CYCLES OF CODIMENSION 3 ON A PROJECTIVE QUADRIC

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Let X be a nonsingular quadratic hypersurface in a projective space over an arbitrary field (of characteristic not two) and let $CH^p X$ be a Chow group of codimension p, that is, a group of classes of codimension p cycles on X with respect to rational equivalency. It is proved that torsion in $CH^3 X$ is either trivial or is a second order group. Torsion in $CH^p X$, when $p \neq 3$, was studied earlier in RZhMat 1990, 9 A334 and 10 A389.

Let X be a nonsingular quadratic hypersurface in a projective space over field F and suppose that the characteristic of F is not equal to two; let $CH^* X$ be a graded (with respect to the codimension of cycles) Chow ring of variety X [5]. This paper is devoted to proving the following theorem.

THEOREM. The torsion subgroup of group $CH^3 X$ is either trivial or isomorphic to $\mathbb{Z}/2$.

Thus, this paper lies within the framework of the intensive study of the Chow ring of a quadric undertaken in recent years. The impetus for this study was furnished by the calculation of the K-theory of quadrics carried out by Swan [10], which provided approaches to the calculation of the Chow ring in certain special cases and offered hope for a nice solution in the general case. Specific results have since been obtained by Merkur'ev, Rost, Swan, the author, and others [2], [4], [8], [11].

The problem of describing $CH^* X$ is reducible to the problem of describing for every p a torsion subgroup $TCH^p X \subset CH^p X$. Simple arguments show that groups $TCH^0 X$ and $TCH^1 X$ are trivial. Theorem 1.2, proved in [2], gives a complete description of $TCH^2 X$; in this case, it turns out, it is either isomorphic to $\mathbb{Z}/2$ or trivial. However, hopes for a nice solution in the general case have not been realized: as was shown by Merkur'ev and the author [4], for $p \geq 4$ a group $TCH^p X$, under a suitable choice of field F and quadric X over it, will contain arbitrarily many different second order elements. The final group for which the answer could still be simple is $TCH^3 X$. The theorem proved here about $CH^3 X$ strengthens this last hope, though it must be said that it has not been possible to verify in the general case for which quadrics $TCH^3 X$ is trivial and for which it is not. This problem has been solved only for quadrics of dimension not greater than five [2].

The calculations in the Grothendieck ring of quadric X given in §2 are at the basis of the proof of the theorem about $CH^3 X$. These calculations became possible, of course, only after the appearance of [10]. Theorems 2.13 and 2.15, the latter in a somewhat modified form, can also be found in [2], although with less detailed and less complete proofs.

Throughout the letter X denotes a projective quadric defined by a nonsingular quadratic form φ over F, that is, a hypersurface in a projective space over F given by the equation $\varphi = 0$. In the second part of §2 it is assumed that φ is anisotropic. The notation connected with the Grothendieck ring of X is introduced in §2. The standard notation and theorems from [6] are used for the quadratic forms (which in the text are sometimes simply referred to as "forms"). In particular, a form of type $\langle 1, -a_1 \rangle \otimes \langle 1, -a_2 \rangle \otimes \ldots \otimes \langle 1, -a_n \rangle$ is called a Pfister n-form and is denoted by $\langle\langle a_1, a_2, \ldots, a_n \rangle\rangle$. Forms φ_1 and φ_2 over F are called proportional if $\varphi_1 \simeq c \varphi_2$ for some $c \in F^*$. By $I^n(F)$ we denote the n-th degree of the ideal of even-dimensional forms I(F) in Witt ring W(F) of quadratic forms over F. By i(φ) we denote the Witt index of φ. An important part is played by the invariant $\mathfrak{s}(\varphi)$ defined in 2.4. A form is called split (totally split) if it is isotropic (correspondingly, has maximal Witt index possible for its dimension). We say that extension E/F (totally) splits φ, where φ is a quadratic form over F, if form φ_E is (totally) split.

1. Necessary Information on the Chow Groups of a Projective Quadric. The proofs of both assertions of this section are found in [2].

Proposition 1.1. Let form φ be isotropic, $\varphi = \mathbb{H} \perp \psi$, and let Y be a projective quadric corresponding to form ψ (note that $\dim Y = \dim X - 2$). Then for $p = 1, 2, \ldots, \dim X - 1$ group $CH^p X$ is canonically isomorphic to group $CH^{p-1} Y$.

This proposition reduces the problem about the Chow group of an arbitrary quadric to the case of a quadric given in an anisotropic form.

THEOREM 1.2. If form φ has dimension greater than four and is proportional to the subform of any anisotropic Pfister 3-form, then $TCH^2X \simeq \mathbb{Z}/2$. Otherwise, $TCH^2X = 0$.

Note that the second order nontrivial element arising in CH^2X is constructed in [2].

2. The Grothendieck Ring. Let $K = K(X)$ be the Grothendieck group of quadric X. The variety X is nonsingular, therefore $K = K_0'(X) \simeq K_0(X)$ [7]. The tensor product of locally free sheaves (O_X-modules) induces on $K_0(X)$ a ring structure and group $K'_0(X)$ possesses the so-called topological filtration [7]. Therefore K is a ring with filtration, moreover we know that these structures are compatible. Let us introduce the notation for filtration: $O = K^{(d+1)} \subset K^{(d)} \subset \ldots \subset K^{(0)} = K$

$(d = \dim X)$ where $K^{(p)} \subset K$ is a subgroup generated by sheaf classes the codimension of whose support is greater than or equal to p. In addition, let $K_{(p)} = K^{(d-p)}$, $G^p K = G_{d-p} K = K^{(p/p+1)}$ and suppose that $TG^p K \subset G^p K$ is a torsion subgroup.

Any simple cycle $Z \subset X$ defines an element in $K(X)$ to be the class of structure sheaf O_E extended by zero on X. The arising mapping $CH^*X \longrightarrow G^*K(X)$ s, as is known, an epimorphism of a graded ring.

THEOREM 2.1 [2]. The kernel of epimorphism $CH^p X \longrightarrow G^p K(X)$ is contained in $TCH^p X$ for any p and is trivial for $p \leq 3$.

The remainder of this section is devoted to the study of groups $G^p K$; the main result is formulated in Theorems 2.13 and 2.15.

K-theory of a quadric and, in particular, the Grothendieck group $K(X)$ have been calculated by Swan [10]. He has constructed a canonic sheaf U on X: a locally free O_X-module with the structure of the right C_0-module, where $C_o = C_o(\varphi)$ is the even part of Clifford algebra of φ, and proved

THEOREM 2.2 [10]. The group homomorphism $\mathbb{Z}^d \oplus K_o(C_o(\varphi)) \longrightarrow K(X)$, taking the standard generators of group $\mathbb{Z}^d$, respectively, into $[O_X], [O_X(-1)], \ldots, [O_X(-d+1)]$ and class $[M] \in K_o(C_o(\varphi))$ of left module M over algebra $C_o(\varphi)$ into the class of tensor product $\mathcal{U} \otimes_{C_o} M$ is an isomorphism.

Proposition 2.3 [1], [6]. If $\varphi \notin I^2(F)$, then C_0 is a simple F-algebra (central if $\varphi \notin I(F)$; if, however, $\varphi \in I^2(F)$, then $C_o \simeq A \times A$, where A is a simple central F-algebra, moreover the complete Clifford algebra $C(\varphi)$ is isomorphic to $M_2(A)$.

Definition 2.4. Let us define the number $s = s(\varphi)$ in the following way. If $\varphi \notin I^2(F)$, then, according to 2.3, $C_o \simeq M_{2^s}(\mathcal{D})$ for some division ring D and nonnegative integer s. If, however, $\varphi \in I^2(F)$, then let s be a number such that $A \simeq M_{2^s}(\mathcal{D})$, where $A \times A \simeq C_o$.

From Definition 2.4 and Proposition 2.3 there immediately follows

Proposition 2.5. There exist canonic isomorphisms $K_o(C_o) \simeq \mathbb{Z}$ if $\varphi \notin I^2(F)$, and $K_o(C_o) \simeq \mathbb{Z}^2$, if $\varphi \in I^2(F)$. Here the class of algebra C_0 in $K_0(C_0)$ is equal to 2^s in the first case and to $K_o(C_o)$ in the second.

Note that from Theorem 2.2 and Proposition 2.5 we obtain

COROLLARY 2.6. The group $K(X)$ is torsion free.

Let us denote by h an element of $1 - [O_X(-1)] \in K^{(1)}(X)$ which is the class of the total section of X by a hypersurface in a projective space.

LEMMA 2.7 [2]. Let U be a Swan sheaf on X. Then $[\mathcal{U}(d)] = h^d + 2h^{d-1} + \ldots + 2^{d-1}h + 2^d$ in group $K(X)$.

From Proposition 2.5, in particular, we can see that the class of the Swan sheaf in group K is divisible by 2^s. Consequently, the sum $h^d + 2h^{d-1} + \ldots + 2^d$ from Lemma 2.7 is also divisible by 2^s and, therefore, the following definition is correct.

Definition 2.8. For $i = 0, 1, \ldots, s - 1$ we define elements $l_i \in K$ by equalities $l_i = \frac{1}{2^{i+1}}(h^d + 2h^{d-1} + \ldots + 2^i h^{d-i})$;

in addition, for convenience, we let $l_{-1} = 0$.

The geometrical meaning of l_i is as follows: if form φ defining X is totally split, then l_i is the class of any i-dimensional projective space P_F^i contained in X.

LEMMA 2.9. For all $i = 0, 1, \ldots, s-1$ the following equalities hold:

$$h l_i = l_{i-1} \quad ; \quad 2 l_i - l_{i-1} = h^{d-i}.$$

Proof. It is obvious from the definition of l_i.

LEMMA 2.10. Let p_i be the dimension of element l_i, that is, a number such that $\ell_i \in K_{(p_i)} \smallsetminus K_{(p_i-1)}$. Then $p_o < p_1 < \ldots < p_{s-1}$ and if X has no rational points, then, in addition, $p_i > i$ for all i.

Proof. The dimension of l_{i-1} is less than that of l_i since $\ell_{i-1} = h \ell_i$, $h \in K^{(1)}$ and multiplication in K is compatible with filtration. If the quadric contains no rational points, then $\ell_* \notin K_{(o)}$ [2], [9], [11]; hence $p_0 > 0$ and, consequently, by virtue of the first assertion of the lemma, $p_i > i$ for all i.

In the remainder of this section we assume that form φ defining X is anisotropic.

LEMMA 2.11. The element $\overline{t_i}$ has order two in group $G_{p_i} K$.

Proof. Obviously, $\overline{t_i} \neq o$. On the other hand, according to Lemma 2.9, element $2l_i$ is equal to $h^{d-i} + \ell_{i-1}$ and, consequently, lies in $K_{(p_i-1)}$ since the dimension of l_{i-1} is less than the dimension of l_i and the dimension of h^{d-i}, the number i, is less than p_i according to Lemma 2.10.

Definition 2.12. Let us define torsion of the first kind to be the graded subgroup $\overline{I}_* \subset TG_* K$ defined by the formula

$$\overline{I}_p = \begin{cases} (\mathbb{Z}/2) \cdot \overline{t_i}, & \text{if } p = p_i, \text{ where } i = o, 1, \ldots, s-1; \\ o \text{ for the remaining } p. \end{cases}$$

Torsion of the second kind II_* is defined to be the factor-group $TG_* K$ by I_*.

Let $H \subset K$ be a subring generated by element h. Clearly, $H \simeq \mathbb{Z}[h]/(h^{d+1})$, moreover the filtration on H induced with K is a filtration with respect to the powers of h: $H^{(p)} = \mathbb{Z} \cdot h^p \oplus \mathbb{Z} \cdot h^{p+1} \oplus \ldots \oplus \mathbb{Z} \cdot h^d$. Since $h = 1 - [O_x(-1)]$, subring H is also generated by element $[O_x(-1)]$ and contains $[O_x(n)]$ for all integer n.

THEOREM 2.13. If $\varphi \notin I^2(F)$, then $II_* = 0$, that is, I_* coincides with $TG_* K$ (groups I_* and II_* are defined in 2.12). The following lemma is at the basis of the proof of Theorem 2.13.

LEMMA 2.14. If $\varphi \notin I^2(F)$, then K/H is a group of order 2^s.

Proof. Let us consider the composition

$$\mathbb{Z}^d \oplus K_o(C_o(\varphi)) \xrightarrow{\sim} K \xrightarrow{[O_x(1)]} K \longrightarrow K/H$$

of Swan isomorphism 2.2 of a d-fold twist and a natural mapping on a factor-group. The term $\mathbb{Z}^d$ lies in the kernel since $[O_x(n)] \in H$. We obtain the epimorphism $\alpha : \mathbb{Z} \simeq K_o(C_o(\varphi)) \rightarrow K/H$. According to Lemma 2.7, $[U(d)] \in H$; therefore $\alpha(2^s) = 0$ and, in addition, $\alpha(2^{s-1}) \neq o$ since $[U(d)] = h^d + 2h^{d-1} + \ldots + 2^d$ is not divisible by two in H. Consequently, $Ker\alpha = 2^s\mathbb{Z}$. The lemma is proved.

Proof of Theorem 2.13. Let us endow factor-group K/H induced with K with a filtration. Let $G_*(H)$ and $G_*(K/H)$ be associative graded groups. The exact sequence $o \longrightarrow H \longrightarrow K \longrightarrow K/H \longrightarrow o$, since the filtrations of the extreme terms are lowered from the middle one, induces the exact sequence of graded groups $o \longrightarrow G_* H \longrightarrow G_* K \longrightarrow G_*(K/H) \longrightarrow o$. The group $G_* H$ is torsion free and $G_*(K/H)$, according to Lemma 2.14, is a group of order 2^s. Consequently, the order of torsion in $G_* K$ divides 2^s. Since the constructed torsion subgroup of the first kind $\overline{I}_* \subset TG_* K$ has order 2^s, we conclude that $\overline{I}_* = TG_* K$.

THEOREM 2.15. Let $\varphi \in I^2(F)$. Then: 1) II_p is a cyclical 2-group for each p; $II_p = 0$ for $p \leq m$ ($m = d/2$); 2) if $\underline{II}^o = \underline{II}^1 = \ldots = \underline{II}^p = o$ for some $p < m$, then $\overline{I}^o = \overline{I}^1 = \ldots = \overline{I}^p = \overline{I}^{p+1} = o$.

Proof. Let P be any (of the two nonisomorphic) simple $C_o(\varphi)$ modules; we call element $u = [U \otimes_c P] \in K$ a Swan generator. Denote by L a subgroup of $H + \mathbb{Z} \cdot \frac{[U]}{2^s} = H + \mathbb{Z} \cdot \ell_{s-1}$ in K with an induced filtration.

LEMMA 2.16. 1) The factor-group K/L is an infinite cyclical group generated by element u. 2) The associated graded group $G_* L$ is the direct sum $G_* H \oplus \overline{I}_*$.

Proof. Assertion 1) follows from Swan's Theorem 2.2 and 2) is analogous to 2.13.

Let us continue proving Theorem 2.15. Let us endow factor-group K/L with an induced filtration and consider the associated graded group $G_*(K/L)$. From the exact sequence $o \longrightarrow G_* L \longrightarrow G_* K \longrightarrow G_*(K/L) \longrightarrow o$ we obtain isomorphism $G_*(K/L) \simeq G_* K / (G_* H \oplus \overline{I}_*)$. Consequently, torsion of the second kind II_* is isomorphic to the torsion subgroup in $G_*(K/L)$. Since K/L is cyclical, II_p is also a cyclical group for every p. The factor-group $G_m K/G_m H$ contains an element of infinite order [2], [9]; in view of the existence of isomorphism $G_m(K/L) \simeq G_m K / (G_m H \oplus \overline{I}_*)$, we can also say the same of $G_m(K/L)$. Therefore, $G_m(K/L)$ is an infinite cyclical group and $G_p(K/L)$ is trivial for $p < m$. Consequently, $II_p = 0$ for $p \leq m$.

Let us now turn to the proof of the second assertion of the theorem. Let $\overline{X} = X_E$, where E/F is an extension totally splitting φ, and suppose that $\ell_m \in K_{(m)}(\overline{X})$ is a class of some m-dimensional projective space $\mathbb{P}_E^m$ contained in $\overline{X}$. Note that l_m, in contrast to l_i for $i < m$, is not a linear combination of the powers of h (with rational coefficients) and depends on the choice of $\mathbb{P}_E^m \subset \overline{X}$.

LEMMA 2.17. The product $h l_m$ is equal to l_{m-1}.

Proof. Applying projection formula to closed inclusion $i : \mathbb{P}^m \hookrightarrow \overline{X}$ we obtain $h \ell_m = h \cdot i_*[\mathbb{P}^m] = i_*(i^*(h)\cdot[\mathbb{P}^m])$ $i_*[\mathbb{P}^{m-1}]$. Since $i_*[\mathbb{P}^{m-1}] = \ell_{m-1}$ the lemma is proved.

Swan's Theorem 2.2 implies that $2^{m-1} K(\overline{X}) \subset K(X)$. Denote element $2^{m-1}\ell_m \in K$ by l.

LEMMA 2.18. The inequalities

$$\dim \ell \geq m \quad ; \quad \dim 2^s \ell = m \quad ; \quad \dim \ell_{s-1} < \dim \ell$$

hold in K.

Proof. Since $\bar{\ell}_m \in G_m K(\overline{X})$ is an element of infinite order [2], [9], we have diml $\geq$ m and, moreover, $\dim c\ell \geq m$ for any integer c. If extension E/F, for which $\overline{X} = X_E$, is chosen so that its degree is divisible by 2^m, then applying norm mapping $N_{E/F} : K_{(m)}(\overline{X}) \longrightarrow K_{(m)}$ to element l_m we obtain $2^m \ell_m = 2^s \ell \in K_{(m)}$. Finally, $h\ell = h (2^{m-1}\ell_m) = 2^{m-1}\ell_{m-1} \equiv \ell_{s-1} \pmod{H_{(m-1)}}$, whence $\dim \ell_{s-1} < \dim \ell$. The lemma is proved.

Let I_p be a nontrivial component of torsion of the first kind with maximal dimension and suppose that $p \geq m$. To prove assertion 2) of Theorem 2.15, it is enough to find a number $q > p$ for which $TG_q K \neq 0$. From the definition of group I_* it is clear that $p = \dim \ell_{s-1}$. Let q = diml. Then $q > p$ according to Lemma 2.18 and $G_q K$ contains a nontrivial element ℓ, moreover $2^s \ell = 0$, again according to the same lemma. Consequently, $TG_q K \neq 0$ and the proof of Theorem 2.15 is completed.

Remark 2.19. We can show under the hypotheses of Theorem 2.15 that the exact sequence $0 \longrightarrow I_* \longrightarrow TG_* K \longrightarrow II_* \longrightarrow 0$ splits.

3. The Main Theorem.
THEOREM 3.1. For any projective quadric X the group $TCH^3 X$ is 0 or $\mathbb{Z}/2$.

We preface the proof of the theorem with several auxiliary lemmas.

LEMMA 3.2. For any six-dimensional anisitropic quadratic form ψ of determinant -1 the invariant $s(\psi)$ is equal to zero.

Proof. If $s(\psi) > 0$, then the index of the complete Clifford algebra $C(\psi)$ does not exceed one. Consequently, for some quadratic extension E/F algebra $C(\psi)_E \simeq C(\psi_E)$ is isomorphic to matrix algebra, that is $\psi_E \in I^3(E)$ [3]. Since $\dim \psi_E = 6 < 2^3$, the ψ_E is hyperbolic [6], whence $\psi \simeq \langle\!\langle a \rangle\!\rangle \otimes \rho$ for some a $\in F^*$ and a three-dimensional form ρ [6]. Calculating the determinants, we obtain $a = -\det\psi = 1$, which contradicts the anisotropic nature of ψ.

LEMMA 3.3. If $s(\psi) = 2$, where ψ is an eight-dimensional anisotropic form of determinant 1, then ψ contains a subform proportional to a Pfister 2-form.

Proof. Let E/F be a quadratic extension splitting ψ : $\psi_E \simeq \mathbb{H} \perp \psi'$. Then $s(\psi') = s(\psi_E) - 1 \geq 1$ and, consequently, according to Lemma 3.2, ψ' is isotropic. Therefore $i(\psi_E) \geq 2$, whence ψ contains a subform proportional to a Pfister 2-form [6].

LEMMA 3.4. Under the hypotheses of the previous lemma, ψ is divisible by $\langle\langle a \rangle\rangle$ for some $a \in F^*$.

Proof. According to 3.3, $\psi \simeq f \cdot \langle\!\langle b_1, b_2 \rangle\!\rangle \perp g \cdot \langle\!\langle c_1, c_2 \rangle\!\rangle$, whence

$$[C(\psi)] = \left[\left(\frac{b_1, b_2}{F} \right) \otimes \left(\frac{c_1, c_2}{F} \right) \right] \in Br(F).$$

Since $s(\psi) = 2$, the above tensor product of quaternion algebras is not a division ring, that is, the form $\langle b_1, b_2, -b_1 b_2, -c_1, -c_2, c_1 c_2 \rangle$ is isotropic [6]. The latter means that there exists an element a $\in F^*$ which is the value of the pure subforms of both Pfister 2-forms. Therefore $\langle\!\langle b_1, b_2 \rangle\!\rangle$ and $\langle\!\langle c_1, c_2 \rangle\!\rangle$ are divisible by $\langle\langle a \rangle\rangle$ [6] and, consequently, ψ is divisible by $\langle\langle a \rangle\rangle$.

LEMMA 3.5. Let ψ be an eight-dimensional quadratic form over F and suppose that for some quadratic extension L/F form ψ_L is proportional to a Pfister 3-form. Then there exists a quadratic extension L'/F such that $i(\psi_{L'}) \geq 3$.

Proof. If ψ is isotropic, then ψ_L is hyperbolic and we can take L itself for L'. In the anisotropic case let ρ be some seven-dimensional subform of ψ and let $\psi' = \rho \perp \langle \det\rho \rangle$. If ψ' is isotropic, then L/F splits and, hence, totally splits and

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we again take $L' = L$. If, however, Ψ' is anisotropic, then we use Lemma 3.4. Since $s(\Psi) \geq 2$, we conclude that Ψ' totally splits in some quadratic extension L'/F. Clearly, for this extension $i(\Psi_L) \geq 3$.

Proof of the Theorem. If form φ defining quadric X is isotropic, then, by virtue of Proposition 1.1, CH^3X is isomorphic to group CH^2 of some quadric and we can apply Theorem 1.2. Next we assume that φ is anisotropic.

According to 2.1, $CH^3X \simeq G^3K(X)$. If $\varphi \notin I^2(F)$, then according to Theorem 2.13 $TG^3K(X)$ is 0 or $\mathbb{Z}/2$. Next we assume that $\varphi \in I^2(F)$.

If the even number $\dim\varphi$ does not exceed eight, then, in view of 2.15, torsion of the second kind does not exist in codimension three; consequently, the theorem is also proved in this case. Next we assume that $\dim\varphi \geq 10$.

Groups $G^0K(X)$, $G^1K(X)$, $G^2K(X)$ are torsion free (the latter by Theorem 1.2); consequently, according to Theorem 2.15, $I^3 = 0$ and $TG^3K(X) = \overline{H}^3$ is a cyclical group. It remains to prove that this group is annihilated by multiplication by two. For this let us consider an arbitrary quadratic extension L/F. Composition $TCH^3X \xrightarrow{res} TCH^3X_L \xrightarrow{N} TCH^3X$ coincides with multiplication by two and the proof of the theorem is concluded as soon as we verify that the mean group is trivial for a suitable choice of L/F. Suppose that L/F splits φ: $\varphi_L \simeq H \perp \psi_L$, where ψ is a subform of φ, and let Y be a quadric corresponding to ψ. Then by 1.1, $CH^3X_L \simeq CH^2Y_L$. Since $\dim\psi = \dim\varphi-2 \geq 8$, according to Theorem 1.2, group TCH^2Y_L can be nontrivial only if ψ_L is proportional to a Pfister 3-form. Assume that it is so. Then according to Lemma 3.5 there exists a quadratic extension L'/F for which $i(\psi_L) \geq 3$. Since ψ is a subform of φ so is $i(\varphi_L) \geq 3$. Hence, by 1.1, $TCH^3X_{L'} = 0$. The theorem is proved.

REFERENCES

1. N. Bourbaki, Algebra: Modules, Rings, Forms [Russian translation], Nauka, Moscow (1966).
2. N. A. Karpenko, "Algebro-geometric invariants of quadratic forms," Algebra i Analiz, **2**, No. 1, 141-162 (1990).
3. A. S. Merkur'ev, "On the norm residue symbol of degree 2," Dokl. Akad. Nauk SSSR, **261**, No. 3, 542-547 (1981).
4. N. A. Karpenko and A. S. Merkur'ev, "The Chow groups of projective quadrics," Algebra i Analiz, **2**, No. 3, 218-235 (1990).
5. R. Hartshorne, Algebraic Geometry, Springer-Verlag, New York (1977).
6. T.-Y. Lam, The Algebraic Theory of Quadratic Forms, Benjamin, Reading, Massachusetts (1973).
7. D. Quillen, Higher Algebraic K-theory, I., Lecture Notes in Mathematics **341**, Springer, New York (1973), pp. 77-139.
8. M. Rost, Some New Results on the Chow Groups of Quadrics, Preprint, Regensburg (1990).
9. R. G. Swan, "Vector bundles, projective modules, and the K-theory of spheres," Ann. Math. Stud., **113**, 432-522 (1987).
10. R. G. Swan, "K-theory of qaudric hypersurfaces," Ann. Math. (2), **122**, No. 1, 113-154 (1985).
11. R. G. Swan, "Zero cycles on quadric hypersurfaces," Proc. Amer. Math. Soc., **109**, No. 1, 43-46 (1989).