)$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}_x & \longrightarrow & \mathcal{F}'_x \\ \downarrow & & \downarrow \\ \mathcal{F}_y & \longrightarrow & \mathcal{F}'_y \end{array}$$

Now everything is prepared to prove

PROPOSITION 6.3. *Let A be a central simple algebra over a field F, X its Severi–Brauer variety, and Y a generalized Severi–Brauer variety* $\mathrm{SB}(n, A)$ *with some* $n \geqslant 0$ *(see Remark 6.6).*

The product $X \times Y$ considered over X (via the first projection) is isomorphic (as a scheme over X) to the Grassmann bundle $\mathbb{\Gamma}(n, \mathcal{V})$ 'of n-dimensonal subspaces' of the canonical vector bundle $\mathcal{V}$ on X (see, e.g., [28, page 94] for a definition of the canonical vector bundle on a Severi–Brauer variety).

Proof. It suffices to show that for every $R \in F\text{-}\mathfrak{alg}$ and every $x \in X(R)$ there is a natural bijection of the sets $(X \times Y)_x$ and $\mathbb{\Gamma}(n, \mathcal{V})_x$. First of all we give descriptions of the sets of R-points of the varieties under consideration (these descriptions are in fact the most natural definitions of the varieties, see, e.g., [16]).

The set $Y(R)$ consists of *left* ideals J of the R-algebra $A_R = A \otimes_F R$ having the following two properties:

(1) the exact sequence of A_R-modules

$$0 \to J \to A_R \to A_R/J \to 0$$

splits (in particular, J is a projective R-module),
(2) $\mathrm{rk} J = n$ where $\mathrm{rk} J$ is the R-rank of J devided by deg A.

Analogously, the set $X(R)$ consists of right ideals I of A_R such that the sequence $0 \to I \to A_R \to A_R/I \to 0$ splits and $\mathrm{rk} I = 1$. For the rest of the proof, we fix R, an ideal I like that, and we set $x = I \in X(R)$. Note that $A_R = \mathrm{End}_R I$.

The fiber $\mathcal{V}_x$ of $\mathcal{V}$ over x is I; $\mathbb{\Gamma}(n, \mathcal{V})_x$ is the set of R-submodules N of I such that the sequence $0 \to N \to I \to I/N \to 0$ splits and $\mathrm{rk}_R N = n$.

Now it is clear that the Morita theory ([5, Theorem 4.29]) gives a canonical bijection of the sets $(X \times Y)_x = Y(R)$ and $\mathbb{\Gamma}(n, \mathcal{V})_x$: $N \in \mathbb{\Gamma}(n, \mathcal{V})_x$ corresponds to the left ideal $\mathrm{Hom}_R(I, N)$ of $(\mathrm{End}_R I)^{\mathrm{op}} = A_R$, where $(\mathrm{End}_R I)^{\mathrm{op}}$ is the opposite algebra.

COROLLARY 6.4. *In the condition of the proposition, put* $Y_m = \mathrm{SB}(n, A^{\otimes m})$ *for any* $m > 0$. *Then* $X \times Y_m$ *is a Grassmann bundle over* X.

Proof. Two ways of proving are possible: one can adopt the proof of the proposition to this new setting, or one can argue as follows.

Put $X_m = \mathrm{SB}(A^{\otimes m})$ and consider the morphism of varieties $X \to X_m$ given by the following natural transformation of their functors of points: for every $R \in F$-$\mathfrak{alg}$ the map $X(R) \to X_m(R)$ puts an ideal $I \in X(R)$ to its mth tensor (over R) power $I^{\otimes m} \in X_m(R)$. In the Cartesian square

$$\begin{array}{ccc} X \times Y_m & \longrightarrow & X_m \times Y_m \\ \downarrow & & \downarrow \\ X & \longrightarrow & X_m \end{array}$$

the right arrow is a Grassmann bundle by the proposition. Hence the left arrow is a Grassmann bundle as well. □

Remark 6.5. It is possible to 'spread out' the statement of the corollary a little bit replacing $A^{\otimes m}$ by any Brauer equivalent central simple F-algebra.

Remark 6.6. In contrast to [4, Section 2] and in contrast to the definition of $\mathrm{SB}(A)$, we define here $\mathrm{SB}(n, A)$ to be the variety of rank n *left* ideals in A. This variety is canonically isomorphic to the variety of rank deg $A - n$ *right* ideals in A [18, Section1 of Chapter I]. Of course, it is also isomorphic to the variety of rank n *right* ideals in the opposite algebra A^{op}.

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