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A criterion of decomposability for degree 4 algebras with unitary involution

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Abstract

Let A be a degree 4 central simple algebra endowed with a unitary involution σ. We prove that (A, σ) is decomposable if and only if its discriminant (i.e. the Brauer class of its discriminant algebra) is trivial.

MSC: 16K20; 16W10; 11E88

0. Introduction

Let K be an arbitrary field. A central simple K-algebra A endowed with an involution σ (not necessarily trivial on K) is said to be decomposable if A is tensor product over K of two proper subalgebras stable under σ.

The study of the decomposability of algebras with involution is a classical question (see e.g. [3, 12]). Let us assume that the algebra A is of degree 4. Rowen [11, Theorem B] proved that in the symplectic case (A, σ) is always decomposable. In the orthogonal case, Knus et al. [7, Theorem 3.1] (see also [6, Corollary 5.4]) gave a

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criterion of decomposability using the classical notion of *discriminant* of an orthogonal involution (also called *pfaffian discriminant*): (A,σ) is decomposable if and only if the discriminant of σ is trivial.

The purpose of this paper is to give an analogous criterion in the unitary case using the *discriminant algebra* of (A,σ) introduced in [5] (see Theorem 3.1). When the base field is of characteristic different from 2, the criterion can also be formulated in terms of the *determinant class modulo* 2, an invariant of unitary involutions introduced in [10] (see Remark 3.2).

1. Discriminant algebra, discriminant and determinant class modulo 2

Let A be a central simple K-algebra endowed with an involution σ. The involution is said to be of the first kind if it is trivial on K and of the second kind, or of unitary type, otherwise. We refer the reader to [13, Chap. 8] or [5, Chap. 1] for more detailed information on involutions.

To deal with the characteristic 2 case, we need to replace orthogonal involutions by quadratic pairs. One will find in [5, Section 5] an explanation of this fact and the definition of a quadratic pair. In characteristic $\neq 2$, this notion is equivalent to the notion of orthogonal involution.

We assume in this section that A is of even degree $n=2m$ and that σ is of unitary type. We denote by k the subfield of K fixed by σ. Thus K/k is a separable quadratic extension.

In [5, Section 10], Knus et al. associate to (A,σ) a degree $\binom{n}{m}$ central simple k-algebra $\mathscr{D}(A,\sigma)$ called the *discriminant algebra* of (A,σ). If m is even, then the discriminant algebra is endowed with a quadratic pair.

The Brauer class $[\mathscr{D}(A,\sigma)]$ of the discriminant algebra will be called the *discriminant* of (A,σ). This terminology is motivated by the following remark. If the exponent of A divides m, then the algebra $\mathscr{D}(A,\sigma)\otimes_k K$ is split. Indeed, it is isomorphic by [5, Proposition 10.30] to $\lambda^m A$, which is Brauer equivalent to $A^{\otimes m}$. Hence, in this case, the discriminant of (A,σ) is an element of the relative Brauer group $\mathrm{Br}(K/k)$ which is canonically isomorphic to the quotient $k^*/N_{K/k}(K^*)$, where $N_{K/k}$ is the norm homomorphism.

If the base field has characteristic different from 2, the second named author attached to (A,σ) an invariant called *determinant class modulo* 2, which can be defined either using Galois cohomology, or via the trace forms of (A,σ) (see [10] or [9]). This invariant, denoted by $D(A,\sigma)$, takes values in the 2-part $\mathrm{Br}_2(k)$ of the Brauer group of k.

It follows from [5, Section 11] that (in the characteristic $\neq 2$ case) the Brauer class of $\mathscr{D}(A,\sigma)$ is $D(A,\sigma)+m[(-1,\alpha)]$, where α is an element of k^* such that $K=k(\sqrt{\alpha})$ and where $[(-1,\alpha)]$ is the Brauer class of the quaternion k-algebra $(-1,\alpha)$. In particular, if the degree of A is divisible by 4, then the determinant class modulo 2 coincides with the discriminant of (A,σ).

2. Clifford algebra

Let k be a field (of arbitrary characteristic) and let (V, q) be a non-degenerate quadratic space over k (see [4, Definition 1.5 of Chap. I] or [13, Definition 6.9 of Chap. 1]). We recall that the *Clifford algebra* $C(q)$ of the quadratic space (V, q) is by definition the tensor algebra of the vector space V modulo the two-sided ideal generated by the products $v \otimes v - q(v)$, where v ranges over V. For the properties of the Clifford algebra listed below we refer to [13, Chap. 9], [4, Chap. 2], and [5, Section 8.A].

First of all $C(q)$ is a $\mathbb{Z}/2$-graded algebra; its 0th component is a subalgebra called *even Clifford algebra* and denoted by $C_0(q)$. The dimension of $C(q)$ over k equals 2^n and the dimension of $C_0(q)$ is 2^{n-1}, where $n = \dim_k V$.

We can consider any vector $v \in V$ as an element of $C(q)$. Clearly, the algebra $C(q)$ is generated by these elements v $(v \in V)$, while $C_0(q)$ is the subalgebra generated by the products $v \cdot v'$ $(v, v' \in V)$. Moreover, if $e_1, \ldots, e_n \in V$ is a basis of the vector space V, then the elements e_i $(1 \le i \le n)$ generate $C(q)$ while the products $e_i \cdot e_j$ generate $C_0(q)$. The generators $e_i \in C(q)$ are subject to the relations $e_i^2 = q(e_i)$ $(1 \le i \le n)$ and $e_i e_j + e_j e_i = b_q(e_i, e_j)$ $(1 \le i, j \le n, i \ne j)$, where b_q is the bilinear form associated to q (see [4, Definition 1.5 of Chap. I]). If $\operatorname{char} k \ne 2$, then there exists an orthogonal basis $e_1, \ldots, e_n$ of (V, q) and the previous relations become $e_i^2 = q(e_i)$ and $e_i e_j = -e_j e_i$.

There is a unique involution on $C(q)$ which leaves all the elements v invariant. It is called the *canonical involution* on the Clifford algebra.

The definition of the *even* Clifford algebra for a quadratic space can be generalized to the case of a central simple algebra B endowed with a quadratic pair τ (see [5, Section 8.B] for the definition). What is obtained in this way is denoted by $C(B, \tau)$ and called the *Clifford algebra* of (B, τ). Note that $C(B, \tau)$ is an algebra endowed with an involution. It generalizes the classical notion of even Clifford algebra in the following sense:

Proposition 2.1 (Knus et al. [5, Proposition 8.8]). *Let (V, q) be a quadratic space over k (of even dimension if $\operatorname{char} k = 2$). Let $B = \operatorname{End}_k(V)$ and let τ be the quadratic pair associated to q. Then there is an isomorphism of algebras with involutions $C(B, \tau) \simeq C_0(q)$.*

Denote by $\mathscr{B}$ the set of isomorphism classes of degree 6 central simple k-algebras endowed with a quadratic pair. Let $\mathscr{A}_1$ be the set of isomorphism classes of central simple k-algebras of degree 4 over a quadratic field extension of k endowed with a unitary involution trivial on k. Let $\mathscr{A}_2$ be the set of isomorphism classes of algebras $A_2 \times A_2^{op}$ endowed with the exchange involution, where A_2 is a central simple k-algebra of degree 4 and A_2^{op} its opposite algebra. We put $\mathscr{A} = \mathscr{A}_1 \cup \mathscr{A}_2$. A proof of the following theorem in the characteristic $\ne 2$ case is given also in [8, Proposition 4].

Theorem 2.2 (Knus et al. [5, Section 15.D]). *The map $\mathscr{B} \to \mathscr{A}$ given by $(B,\tau) \mapsto C(B,\tau)$ is a bijection. The inverse map $\mathscr{A} \to \mathscr{B}$ is given by $(A,\sigma) \mapsto \mathscr{D}(A,\sigma)$ (see* [5, *Section* 10] *for a definition of $\mathscr{D}$ on $\mathscr{A}_2$).*

3. The criterion

Let K be a field, and let A be a central simple algebra of degree 4 over K endowed with an involution σ of unitary type. Let $k \subset K$ be the subfield of σ-invariant elements of K. Here is the criterion:

Theorem 3.1. *The algebra with involution (A,σ) is decomposable if and only if its discriminant is trivial.*

Remark 3.2. Assume that $\operatorname{char} K \neq 2$. As we already noticed it in Section 1, the discriminant of (A,σ) coincides with $D(A,\sigma)$. Hence, (A,σ) is decomposable if and only if its determinant class modulo 2 is trivial.

Remark 3.3. If the exponent of A is equal to 4, then the algebra itself cannot be decomposed into a tensor product of two proper subalgebras over K. Hence, we may assume that the exponent of A divides 2. As mentioned in Section 1, the discriminant of (A,σ) is then represented by an element of k^*, determined modulo $N_{K/k}(K^*)$.

Remark 3.4. The criterion is also true for the algebra $A = A_2 \times A_2^{op}$ with the exchange involution, where A_2 is a central simple k-algebra of degree 4. In this situation, it is equivalent to a theorem of Albert [1, p. 369] (see also [5, Section 16.1]).

Proof of Theorem 3.1. Let K, A, σ, and k be as introduced in the beginning of this section. We reproduce here a proof of the "only if" part which is already contained in [10, 9] but slightly modified so that it covers the characteristic 2 case. It relies on the following lemma due to Albert:

Lemma 3.5 (Albert, see [2, Theorem 10.21] or Knus et al. [5, Proposition 2.22]). *Let Q be a quaternion algebra over K, endowed with a unitary involution σ_Q trivial on k. Then there exists a quaternion k-algebra Q_0 such that $Q = Q_0 \otimes_k K$ and $\sigma_Q = \tau \otimes \bar{\ }$, where τ is the canonical involution on Q_0 and $\bar{\ }$ the non-trivial automorphism of K/k.*

Let us assume that the algebra with involution (A,σ) is decomposable. Since A is of degree 4, it actually decomposes as a tensor product of two quaternion algebras. Hence, because of Albert's lemma, there exist two quaternion algebras Q_1 and Q_2 over k such that $A = (Q_1 \otimes_k Q_2) \otimes_k K$, and $\sigma = (\tau_1 \otimes \tau_2) \otimes \bar{\ }$, where τ_i is the canonical involution on Q_i. Therefore, it follows from [5, Proposition 10.33] that the discriminant of (A,σ), which lies in $k^*/N_{K/k}(K^*)$ (see Remark 3.3), is given by the discriminant of the orthogonal involution $\tau_1 \otimes \tau_2$ (see also [9, Proposition 11]). By the decomposability criterion of Knus–Parimala–Sridharan, the involution $\tau_1 \otimes \tau_2$ has trivial discriminant and we get $[\mathscr{D}(A,\sigma)] = 0$.

Before proving the converse, let us prove the following lemma:

Lemma 3.6. *Let (V,q) be a six-dimensional quadratic k-space. There is an isomorphism of algebras with involutions $C_0(q) \simeq Q_1 \otimes_k Q_2 \otimes_k K$ where Q_1 and Q_2 are quaternion k-algebras, K is a quadratic étale k-algebra, and all three are endowed with their canonical involution.*

Proof. We break up the proof into two cases: $\operatorname{char} k = 2$ and $\operatorname{char} k \neq 2$. The proof we give in characteristic 2 was suggested by Wadsworth. Let us assume first that $\operatorname{char} k \neq 2$. Let $e_1, \dots, e_6$ be any orthogonal basis of V, and let $a_i = q(e_i)$. As explained in Section 2, the Clifford algebra $C = C(q)$ is generated by the elements $e_1, e_2, \dots, e_6$, which are subject to the relations $e_i e_j = -e_j e_i$ and $e_i^2 = a_i$ for all i, j with $i \neq j$. The canonical involution on C, which we denote by τ, is determined by $\tau(e_i) = e_i$ $(i = 1, \dots, 6)$. The even Clifford algebra $C_0 = C_0(q)$ is the subalgebra of C generated by the pairwise products of $e_1, \dots, e_6$; the canonical involution τ_0 on C_0 is the restriction of τ to C_0. In particular, we have $\tau_0(e_i e_j) = -e_i e_j$ (for $i \neq j$).

Consider the étale k-algebra $K = k[t]/(t^2 + a_1 \dots a_6)$ and the quaternion k-algebras $Q_1 = (-a_1 a_2, -a_1 a_3)$, $Q_2 = (-a_4 a_5, -a_4 a_6)$. By definition, Q_1 is generated by elements i_1 and j_1 subject to the relations $i_1^2 = -a_1 a_2$, $j_1^2 = -a_1 a_3$, and $i_1 j_1 = -j_1 i_1$. The canonical involution on Q_1 is determined by $i_1 \mapsto -i_1$, $j_1 \mapsto -j_1$. Analogously, Q_2 is generated by i_2 and j_2 subject to the relations $i_2^2 = -a_4 a_5$, $j_2^2 = -a_4 a_6$, and $i_2 j_2 = -j_2 i_2$. The canonical involution on Q_2 is determined by $i_2 \mapsto -i_2$, $j_2 \mapsto -j_2$. We consider the k-algebra K with the involution $t \mapsto -t$. One can easily check that the following three homomorphisms of k-algebras:

$$\begin{array}{lll} Q_1 \to C_0, & Q_2 \to C_0, & K \to C_0 \\ i_1 \mapsto e_1 e_2 & i_2 \mapsto e_4 e_5 & t \mapsto e_1 \dots e_6 \\ j_1 \mapsto e_1 e_3 & j_2 \mapsto e_4 e_6 & \end{array}$$

are well-defined homomorphisms of algebras with involutions and that their images commute. Hence, they induce a homomorphism

$$\psi : Q_1 \otimes_k Q_2 \otimes_k K \to C_0.$$

If $K \neq k \times k$, the algebra $Q_1 \otimes_k Q_2 \otimes_k K$ is simple. If $K = k \times k$, it is not simple anymore, but one can easily check it is indeed simple as an algebra with involution, in the sense it has no non-trivial σ-stable ideal. Since ψ is a homomorphism of algebras with involution, this proves it is injective in both cases. By a dimension argument, we get that ψ is an isomorphism.

Assume now that $\operatorname{char} k = 2$. The quadratic space (V,q) can be decomposed into an orthogonal sum $(V,q) = [a_1, a_1'] \perp [a_2, a_2'] \perp [a_3, a_3']$, where $a_i, a_i' \in k^*$ and $[a, a']$ denotes the quadratic form $ax^2 + xy + a'y^2$ (see [4, Proposition 3.4 of Chap. I] or [12, Section 4 of Chap. 9]). Let $e_1, e_1', e_2, e_2', e_3, e_3'$ be the corresponding basis of V. As explained in Section 2, the Clifford algebra $C = C(q)$ is generated by the elements $e_1, e_1', e_2, e_2', e_3, e_3'$, which are subject to the relations $e_ie_j + e_je_i = 0$, $e_ie_j' + e_j'e_i = 0$, $e_i'e_j' + e_j'e_i' = 0$, $e_ie_i' + e_i'e_i = 1$, $e_i^2 = a_i$, and $e_i'^2 = a_i'$ for all i, j with $i \neq j$. The canonical involution on C, which we denote by τ, is determined by $\tau(e_i) = e_i$ and $\tau(e_i') = e_i'$ $(i = 1, 2, 3)$. The even Clifford algebra $C_0 = C_0(q)$ is the subalgebra of C generated by the pairwise products of $e_1, e_1', e_2, e_2', e_3, e_3'$; the canonical involution τ_0 on C_0 is the restriction of τ to C_0.

Consider the étale k-algebra $K = k[t]/(t^2 + t + a_1a_1' + a_2a_2' + a_3a_3')$ and the quaternion k-algebras $Q_1 = [a_1a_1', a_1a_3)$, $Q_2 = [a_2a_2', a_2a_3)$. By definition (see [13, Section 11 of Chap. 8] or [5, Section 2.C]), Q_1 is generated by elements u_1 and v_1 subject to the relations $u_1^2 + u_1 = a_1a_1'$, $v_1^2 = a_1a_3$, and $u_1v_1 = v_1u_1 + v_1$. The canonical involution on Q_1 is determined by $u_1 \mapsto u_1 + 1$, $v_1 \mapsto v_1$. Analogously, Q_2 is generated by u_2 and v_2 subject to the relations $u_2^2 + u_2 = a_2a_2'$, $v_2^2 = a_2a_3$, and $u_2v_2 = v_2u_2 + v_2$. The canonical involution on Q_2 is determined by $u_2 \mapsto u_2 + 1$, $v_2 \mapsto v_2$. We consider the k-algebra K with the involution $t \mapsto t + 1$. A straightforward computation shows that the homomorphisms of k-algebras

$$\begin{array}{lll} Q_1 \to C_0, & Q_2 \to C_0, & K \to C_0 \\ u_1 \mapsto e_1e_1' & u_2 \mapsto e_2e_2' & t \mapsto e_1e_1' + e_2e_2' + e_3e_3' \\ v_1 \mapsto e_1e_3 & v_2 \mapsto e_2e_3 & \end{array}$$

are well-defined homomorphisms of algebras with involutions and that their images commute. Hence, they induce a homomorphism

$$\psi : Q_1 \otimes_k Q_2 \otimes_k K \to C_0.$$

By the same argument as above, this homomorphism ψ is an isomorphism. □

Let us now end the proof of Theorem 3.1. Since the degree of A is 4 (in particular, divisible by 4), the algebra $\mathscr{D}(A, \sigma)$ is endowed with a quadratic pair τ. We assume that the discriminant algebra of (A, σ) is split, i.e. isomorphic to $\mathrm{End}_k(V)$ for some six-dimensional k-vector space V. Thus τ is the quadratic pair associated to some quadratic form q on V (see [13, Theorem 7.4 of Chap. 8] for the char $k \neq 2$ case and [5, Proposition 5.11] for the char $k = 2$ case). By Theorem 2.2, the algebra with involution (A, σ) is isomorphic to the Clifford algebra $C(\mathrm{End}_k(V), \tau_q)$; the latter is isomorphic to $C_0(q)$ by Proposition 2.1. By Lemma 3.6, the algebra with involution $C_0(q)$ is decomposable. □

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