CHOW GROUPS OF SOME GENERICALLY TWISTED FLAG VARIETIES

NIKITA A. KARPENKO

Abstract. We classify the split simple affine algebraic groups G of types A and C over a field with the property that the Chow group of the quotient variety E/P is torsion-free, where $P \subset G$ is a special parabolic subgroup (e.g., a Borel subgroup) and E is a generic G-torsor (over a field extension of the base field). Examples of G include the adjoint groups of type A. Examples of E/P include the Severi-Brauer varieties of generic central simple algebras.

1. Introduction

Let G be a split semisimple affine algebraic group over a field k and let P be a parabolic subgroup of G. The quotient G/P is a smooth projective algebraic k-variety sometimes called a *flag variety* of G. The variety G/P is (absolutely) *cellular* (in the sense of [4, §66]). In particular, its Chow group $\mathrm{CH}(G/P)$ is torsion-free.

Given a G-torsor E over k, the quotient variety E/P is a *twisted flag variety*, a twisted form of G/P. The Chow group $\mathrm{CH}(E/P)$ may have a large torsion subgroup and is far from being understood. The situation is still the same when we restrict our attention to the case of a *special* parabolic subgroup P. Recall that P is *special*, if any P-torsor over any field extension of k is trivial. (For instance, any Borel subgroup of G is special parabolic.) For any special parabolic P, every G-torsor E over k splits over the function field $k(E/P)$ (see [10, Lemma 6.5]), showing that E/P is a generically cellular variety, i.e., becomes cellular over its own function field.

Let now E be a *generic* G-torsor. By this we mean a G-torsor over a certain field extension F/k, obtained by the following construction (see Remark 2.3). We fix an imbedding of G into the general linear group GL_N for some N. This makes GL_N a G-torsor over the quotient variety $S := \mathrm{GL}_N/G$. We define F to be the function field $k(S)$ and we define the generic G-torsor E to be the G-torsor over F given by the generic fiber of $\mathrm{GL}_N \to S$.

For any other G-torsor E' over any field extension k'/k, there exists a k'-point of S such that E' is isomorphic to the fiber of $\mathrm{GL}_N \to S$ over the point. Moreover, for infinite k', the set of such k'-points is dense in S, [18, §5.3]. This suggests that E, being the generic fiber of $\mathrm{GL}_N \to S$, is the most complicated G-torsor and that the variety E/P, which we call a *generically twisted flag variety*, is the most complicated twisted flag variety (for given G and P). Nevertheless, the Chow group $\mathrm{CH}(E/P)$ for a generic E turns out to be

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more accessible than in general. In this paper, we classify the split simple affine algebraic groups G of types A and C over a field with the property that the Chow group $\mathrm{CH}(E/P)$ of E/P is torsion-free – see Theorems 3.1 and 4.1. Examples of G include adjoint groups of type A (Theorem 3.7). Examples of E/P include the Severi-Brauer varieties of generic central simple algebras.

An application to computation of the *topological* (also called *geometrical*) filtration on the Grothendieck ring of twisted flag varieties is provided as well as some other applications – see Corollaries 3.9, 3.10, 3.14.

For G of type B_n, an analogue of Theorems 3.1 and 4.1 is known. Note that G is isomorphic to Spin_{2n+1} (the simply connected case) or to O^+_{2n+1} (the adjoint case). Since $\mathrm{B}_n = \mathrm{C}_n$ for $n = 1, 2$, let us assume that $n \geq 3$. By [15] (see also [19]), $\mathrm{CH}(E/P)$ is torsion-free for $G = \mathrm{O}^+_{2n+1}$. And it is easy to see that $\mathrm{CH}^2(E/P)$ contains an element of order 2 for $G = \mathrm{Spin}_{2n+1}$.

For the type D_n (with $n \geq 4$), $\mathrm{CH}(E/P)$ is torsion-free if $G = \mathrm{O}^+_{2n}$ (see [15] or [19]), and $\mathrm{CH}^2(E/P)$ has an element of order 2 for $G = \mathrm{Spin}_{2n}$. However, the analysis of the remaining projective orthogonal and semi-spinor groups has not been completed so far.

For G of type G_2 and *any* non-split G-torsor E over a field, $\mathrm{CH}^2(E/P)$ has an element of order 2, see, e.g., [20]. See [20] as well for some other computations concerning Chow groups of some other twisted flag varieties.

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2. Generic torsors

For G as in the introduction and P a parabolic subgroup of G, we consider a generically twisted flag variety E/P, where E is the generic G-torsor over F obtained out of an imbedding $G \hookrightarrow \mathrm{GL}_N$ for some N. Here F is the function field $k(S)$ of the k-variety $S := \mathrm{GL}_N/G$.

We consider the pull-back homomorphism of *P-equivariant Chow groups* (see [3])

$$\mathrm{CH}_P \operatorname{Spec} F \to \mathrm{CH}_P E$$

with respect to the (P-equivariant) structure morphism $E \to \operatorname{Spec} F$ of the F-variety E (where P acts on $\operatorname{Spec} F$ trivially). Note that the P-equivariant Chow group $\mathrm{CH}_P E$ coincides with the ordinary Chow group of E/P. The following statement is proved but not formulated in [10]:

Lemma 2.1. *The homomorphism* $\mathrm{CH}_P \operatorname{Spec} F \to \mathrm{CH}(E/P)$ *is surjective.*

Proof. The variety GL_N is a GL_N-equivariant open subvariety of the affine space $\operatorname{End} k^N$. It is enough to prove that the composition

$$\mathrm{CH}_P \operatorname{Spec} k \to \mathrm{CH}_P \operatorname{Spec} F \to \mathrm{CH}(E/P) = \mathrm{CH}_P E$$

with the change of field homomorphism $\mathrm{CH}_P \operatorname{Spec} k \to \mathrm{CH}_P \operatorname{Spec} F$ is surjective. The homomorphism $\mathrm{CH}_P \operatorname{Spec} k \to \mathrm{CH}_P E$ decomposes as

$$\mathrm{CH}_P \operatorname{Spec} k \to \mathrm{CH}_P \operatorname{End} k^N \to \mathrm{CH}_P \mathrm{GL}_N \to \mathrm{CH}_P E.$$

The first homomorphism here is the pull-back with respect to the structure morphism of the k-variety $\operatorname{End} k^N$; it is an isomorphism by homotopy invariance of equivariant Chow groups. The second and the third homomorphisms are pull-backs with respect to the open imbedding $\mathrm{GL}_N \hookrightarrow \operatorname{End} k^N$ and the localization morphism $E \to \mathrm{GL}_N$; they are surjective by localization property of equivariant Chow groups. □

Example 2.2. For the quotient $G := \mathrm{SL}_n/\mu_m$ of the special linear group SL_n by the central subgroup μ_m of the m-roots of unity, where $m \geq 1$ is a divisor of $n \geq 2$, any G-torsor over k gives rise to a central simple k-algebra of degree n and exponent m. We refer to the algebra A, corresponding to a generic G-torsor, as a *generic central simple algebra of degree n and exponent m.* In the decomposition $n = n_1 n_2$ with $n_1 \geq 1$ having the same prime divisors as m and with n_2 relatively prime to m, the factor n_1 is the index of A. Let P be a parabolic subgroup in G with conjugacy class corresponding to the subset of the Dynkin diagram of G obtained by removing the first vertex. The variety E/P is the Severi-Brauer variety X of A. It is shown in [10, §8.1] that the graded ring $\mathrm{CH}_P \operatorname{Spec} F$ is generated by some homogeneous elements with at most one element in every codimension. Therefore, by Lemma 2.1, the Chow ring $\mathrm{CH}\, X$ is generated by some homogeneous elements with at most one element in every codimension.

In the particular case of $G := \mathrm{PGL}_n = \mathrm{SL}_n/\mu_n$, we refer to A as a *generic central simple algebra of degree n.* The index and exponent of such A are equal to n as well.

Remark 2.3. The construction of a generic G-torsor we use in this paper is a particular case of the construction of [18, Example 5.4], which nowadays became more common. For two generic G-torsors E and E' over fields F/k and F'/k produced by this more general construction, there is a canonical construction of a field L/k, containing both F/k and F'/k, and of an isomorphism $E_L \simeq E'_L$ such that the extensions L/F and L/F' are purely transcendental. Since Chow groups do not change under purely transcendental base field extensions, we get a canonical isomorphism $\mathrm{CH}(E/P) \simeq \mathrm{CH}(E'/P)$ for any P. Thanks to A. Merkurjev for pointing this out.

The relationship between $\mathrm{CH}(E/P)$ and $\mathrm{CH}(E/P')$ for different special parabolic subgroups $P, P' \subset G$ is explained in the proof of Lemma 3.6.

Example 2.4. For any split semisimple G, a generic G-torsor E, and a Borel subgroup $B \subset G$, the topological filtration on the Grothendieck ring $K(E/B)$ coinsides with the gamma filtration. Indeed, by [3, Proposition 6], the graded ring $\mathrm{CH}_B \operatorname{Spec} F$ is identified with the symmetric algebra $S(\hat{T})$ of the character group $\hat{T}$ of a maximal split torus $T \subset B$. It follows that the ring $\mathrm{CH}_B \operatorname{Spec} F$ is generated by elements of codimension 1. By Lemma 2.1, this implies that the ring $\mathrm{CH}(E/B)$ is generated by elements of codimension 1. Therefore the ring $\mathrm{CH}(E/B)$ is generated by Chern classes. In particular, the associated graded ring of the topological filtration on $K(E/B)$ is generated by Chern classes, which precisely means that the topological filtration coincides with the gamma filtration, see [9, Remark 2.17].

The above considerations also show that the ring $\mathrm{CH}(E/B)$ is finitely generated. In particular, its torsion subgroup $\operatorname{Tors} \mathrm{CH}(E/B)$ is finite.

3. Type A_{n-1}

Let $n \geq 2$. Any split simple affine algebraic group G of type A_{n-1} over any field k is isomorphic to the quotient SL_n / μ_m, where $m \geq 1$ is a divisor of n. Here is the main result of this section:

Theorem 3.1. *For $G := \mathrm{SL}_n / \mu_m$ (with n and m as above) over any field k, let $P \subset G$ be a special parabolic subgroup and let E be a generic G-torsor over a field extension F/k. The group $\mathrm{CH}(E/P)$ is torsion-free if and only if the g.c.d. $(m, n/m)$ of m and n/m is bounded by 2. Moreover, for every odd prime divisor p of $(m, n/m)$ as well as if p^2 divides $(m, n/m)$ for $p = 2$, the group $\mathrm{CH}^2(E/P)$ contains an element of order p.*

We will prove Theorem 3.1 after some preparation work. The most significant cases of torsion-free $\mathrm{CH}(E/P)$ are the cases of $G = \mathrm{PGL}_n = \mathrm{SL}_n / \mu_n$ and $G = \mathrm{SL}_{2^r} / \mu_{2^{r-1}}$ (for any $r \geq 1$). Since SL_n is special, the case of $G = \mathrm{SL}_n$ is trivial. We start with a result covering the case of $G = \mathrm{PGL}_n$:

Proposition 3.2. *Let F be a field and A a central simple F-algebra. Assume that the Chow ring $\mathrm{CH}\, X$ of the Severi-Brauer variety X of A is generated (as a ring) by some homogeneous elements with at most one element in every codimension. Then the group $\mathrm{CH}\, X$ is p-torsion-free for every prime number p such that the p-primary parts of the exponent and the index of A coincide.*

Remark 3.3. According to Example 2.2, Proposition 3.2 applies to any generic central simple algebra A of any given degree (without restriction on its exponent), implying that the Chow ring of the Severi-Brauer variety of A is torsion-free.

Remark 3.4. In the case where $\exp A = \operatorname{ind} A$, Proposition 3.2 provides a complete description of the ring $\mathrm{CH}\, X$. Indeed, for any $n \geq 1$ and any central simple F-algebra A of degree n, the kernel of the change of field homomorphism

$$\mathrm{CH}\, X \to \mathrm{CH}\, X_L = \mathrm{CH}\, \mathbb{P}^{n-1} = \mathbb{Z}[H]/(H^n),$$

given by any splitting field L/F of the algebra, where H corresponds to the hyperplane class in $\mathrm{CH}\, \mathbb{P}^{n-1}$, is the torsion subgroup of $\mathrm{CH}\, X$. Moreover, by [8, Theorem 1], if $\exp A = \operatorname{ind} A =: d$, then for any $0 \leq j \leq n-1 = \dim X$, the image of $\mathrm{CH}^j X$ in $\mathrm{CH}^j \mathbb{P}^{n-1} = \mathbb{Z}$ is generated by $d/(j, d)$.

Proof of Proposition 3.2. Let n be the degree of A. Let $x_i \in \mathrm{CH}^i(X)$, $i = 0, 1, \ldots, n-1$, be elements generating the ring $\mathrm{CH}\, X$.

We fix an arbitrary prime number p such that the p-primary parts of the exponent and the index of A coincide. For the remainder of the proof, we switch to the Chow groups $\mathrm{CH} \otimes \mathbb{Z}_{(p)}$ with coefficients in $\mathbb{Z}_{(p)}$ – the localization of $\mathbb{Z}$ at the prime ideal (p) generated by p. To prove Proposition 3.2 it suffices to show that the group $\mathrm{CH}\, X \otimes \mathbb{Z}_{(p)}$ is torsion-free.

Let p^r be the p-primary part of $\operatorname{ind} A$. By Lemma 3.5, we only need to check that $\mathrm{CH}^j X \otimes \mathbb{Z}_{(p)}$ is torsion-free for $j < p^r$.

Let L/F be a finite Galois field extension splitting A. Let L_r be the intermediate field corresponding to a p-Sylow subgroup of $\mathrm{Gal}(L/F)$ so that $[L_r : F]$ is prime to p and

$[L:L_r]$ is a p-power. Let L_0 be a minimal subfield of L containing L_r and splitting A. We have $[L_0:L_r]=p^r$. By [5, Theorem 4.2.1], there is a chain of subfields

$$L_r \subset L_{r-1} \subset \cdots \subset L_0$$

with $[L_{i-1}:L_i]=p$ for every $i=r,\dots,1$. Note that $\operatorname{ind} A_{L_i}=p^i$ for $i=0,1,\dots,r$.

We claim that for any $j=1,\dots,p^r-1$, the norm map

$$N_i^j : \mathrm{CH}^j X_{L_i}\otimes\mathbb{Z}_{(p)} \to \mathrm{CH}^j X\otimes\mathbb{Z}_{(p)}$$

is surjective, where $i=v_p(j)$ and v_p is the p-adic valuation. Since $\operatorname{ind} A_{L_i}=p^i$ divides j, we have $\mathrm{CH}^j X_{L_i}=\mathbb{Z}$ (by [7, Corollary 1.3.2]). More precisely, $\mathrm{CH}^j X_L=\mathrm{CH}^j\mathbb{P}^{n-1}=\mathbb{Z}$, where $1\in\mathbb{Z}$ corresponds to the class in $\mathrm{CH}^j\mathbb{P}^{n-1}$ of a linear subspace in $\mathbb{P}^{n-1}$ of codimension j, and the change of field homomorphism $\mathrm{CH}^j X_{L_i}\to\mathrm{CH}^j X_L$ is an isomorphism. Therefore the claim implies that $\mathrm{CH}^j X\otimes\mathbb{Z}_{(p)}$ is torsion-free.

We prove the claim by induction on j. Given an arbitrary positive $j\le p^r-1$, we assume that the claim holds in positive codimensions $<j$. We first check that every element of $\mathrm{CH}^j X\otimes\mathbb{Z}_{(p)}$ which is a polynomial in $x_1,\dots,x_{j-1}$ (without x_j) is in the image of the norm map N_i^j. It suffices to consider the case where the polynomial is a monomial. Since the degree j of the monomial is not divisible by p^{i+1}, the monomial contains a factor x_k for some $k\in\{1,\dots,j-1\}$ not divisible by p^{i+1}. Since $v_p(k)\le i$, it follows by the induction hypothesis that x_k is in the image of N_i^k. Therefore, by the projection formula [4, Proposition 56.8], the monomial is in the image of N_i^j.

To finish the proof of the claim (and therefore the proof of Proposition 3.2), it suffices to check that x_j is also in the image of N_i^j. For this we decompose the element $N_i^j(1)\in\mathrm{CH}^j X\otimes\mathbb{Z}_{(p)}$, where 1 is the generator of $\mathrm{CH}^j X_{L_i}\otimes\mathbb{Z}_{(p)}=\mathbb{Z}_{(p)}$, in a linear combination of the degree-j monomials in $x_1,x_2,\dots,x_j$ and check that the coefficient $\lambda\in\mathbb{Z}_{(p)}$ at the monomial x_j is invertible.

Let us observe that $v_p(N_i^j(1)_L)=v_p([L_i:F])=r-i$. On the other hand, if λ is not invertible, then $(\lambda x_j)_L$ is divisible by p^{r-i+1} because x_L is divisible by p^{r-i} for any element $x\in\mathrm{CH}^j X$, see Remark 3.4[1] Also M_L is divisible by p^{r-i+1} for any monomial $M\in\mathrm{CH}^j X$ in $x_1,\dots,x_{j-1}$ because M contains x_k with some k not divisible by p^{i+1}: x_{kL} is then divisible by p^{r-i}; in the same time M necessarily contains another factor x_l with some $l=1,\dots,j-1$ ($l=k$ is also possible). Our assumption that $j<p^r$ ensures that l is not divisible by p^r so that x_{lL} is divisible by p. □

Here is the lemma used in the proof of Proposition 3.2:

Lemma 3.5. *Let A be a central simple algebra over a field F of degree $n\ge 1$. Let p be a prime number and p^r the p-primary part of* $\operatorname{ind} A$. *Let X be the Severi-Brauer variety of A. For any integer $0\le j\le \dim X=n-1$, the group $\mathrm{CH}^j X\otimes\mathbb{Z}_{(p)}$ is isomorphic to the group $\mathrm{CH}^{j'} X\otimes\mathbb{Z}_{(p)}$, where $0\le j'\le p^r-1$ is the remainder after division of j by p^r.*

Proof. Let A_p be the p-primary part of the underlying division algebra of A (so that $\operatorname{ind} A_p=p^r$). Let X_p be the Severi-Brauer variety of A_p.

[1]This is the only place in the proof where we use the fact that the p-primary part of the exponent of A coincides with the p-primary part of its index.

Let L/F be a finite Galois field extension splitting the algebra A. Let K/F be the subextension corresponding to a p-Sylow subgroup of $\mathrm{Gal}(L/F)$. Therefore the degree of K/F is prime to p, the degree of L/K is a p-power, and the algebra A_K is isomorphic to a matrix algebra over A_{pK}.

Below we work in the category of Chow motives, [4, §64]: first with integral coefficients, then with coefficients in $\mathbb{Z}_{(p)}$. The integral Chow motive $M(X_K)$ of the K-variety X_K is isomorphic to the direct sum of shifts of the Chow motive of X_{pK} with the shifting numbers of the summands being the multiples of p^r from 0 to $n - p^r$, [7, Corollary 1.3.2]:

$$M(X_K) \simeq \bigoplus_{i=0}^{n/p^r-1} M(X_{pK})\{ip^r\}.$$

We switch to the Chow motives with coefficients in $\mathbb{Z}_{(p)}$. Let f be the above isomorphism after the switch. We apply the norm $N_{K/F}$ to f and divide the result by $[K : F] \in \mathbb{Z}_{(p)}^{\times}$. This way we get a morphism $g : M(X) \to \bigoplus_{i=0}^{n/p^r-1} M(X)\{ip^r\}$ with the property that $g_L = f_L$. In particular, g_L is an isomorphism. It follows by [4, Corollary 92.7 with Remark 92.3], a consequence of Nilpotence Theorem for projective homogeneous varieties, that g is an isomorphism. Thus $\mathrm{CH}^j X \otimes \mathbb{Z}_{(p)} \simeq \mathrm{CH}^{j'} X_p \otimes \mathbb{Z}_{(p)} \simeq \mathrm{CH}^{j'} X \otimes \mathbb{Z}_p$. □

Lemma 3.6. *Let G be a split semisimple linear algebraic group over a field k and let E be a G-torsor over k. If the Chow group $\mathrm{CH}(E/P)$ is torsion-free for at least one special parabolic subgroup P of G, then it is torsion-free for every special parabolic. The same holds with $\mathrm{CH}^2(E/P)$ in place of $\mathrm{CH}(E/P)$.*

Proof. Let P and P' be special parabolic subgroups of G with torsion-free $\mathrm{CH}(E/P)$. Since E splits over $F(E/P)$ (see [10, Lemma 6.5]), the Chow motive of the variety $E/P \times E/P'$ is a direct sum of shifts of the motive of E/P, [14, Corollary 3.4]. Therefore $\mathrm{CH}(E/P \times E/P')$ is torsion-free. In the same time, the Chow motive of $E/P \times E/P'$ is a direct sum of shifts of the motive of E/P' so that $\mathrm{CH}(E/P')$ is torsion-free as well.

The same chain of conclusions goes through for $\mathrm{CH}^2(E/P)$ in place of $\mathrm{CH}(E/P)$, because one shifting number is 0 and the remaining shifting numbers are positive in both motivic decompositions mentioned. (Recall that for any projective homogeneous variety, the groups CH^0 and CH^1 are torsion-free.) □

At this point we already proved Theorem 3.1 for $m = n$, i.e., for $G = \mathrm{PGL}_n$:

Theorem 3.7. *For any field k and any $n \geq 2$, let G be the projective linear group PGL_n over k, let P be a special parabolic subgroup of G, and let E be a generic G-torsor (over a field extension of k). Then the Chow group of the generically twisted flag variety E/P is torsion-free.* □

The Severi-Brauer variety X of a degree-n central simple algebra A is by definition a closed subvariety of the Grassmannian of n-dimensional subspaces in the n^2-dimensional vector space A. The tautological bundle on X has rank n and is the restriction of the tautological bundle on the Grassmannian.

Corollary 3.8. *For any n, let X be the Severi-Brauer variety of a generic central simple algebra of degree n. Then the Chow ring $\mathrm{CH}\, X$ is generated by the Chern classes of the tautological vector bundle on X.*

Proof. Let $\bar{X}$ be X over a splitting field of the algebra. As shown in [10], the image of the change of field homomorphism $\mathrm{CH}\, X \to \mathrm{CH}\, \bar{X}$ is generated by the Chern classes of the tautological vector bundle. Since $\mathrm{CH}\, X$ is torsion-free, the change of field homomorphism $\mathrm{CH}\, X \to \mathrm{CH}\, \bar{X}$ is injective and it follows that $\mathrm{CH}\, X$ itself is generated by the Chern classes of the tautological vector bundle. □

Here is a couple of applications:

Corollary 3.9. *Let X be the Severi-Brauer variety of a central simple algebra A over a field k satisfying* $\operatorname{ind} A = \exp A$. *Then the torsion subgroup* $\operatorname{Tors} \mathrm{CH}\, X \subset \mathrm{CH}\, X$ *splits off canonically as a direct summand of* $\mathrm{CH}\, X$.

Proof. By [7, Corollary 1.3.2], we may assume that A is a division algebra. By specialization, all relations between the Chern classes of the tautological vector bundle on the Severi-Brauer variety of a generic central simple algebra of degree $\deg A$ hold for the Chern classes of the tautological vector bundle on our X. It follows that the subring $C \subset \mathrm{CH}\, X$, generated by these Chern classes, is mapped under the quotient map $\mathrm{CH}\, X \to \mathrm{CH}\, X / \operatorname{Tors} \mathrm{CH}\, X$ isomorphically onto the quotient (see Remark 3.4), whence the statement. □

The following result has been proved in [9] for division algebras of p-primary index. Those assumptions can be dropped:

Corollary 3.10. *Let X be the Severi-Brauer variety of a central simple algebra A over a field k satisfying* $\operatorname{ind} A = \exp A$. *Then the topological filtration on the Grothendieck ring $K(X)$ coincides with the gamma filtration. Moreover, for any finite product Y of any generalized Severi-Brauer varieties of any tensor powers of A, the topological filtration on the Grothendieck ring $K(X_{k(Y)})$ coincides with the gamma filtration.*

Proof. Let $\tilde{X}$ be the Severi-Brauer variety of a generic central simple algebra $\tilde{A}$ of degree $\deg A$ over a field F. Note that $\exp \tilde{A} = \operatorname{ind} \tilde{A} = \deg \tilde{A}$. By Corollary 3.8, the ring $\mathrm{CH}\, \tilde{X}$ is generated by Chern classes. Therefore, the topological filtration on the Grothendieck ring $K(\tilde{X})$ coincides with the gamma filtration. Let T be the generalized Severi-Brauer variety $\mathrm{SB}_{\operatorname{ind} A}(\tilde{A})$ (of right ideals in $\tilde{A}$ of reduced dimension $\operatorname{ind} A$; the usual Severi-Brauer variety $\mathrm{SB}(\tilde{A})$ is $\mathrm{SB}_1(\tilde{A})$ in this notation). By the index reduction formula [12, (5.11)], the index and the exponent of the central simple $F(T)$-algebra $\tilde{A}_{F(T)}$ are equal to $\operatorname{ind} A$. Since the projection $T \times \tilde{X} \to \tilde{X}$ is a Grassmann bundle, the topological filtration on the Grothendieck ring $K(\tilde{X}_{F(T)})$ coincides with the gamma filtration, cf. [9]. Moreover, by [8], since $\operatorname{ind} \tilde{A}_{F(T)} = \exp \tilde{A}_{F(T)}$, the topological filtration on $K(\tilde{X}_{F(T)})$ coincides with the filtration induced by the topological filtration on the Grothendieck ring of $\tilde{X}$ considered over an algebraic closure of $F(T)$.

Turning back to A and X over k, we have three embedded filtrations on $K(X)$: the gamma filtration, which is contained in the topological filtration, which in its turn is contained in the filtration induced by the topological filtration over an algebraic closure of k. By [16], since for any $i \geq 1$, the indexes of the ith tensor powers of the algebras A and $\tilde{A}_{F(T)}$ coincide (cf. [9, Example 3.9]), the rings $K(X)$ and $K(\tilde{X}_{F(T)})$ are identified. Under this identification, both gamma filtrations and both filtration induced from the respective algebraic closures are identified as well. It follows that all three filtrations on $K(X)$ coincide. In particular, the topological filtration on the Grothendieck ring $K(X)$ coincides with the gamma filtration.

From this point, the deduction of the statement on $K(X_{k(Y)})$ is standard, cf. [9]. □

The following statement will be of help in the proof of Proposition 3.12:

Corollary 3.11. *Let A be an arbitrary central simple algebra over a field F and let L be a maximal subfield of the underlying division algebra. Let p be a prime integer. For $i > 0$, let $c_i \in \operatorname{CH}^i X \otimes \mathbb{Z}_{(p)}$ be the ith Chern class of the tautological vector bundle on the Severi-Brauer X variety of A, considered in the Chow group with coefficients in $\mathbb{Z}_{(p)}$. For any $i > 0$ coprime with p, c_i is in the image of the norm map $N_{L/F}$.*

Proof. We fix some $i > 0$ coprime with p and set $n := \deg A$. The image of $1 \in \mathbb{Z} = \operatorname{CH}^i X_L$ under $N_{L/F} : \operatorname{CH}^i X_L \to \operatorname{CH}^i X$ equals $h^i_*(e)$, where $e \in \operatorname{CH}_0 X$ is the class of a closed point of degree $\operatorname{ind} A$ (the canonical generator of the torsion-free group $\operatorname{CH}_0 X$, see [13] or [2]) and $h \in \operatorname{CH}^1(X \times X)$ is the first Chern class of the canonical line bundle on $X \times X$. (In particular, $N_{L/F}(1)$ does not depends on the choice of L.) We need to show that c_i is a multiple of $h^i_*(e)$ (in the Chow group with coefficients in $\mathbb{Z}_{(p)}$).

By Theorem 3.7, c_i is a multiple of $h^i_*(e)$ provided that A is replaced by a generic central simple algebra of degree n (over a field extension of F). Indeed, for generic A, the Chow group with integer coefficients is torsion-free (by Theorem 3.7) and, by Remark 3.4, the image of $\operatorname{CH}^i X \otimes \mathbb{Z}_{(p)}$ in $\operatorname{CH}^i X_L \otimes \mathbb{Z}_{(p)} = \mathbb{Z}_{(p)}$ is generated by the image $[L : F] = \operatorname{ind} A$ of $h^i_*(e)$.

It follows by specialization that c_i is a multiple of $h^i_*(e)$ for our initial A as well. □

Here is the result serving the case of $G = \operatorname{SL}_n / \mu_{n/2}$:

Proposition 3.12. *Let F be a field and let A be a central simple F-algebra such that the 2-primary part of its exponent is equal to the half of the 2-primary part of its index d (this implied that d is divisible by 4) and the index of the tensor power $A^{\otimes(d/4)}$ is divisible by 4. Assume that the Chow ring $\operatorname{CH} X$ of the Severi-Brauer variety X of A is generated (as a ring) by some element of codimension 1 and the Chern classes of the tautological vector bundle. Then the group $\operatorname{CH} X$ is 2-torsion-free.*

Remark 3.13. In the case where $d := \operatorname{ind} A = 2 \exp A$ and $4 \mid \operatorname{ind} A^{\otimes(d/4)}$, Proposition 3.12 provides a complete description of the ring $\operatorname{CH} X$. Indeed, for any $n \geq 1$ and any central simple F-algebra A of degree n, the kernel of the change of field homomorphism $\operatorname{CH} X \to \operatorname{CH} \mathbb{P}^{n-1} = \mathbb{Z}[H]/(H^n)$, given by any splitting field of the algebra, is the torsion subgroup of $\operatorname{CH} X$. Moreover, if $\exp A = d/2$, where $d := \operatorname{ind} A$, and $4 \mid \operatorname{ind} A^{\otimes(d/4)}$, then for any $0 \leq j \leq n-1$ and any prime integer p, the p-adic valuation of a generator of the image of $\operatorname{CH}^j X$ in $\operatorname{CH}^j \mathbb{P}^{n-1} = \mathbb{Z}$ is determined as follows: for odd p it is $v_p(d/(j,d))$; for $p = 2$ it is $v_2(d/(j,d))$ provided that $v_2(j-1) < v_2(d)$ and it is $v_2(d) - 1$ otherwise. This is a consequence of Remark 3.4 (for odd p) and of [9, proof of Proposition 4.9] (for $p = 2$), since by the proof of Lemma 3.5 we only need to consider the case where d is a p-power.

Proof of Proposition 3.12. We obtain a proof of Proposition 3.12 appropriately modifying the proof of Proposition 3.2. Let n be the degree of A. For $i \geq 2$, let $x_i \in \operatorname{CH}^i X$ be the

ith Chern class of the tautological vector bundle on X. As a ring, $\operatorname{CH} X$ is generated by some element $x_1 \in \operatorname{CH}^1 X$ and the elements $x_i \in \operatorname{CH}^i X$, $i = 1, \dots, \dim X = n-1$.

For the remainder of the proof, we switch to the Chow groups with coefficients in $\mathbb{Z}_{(2)}$ – the localization of $\mathbb{Z}$ in the prime ideal generated by 2. To prove Proposition 3.12, it suffices to show that the group $\operatorname{CH} X \otimes \mathbb{Z}_{(2)}$ is torsion-free.

Let 2^r be the 2-primary part of $d = \operatorname{ind} A$. Recall that d is divisible by 4, that is to say, $r \geq 2$. By Lemma 3.5, we only need to check that $\operatorname{CH}^j X \otimes \mathbb{Z}_{(2)}$ is torsion-free for $j < 2^r$.

Let L/F be a finite Galois field extension splitting A. Let L_r be the intermediate field corresponding to a 2-Sylow subgroup of $\operatorname{Gal}(L/F)$ so that $[L_r : F]$ is odd and $[L : L_r]$ is a 2-power. Let L_0 be a minimal subfield of L containing L_r and splitting A. We have $[L_0 : L_r] = 2^r$. By [5, Theorem 4.2.1], there is a chain of subfields

$$L_r \subset L_{r-1} \subset \cdots \subset L_0$$

with $[L_{i-1} : L_i] = 2$ for every $i = r, \dots, 1$. Note that $\operatorname{ind} A_{L_i} = 2^i$ for $i = 0, 1, \dots, r$.

We claim that for any $j = 2, \dots, 2^r - 1$, the norm map

$$N_i^j : \operatorname{CH}^j X_{L_i} \otimes \mathbb{Z}_{(2)} \to \operatorname{CH}^j X \otimes \mathbb{Z}_{(2)}$$

is surjective, where $i = v_2(j)$ and v_2 is the 2-adic valuation. In contrast with the proof of Proposition 3.2, where exponent of A was equal to the index of A, not to its half, the norm map N_0^1 is not surjective; moreover, none of the maps $N_1^1, \dots, N_{r-1}^1$ is surjective. However, and this will be used in the proof below, the image of the change of field homomorphism $\operatorname{CH}^1 X \otimes \mathbb{Z}_{(2)} \to \operatorname{CH}^1 X_{L_{r-1}} \otimes \mathbb{Z}_{(2)}$ coincides with the image of the norm map $N_{L_0/L_{r-1}} : \operatorname{CH}^1 X_{L_0} \otimes \mathbb{Z}_{(2)} \to \operatorname{CH}^1 X_{L_{r-1}} \otimes \mathbb{Z}_{(2)}$. This is so because the change of field homomorphism $\operatorname{CH}^1 X \to \operatorname{CH}^1 X_L = \mathbb{Z}$ is injective and its image is generated by the integer $\exp A$, [1, §2].

Since $\operatorname{ind} A_{L_i} = 2^i$ divides j, we have $\operatorname{CH}^j X_{L_i} = \mathbb{Z}$ (by [7, Corollary 1.3.2]). More precisely, $\operatorname{CH}^j X_L = \operatorname{CH}^j \mathbb{P}^{n-1} = \mathbb{Z}$, where $1 \in \mathbb{Z}$ corresponds to the class in $\operatorname{CH}^j \mathbb{P}^{n-1}$ of a linear subspace in $\mathbb{P}^{n-1}$ of codimension j, and the change of field homomorphism $\operatorname{CH}^j X_{L_i} \to \operatorname{CH}^j X_L$ is an isomorphism. Therefore the claim implies that $\operatorname{CH}^j X \otimes \mathbb{Z}_{(2)}$ is torsion-free.

We prove the claim by induction on j. Given an arbitrary j with $2 \leq j \leq 2^r - 1$, we assume that the claim holds in codimensions $2, \dots, j-1$. We first check that every element of $\operatorname{CH}^j X \otimes \mathbb{Z}_{(2)}$ which is a polynomial in $x_1, \dots, x_{j-1}$ (without x_j), is in the image of the norm map N_i^j. It suffices to consider the case where the polynomial is a monomial. Since the degree j of the monomial is not divisible by 2^{i+1}, the monomial contains the factor x_k for some $k \in \{1, \dots, j-1\}$ not divisible by 2^{i+1}. If $k \neq 1$, then it follows by the induction hypothesis that x_k is in the image of N_i^k; therefore, by the projection formula, the monomial is in the image of N_i^j.

Now assume that $k = 1$. There is at least one more factor x_l with some $l \in \{1, \dots, j-1\}$. If $l \neq 1$, it follows by the induction hypothesis that x_l is in the image of N_{r-1}^l (our assumption that $j < 2^r$ ensures that l is not divisible by 2^r) so that $x_1 x_l = N_{r-1}^l(x_{1L_{r-1}} y)$ for some $y \in \operatorname{CH}^l X_{L_{r-1}}$. Since $x_{1L_{r-1}}$ is in the image of the norm map $N_{L_0/L_{r-1}}$, the product $x_1 x_l$ is in the image of N_0^{l+1} (and therefore in the image of N_i^{l+1} for any i).

It remains to consider the case with $l = 1$. We show that x_1^2 is in the image of N_0^2. The Chow group $\mathrm{CH}^2 X$ coincides with the quotient $K(X)^{(2)}/K(X)^{(3)}$ of the second term of the topological filtration on the Grothendieck ring $K(X)$ by the third term. The second term of the topological filtration coincides with the second term of the gamma filtration. The third topological term contains the third gamma term and the quotient consists of torsion elements, see [9, Proposition 2.14]. Since $4 \mid \operatorname{ind} A^{\otimes(d/4)}$, the quotient of the second gamma term by the third gamma term is torsion-free by [9, Proposition 4.9 with Lemma 3.10] and the proof of Lemma 3.5. It follows that the third gamma term coincides with the third topological term. In particular, the quotient of the topological terms is torsion-free. Therefore the group $\mathrm{CH}^2 X$ is torsion-free as well. So, by Remark 3.13, it is identified with $2^{r-1}\mathbb{Z} \subset \mathbb{Z} = \mathrm{CH}^2 X_{L_0}$. The image of the norm map N_0^2 is $2^r\mathbb{Z}_{(2)}$. And $x_1^2 = 2^{2r-2}$. Since $r \geq 2$, we have $2r - 2 \geq r$ showing that x_1^2 is indeed in the image of N_0^2.

To finish the proof of the claim (and therefore the proof of Proposition 3.12), it suffices to check that x_j is also in the image of N_i^j. For odd j, this holds by Corollary 3.11 (we recall that x_j is the jth Chern class of the tautological vector bundle). For even j, we decompose the element $N_i^j(1) \in \mathrm{CH}^j X$ in a linear combination of the degree-j monomials in $x_1, x_2, \dots, x_j$ and check that the coefficient $\lambda \in \mathbb{Z}_{(2)}$ at the monomial x_j is invertible.

Let us observe that $v_2(N_i^j(1)_L) = v_2([L_i : F]) = r - i$. On the other hand, if λ is not invertible, then $(\lambda x_j)_L$ is divisible by 2^{r-i+1} because x_L is divisible by 2^{r-i} for any element $x \in \mathrm{CH}^j X$, see Remark 3.13. Also M_L is divisible by 2^{r-i+1} for any monomial $M \in \mathrm{CH}^j X$ in $x_1, \dots, x_{j-1}$ because M contains x_k with some k not divisible by 2^{i+1}: x_{kL} is then divisible by 2^{r-i} (even if $k = 1$ – because $i \geq 1$ since j is even); in the same time M necessarily contains another factor x_l with some $l = 1, \dots, j-1$ ($l = k$ is also possible). Our assumption that $j < 2^r$ ensures that l is not divisible by 2^r so that x_{lL} is divisible by 2. □

Proof of Theorem 3.1. Let A be the central simple F-algebra corresponding to the generic G-torsor E. By Lemma 3.6, we may assume that E/P is the Severi-Brauer variety X of A. By [6, Proof of Theorem 1.1], the ring $\mathrm{CH}\, X$ is generated by $\mathrm{CH}^1 X$ and the Chern classes of the tautological vector bundle. This, in particular, implies that the topological filtration on $K(X)$ coincides with the gamma filtration.

We start by assuming that the condition $(m, n/m) \leq 2$ fails. Then the integer $(m, n/m)$ is divisible by an odd prime number p or by 4. In the first case, let us show that the group $\mathrm{CH}^2(E/P)$ has an element of order p. The group $\mathrm{CH}^2 X$ is isomorphic to the quotient $K(X)^{(2)}/K(X)^{(3)}$ of the topological filtration on the Grothendieck group $K(X)$. Let L/F be a finite extension of degree prime to p such that the index of the L-algebra A_L is a p-power. Note that $\operatorname{ind} A_L = p^{v_p(n)}$ and $\exp A_L = p^{v_p(m)}$ so that $\exp A_L < \operatorname{ind} A_L$. The change of field homomorphism $K(X) \otimes \mathbb{Z}_{(p)} \to K(X_L) \otimes \mathbb{Z}_{(p)}$ is an isomorphism of rings with filtrations. The topological filtration on $K(X_L) \otimes \mathbb{Z}_{(p)}$ coincides with the gamma filtration. By [9, Proposition 4.7], the 2nd quotient of the gamma filtration on $K(X_L)$ has an element of order p. So, we get an element of order p in $\mathrm{CH}^2 X$.

Let now assume that 4 divides $(m, n/m)$ and prove that $\mathrm{CH}^2(E/P)$ has an element of order 2. We proceed as above and come to a 2-primary algebra A_L with $\exp A_L < (\operatorname{ind} A_L)/2$. By [9, Proposition 4.9], the 2nd quotient of the gamma filtration on $K(X_L)$ has an element of order 2. So, we get an element of order 2 in $\mathrm{CH}^2 X$.

Finally, let us assume that $(m, n/m) \leq 2$. For an arbitrary prime number p we claim that the p-torsion of $\operatorname{CH} X$ is trivial. If $v_p(m) = 0$, then p does not divide the index of A so that the claim is obvious. Below we assume that $v_p(m) > 0$ in which case $v_p(m) = v_p(n)$ or $p = 2$ and $v_2(m) = v_2(n) - 1$.

If $v_p(m) = v_p(n)$, Proposition 3.2 does the job.

If $p = 2$ and $v_2(m) = v_2(n) - 1$, we are done by Proposition 3.12. Indeed, by [9, Lemma 3.10], there exists a central simple algebra A (over a field extension of k) of degree n and exponent m, satisfying the condition $4 \mid \operatorname{ind} A^{\otimes(d/4)}$ of Proposition 3.12, where $d := \operatorname{ind} A$. Therefore any generic algebra of degree n and exponent m satisfies this condition. □

The following statement is an application proved similarly to Corollaries 3.9 and 3.10:

Corollary 3.14. *Let X be the Severi-Brauer variety of a central simple k-algebra A such that $d := \operatorname{ind} A = 2 \exp A$ and $4 \mid \operatorname{ind} A^{\otimes(d/4)}$. Then the torsion subgroup* $\operatorname{Tors} \operatorname{CH} X$ *splits off canonically as a direct summand of* $\operatorname{CH} X$*. Besides, the topological filtration on the Grothendieck ring $K(X)$ coincides with the gamma filtration. Moreover, for any finite product Y of any generalized Severi-Brauer varieties of any tensor powers of A, the topological filtration on the Grothendieck ring $K(X_{k(Y)})$ coincides with the gamma filtration.* □

4. Type C_n

A split simple group G over k of type C_n ($n \geq 1$) is isomorphic to Sp_{2n} (the simply connected case) or PGSp_{2n} (the adjoint case). The group Sp_{2n} is special. For this reason, we only treat the adjoint case $G = \mathrm{PGSp}_{2n}$ here below.

The set of isomorphism classes of G-torsors over k is identified with the set of isomorphism classes of central simple k-algebras of degree $2n$ endowed with a symplectic involution. Let E be a G-torsor over k and let A be a corresponding k-algebra. Since A possesses a k-linear involution, the exponent of A is 2 or A is split. The index of A is a 2-power, a divisor of the 2-primary part of $2n$. If E is a generic G-torsor (over $F \supset k$), then $\exp A = 2$ and $\operatorname{ind} A$ is the 2-primary part of $2n$.

Let $P \subset G$ be a parabolic subgroup of type C_{n-1}. Then P is special and the variety E/P can be viewed as the variety of isotropic right ideals in A of reduced dimension 1. But every right ideal of reduced dimension 1 is isotropic with respect to any symplectic involution on A, therefore E/P, which is a priori a closed subvariety in the Severi-Brauer variety $\mathrm{SB}(A)$, coincides with $\mathrm{SB}(A)$.

If n is not divisible by 4, then $\operatorname{ind} A$ divides 4 and it follows that the group $\operatorname{CH} X$ of $X := \mathrm{SB}(A)$ is torsion-free. In more details, $\operatorname{CH} X$ is a direct sum of shifted copies of $\operatorname{CH} X'$, where X' is the Severi-Brauer variety of a degree-4 central simple algebra A' Brauer-equivalent to A. For $i \leq 2$ the group $\operatorname{CH}^i X'$ coincides with the ith quotient of the topological filtration on $K(X')$ which is torsion-free (for $i = 2$, see, e.g., [9, Proposition 4.9]). The group $\operatorname{CH}^3 X' = \operatorname{CH}_0 X'$ is torsion-free by [2] (originally proved in [13]).

For any n and generic E (over $F \supset k$), it follows by Corollary 3.10 and specialization that the topological filtration on $K(X)$ coincides with the gamma filtration. Indeed, over a suitable field extension k''/k, there exists a central division algebra A'' with $2n = \deg A'' = \operatorname{ind} A'' = \exp A''$. Taking for Y in Corollary 3.10 the Severi-Brauer variety of the tensor square of A'' and setting $k' := k''(Y)$, $A' := A''_{k'}$, we get that for $X' := \mathrm{SB}(A')$, the topological filtration on $K(X')$ coincides with the gamma filtration. By the index reduction formula for Severi-Brauer varieties [17] (see also [12, (5.11)]]), the index of the algebra A' is the 2-primary part of $2n$ and its exponent is 2. In particular, A' admits a symplectic involution, [11, Theorem 3.1(1) and Corollary 2.8(2)]. The pair, consisting of the algebra with a fixed symplectic involution on it, is given by a G-torsor E' over k'. Using specialization, we identify $K(X)$ with $K(X')$. Under this identification, the gamma filtration on $K(X)$ is identified with the gamma filtration on $K(X')$ while each term of the topological filtration on $K(X)$ is identified with a subgroup of the corresponding term of the topological filtration on $K(X')$. Since each term of the topological filtration on $K(X)$ contains the corresponding term of the gamma filtration, both filtrations on $K(X)$ coincide.

By [9, Proposition 4.9], if n is divisible by 4, the second quotient of the gamma filtration contains an element of order 2. We have proven

Theorem 4.1. *For $G := \mathrm{PGSp}_{2n}$ ($n \geq 1$) over any field k, let $P \subset G$ be a special parabolic subgroup and let E be a generic G-torsor over a field extension F/k. The group $\mathrm{CH}(E/P)$ is torsion-free if and only if n is not divisible by 4. Moreover, if n is divisible by 4, the group $\mathrm{CH}^2(E/P)$ contains an element of order 2.* □

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Mathematical & Statistical Sciences, University of Alberta, Edmonton, CANADA
E-mail address: `karpenko` *at* `ualberta.ca`, *web page*: `www.ualberta.ca/~karpenko`