A DESCENT OF MOTIVIC ISOMORPHISMS

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Let F be a field. By *motive* of a smooth projective algebraic variety over F we simply mean its classical integral Chow motive of Grothendieck.

In view of the example of a geometrically connected projective homogeneous variety without rational points, possessing a 0-cycle of degree 1, recently constructed by Parimala, the following observation seems to be of interest:

Theorem 0.1. *Let X and X' be projective homogeneous varieties over F (under some linear algebraic groups G and G', defined over F, which may coincide as well as be different from each other). Let Y be a geometrically connected smooth projective variety possessing a 0-cycle of degree 1.*

If over the function field $F(Y)$ the motives of the varieties $X_{F(Y)}$ and $X'_{F(Y)}$ are isomorphic, then the motives of X and X' are isomorphic already over F.

Proof. We write $\bar{F}$ for an algebraic closure of the field F. We refer to elements of Chow groups as to *cycles*. We choose a 0-cycle of degree 1 on Y and denote it by $\mathbf{pt} \in \mathrm{CH}_0(Y)$.

We assume that the motives of the varieties $X_{F(Y)}$ and $X'_{F(Y)}$ are isomorphic. Let $\alpha \in \mathrm{CH}((X \times X')_{F(Y)})$ be a cycle giving such an isomorphism (with CH staying for the total Chow group of the variety). Let $\beta \in \mathrm{CH}(X \times X' \times Y)$ be a cycle mapped to α under the surjective homomorphism

$$g^*\colon \mathrm{CH}(X \times X' \times Y) \to \mathrm{CH}((X \times X')_{F(Y)})$$

given by pull-back with respect to the morphism of F-schemes

$$g\colon (X \times X')_{F(Y)} \to X \times X' \times Y$$

obtained from the generic point morphism of Y by the base change by $X \times X'$. Using the multiplication of cycles on the smooth variety $X \times X' \times Y$, we multiply β by the external product $[X] \times [X'] \times \mathbf{pt} \in \mathrm{CH}(X \times X' \times Y)$. Finally, we set

$$\gamma = pr_*\big(\beta \cdot ([X] \times [X'] \times \mathbf{pt})\big) \in \mathrm{CH}(X \times X')\ ,$$

where

$$pr_*\colon \mathrm{CH}(X \times X' \times Y) \to \mathrm{CH}(X \times X')$$

is the push-forward homomorphism with respect to the projection pr of the product $(X \times X') \times Y$ onto the first factor.

In the remaining part of the proof we will show that $\gamma_{\bar{F}(Y)} = \alpha_{\bar{F}(Y)}$. In particular, $\gamma_{\bar{F}(Y)}$ gives a motivic isomorphism of X and X' over a field extension

Supported in part by IFCPAR project 2501-1 and by the European Community's Human Potential Programme under contract HPRN-CT-2002-00287, KTAGS.

of F (namely, over $\bar{F}(Y)$). Therefore, by [1, cor. 8.4] (which is a recent result on arbitrary projective homogeneous varieties generalizing an old result [4] of M. Rost on projective quadrics), γ gives a motivic isomorphism of X and X' already over F.

We are going to prove that $\gamma_{\bar{F}(Y)} = \alpha_{\bar{F}(Y)}$. In order to simplify the notation, we replace F by $\bar{F}$. We will check a more general relation, namely, for an arbitrary projective homogeneous variety P (replacing the product $X \times X'$) and for an arbitrary cycle $\beta \in \mathrm{CH}(P \times Y)$ (replacing the cycle β used in the above construction of γ), we show that

$$g^*(\beta) = \Big(pr_* \big(\beta \cdot ([P] \times \mathbf{pt})\big)\Big)_{F(Y)} ,$$

where $g\colon P_{F(Y)} \to P \times Y$ is the morphism given by the generic point of Y, while $pr\colon P \times Y \to P$ is the projection.

Since our base field F is now algebraically closed, the variety P is cellular as any projective homogeneous variety over an algebraically closed filed is (see, e.g., [1] or an earlier, may be original, proof given in [3]). Therefore the group $\mathrm{CH}(P \times Y)$ is generated by the external products of cycles on P and Y (see, e.g., [2, §6]), and it suffices to check the relation on β only for $\beta = \pi \times \zeta$ with some homogeneous cycle $\zeta \in \mathrm{CH}(Y)$ (and an arbitrary $\pi \in \mathrm{CH}(P)$). To do so, we consider two complementary cases: the case where the codimension of ζ is positive and the case where ζ is a multiple of $[Y]$.

We start with the second case, where we obviously may assume that $\zeta = [Y]$. Then $g^*(\beta) = \pi_{F(Y)}$. On the other hand, $\beta \cdot ([P] \times \mathbf{pt}) = \pi \times \mathbf{pt}$ and consequently $pr_* \big(\beta \cdot ([P] \times \mathbf{pt})\big) = \deg(\mathbf{pt}) \cdot \pi = \pi$. The second case is done.

In the first case, we clearly have $g^*(\beta) = 0$. In the same time,

$$\beta \cdot ([P] \times \mathbf{pt}) = \pi \times (\zeta \cdot \mathbf{pt}) = 0$$

simply because $\zeta \cdot \mathbf{pt} \in \mathrm{CH}_{<0}(Y) = 0$. Therefore the both sides of the equality under proof are 0. □

References

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