POINCARÉ DUALITY FOR TAUTOLOGICAL CHERN SUBRINGS OF ORTHOGONAL GRASSMANNIANS

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ABSTRACT. Let X be an orthogonal grassmannian of a nondegenerate quadratic form q over a field. Let C be the subring in the Chow ring $\mathrm{CH}(X)$ generated by the Chern classes of the tautological vector bundle on X. We prove Poincaré duality for C. For q of odd dimension, the result was already known due to an identification between C and the Chow ring of certain symplectic grassmannian. For q of even dimension, such an identification is not available.

1. Introduction

Let q be an arbitrary nondegenerate quadratic form over an arbitrary field F of an arbitrary dimension $d = \dim q$ (see [5, §7.A] for the definition of a nondegenerate quadratic form in any characteristic including 2). We take any integer m with $1 \le m \le d/2$ and consider the Chow ring $\mathrm{CH}(X_m)$ of the orthogonal grassmannian variety X_m of totally isotropic m-planes of q. Here we use the affine numbering of grassmannians so that X_1 is the projective quadric given by q.

Assume that the form q is split (it is automatically split if the field F is algebraically closed). Then the group $\mathrm{CH}(X_m)$ is a free abelian group of finite rank. It does not depend on the field F and has a canonical basis given by Schubert classes. The bilinear form $(a, b) \mapsto \deg(ab) \in \mathbb{Z}$ on $\mathrm{CH}(X_m)$, where $\deg\colon \mathrm{CH}(X_m) \to \mathbb{Z}$ is the degree map (vanishing on the homogeneous components of $\mathrm{CH}(X_m)$ other than the component $\mathrm{CH}_0(X_m)$ given by the 0-cycles), turns out to be unimodular in this split case. In other words, $\mathrm{CH}(X_m)$ satisfies Poincaré duality with respect to the bilinear form. Moreover, the Schubert basis is *weakly self-dual*, i.e., dual to a permutation of itself. The unimodularity can also be explained by the fact that the variety X_m is (absolutely) cellular (see [5, §66]). Therefore its Chow motive is split (see [5, Corollary 66.4]). The latter fact implies the Poincaré duality, c.f. [11, Remark 5.6] or [13, Proposition 1.5].

In general (for arbitrary F and nonsplit q), the change of field homomorphism

$$\mathrm{CH}(X_m) \to \mathrm{CH}((X_m)_K), \tag{1.1}$$

for a field extension K/F is neither injective nor surjective. Nevertheless, the ring $\mathrm{CH}(X_m)$ contains a "core" subring C, called the *tautological Chern subring*, generated by the

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Chern classes of the tautological (rank m) vector bundle on X_m. By [9, Theorem 2.1], the homomorphism (1.1) maps C isomorphically onto "itself" – the tautological Chern subring of $\mathrm{CH}((X_m)_K)$. In fact, C is determined by the integers d and m alone and does not depend on F and q. Note that the additive group of C is free of finite rank and that $\deg(C) = 2^m\mathbb{Z}$ (see, e.g., [9, Theorem 2.1] and [16, Statement 2.15]). By [9, Theorem 6.1], if the quadratic form q is *generic*, the subring C coincides with the entire Chow ring of X_m.

Not only the ring C itself, but also the restriction of the degree map $\deg\colon \mathrm{CH}(X_m) \to \mathbb{Z}$ to $C \subset \mathrm{CH}(X_m)$ is determined by d and m. We consider the bilinear form on C given by the modified degree map $2^{-m}\deg\colon C \to \mathbb{Z}$ and show that C satisfies Poincaré duality with respect to it:

Theorem 1.2. *The bilinear form*

$$C \times C \to \mathbb{Z}, \quad (a, b) \mapsto 2^{-m}\deg(a \cdot b)$$

is unimodular. In other terms, every $\mathbb{Z}$-basis for C admits a dual basis.

The idea of considering Poincaré duality for C originated in [15, §4], where it was proven for $m = [d/2]$ and then successfully applied in order to determine the torsion index of spin groups.

The unimodularity means invertibility of the determinant of the Gramm matrix. Therefore, replacing the coefficient ring $\mathbb{Z}$ by an arbitrary field k we get an equivalent version of Theorem 1.2:

Theorem 1.3. *For any field k, the bilinear form on the finite dimensional k-vector space $C \otimes k$, given by $2^{-m}\deg$, is nondegenerate.*

Nondegeneracy in Theorem 1.3 is the precise analogue of unimodularity in Theorem 1.2. Both mean that the induced map of the space to its dual is an isomorphism. But since the coefficient ring in Theorem 1.3 is a field, it is enough to only have injectivity of the map. In other terms, it is enough to check triviality of the radical.

For odd d, Theorems 1.2 and 1.3 have already been shown in [7, Proof of Theorem 4.1] by the argument of [15, §4] – using an identification of C with the entire Chow ring of the m-th symplectic grassmannian Y_m of a nondegenerate alternating bilinear form of dimension $\dim q - 1$. It also identifies the modified degree map on C with the usual degree map on $\mathrm{CH}(Y_m)$. The Schubert classes form a basis for the additive group of $\mathrm{CH}(Y_m)$ and this basis is weakly self-dual.

Possession of a self-dual basis is a stronger property than just the unimodularity. To prove the latter, one can argue (as we already did for $\mathrm{CH}(X_m)$ with split q) without involving the Schubert basis: since the variety Y_m is cellular, its Chow motive is split implying the integral Poincaré duality for $\mathrm{CH}(Y_m)$.

For even d however, an identification like above is not available (aside from the extreme values 1 and $d/2$ of m – see Examples 1.4 and 1.5) and therefore a different approach is needed. A proof of Theorem 1.3 for even d is given in §2.

Example 1.4. For even $d = 2n+2$ and split q, the quadric X_1 has dimension $2n$ and the Schubert basis for $\mathrm{CH}(X_1)$ consists of the powers h^i for $i = 0, 1, \ldots, n-1$ of the class h of a hyperplane section, the classes l_i of linear subspaces of dimensions $i = 0, 1, \ldots, n-1$, and two distinct classes l'_n and l''_n of linear subspaces of dimension n (see [5, §68]). The subring $C \subset \mathrm{CH}(X_1)$ has a weakly self-dual basis consisting of h^i for $i = 0, 1, \dots, 2n$. Note that $h^n = l'_n + l''_n$ and $h^{2n-i} = 2l_i$ for $i = 0, 1, \dots, n-1$. The ring C endowed with the modified degree map $2^{-1}\deg\colon C \to \mathbb{Z}$, is isomorphic to the entire Chow ring of the projective space $\mathbb{P}^{2n}$, endowed with its usual degree map.

Example 1.5. For $m = d/2$ (with even d) and split q, the variety X_m has two connected components isomorphic (to each other as well as) to the $(m-1)$-st grassmannian X'_{m-1} of any nondegenerate $(d-1)$-dimensional subform q' of q. Via this isomorphism, the tautological Chern subring of $\mathrm{CH}(X_m)$ together with its modified (by 2^m) degree map is identified with the tautological Chern subring of $\mathrm{CH}(X'_{m-1})$ endowed with its modified (by 2^{m-1}) degree map. As already discussed, the latter can be identified with the entire Chow ring of the corresponding symplectic grassmannian.

Theorems 1.2 and 1.3 for odd d are crucial for [7, Theorem 4.1] providing an algorithm for computation of indexes of generic $\mathrm{Spin}(d)$-grassmannians. A similar algorithm for even d is not available yet; we expect that our results will help to obtain it.

2. Proof of Theorem 1.3 for even d

Below we provide a proof of the Poincaré duality with coefficients in a field k for even $d = 2n$ by reduction to the duality for $d = 2n+1$.

We choose a nondegenerate $(2n+1)$-dimensional quadratic form q, containing a given nondegenerate $(2n)$-dimensional quadratic form q' as a subform. Writing X'_m for the m-th orthogonal grassmannian of q', we have a closed imbedding $in\colon X'_m \hookrightarrow X_m$. The tautological vector bundle $\mathcal{T}'$ on X'_m is the pull-back of the tautological vector bundle $\mathcal{T}$ on X_m. Therefore the pull-back homomorphism of Chow rings $in^*\colon \mathrm{CH}(X_m) \to \mathrm{CH}(X'_m)$ maps C surjectively onto C'. Note that by [9, Theorem 2.1], $in^*(c_{2n-m}(-\mathcal{T})) = 0$. Here and below we write $c_i(\mathcal{E})$ for the i-th Chern class of a vector bundle $\mathcal{E}$ and we write $c_i(-\mathcal{E})$ for the i-th Chern class of $-\mathcal{E}$ (or, in other terms, for the i-th Segre class of $\mathcal{E}$, see [6, Chapter 3]).

Lemma 2.1. *The class $[X'_m] \in \mathrm{CH}(X_m)$ of $X'_m \subset X_m$ coincides with $c_m(\mathcal{T})$.*

Proof. Let V be the vector space of definition of q and let $V' \subset V$ be the hyperplane on which q' is defined. The grassmannian $\Gamma_m(V')$ of m-planes in V' is a closed subvariety in the grassmannian $\Gamma_m(V)$ and $[\Gamma_m(V')] \in \mathrm{CH}(\Gamma_m(V))$ is the m-th Chern class of the tautological vector bundle on $\Gamma_m(V)$, [6, §14.7]. We have a pull-back square of closed embeddings:

$$\begin{array}{ccc} \Gamma_m(V') & \longrightarrow & \Gamma_m(V) \\ \uparrow & & \uparrow \\ X'_m & \longrightarrow & X_m \end{array}$$

Since $\dim \Gamma_m(V) - \dim \Gamma_m(V') = m = \dim X_m - \dim X'_m$, the statement follows from [5, Corollary 57.20]. □

In particular, it follows that $in_*(C') \subset C$: any element of C' has the form $in^*(x)$ with $x \in C$ and $in_*(in^*(x)) = c_m(\mathcal{T}) \cdot x \in C$ by projection formula.

The push-forward map $in_*\colon \operatorname{CH}(X'_m) \to \operatorname{CH}(X_m)$ satisfies $\deg \circ in_* = \deg'$ with

$$\deg'\colon \operatorname{CH}(X'_m) \to \mathbb{Z}$$

the degree map of X'_m.

We now switch to coefficients k considerations. As in Theorem 1.3, we assume that k is a field.

Let $in_* \otimes k$ be the additive homomorphism $C' \otimes k \to C \otimes k$ given by in_*. By projection formula, Poincaré duality with coefficients k for C' will follow from that for C once we know that $in_* \otimes k$ is injective. Indeed, given any nonzero $a' \in C' \otimes k$, we find $a \in C \otimes k$ non-orthogonal to $in_*(a')$ and get that a' is not orthogonal to $in^*(a)$.

Since $C' = in^*(C)$, the injectivity we need is implied by the following property of C alone (making no reference to C' anymore):

Proposition 2.2. *Any element $f \in C \otimes k$, vanishing under multiplication by $c_m(\mathcal{T})$, is divisible by $c_{2n-m}(-\mathcal{T})$.*

Proof. In the beginning of this proof we allow k to be an arbitrary commutative ring. As already mentioned in §1, C is identified with the Chow ring $\operatorname{CH}(Y_m)$ of the m-th grassmannian of a $2n$-dimensional nondegenerate alternating bilinear form b. Under this identification, the Chern classes in C of the tautological vector bundle on X_m correspond to the respective Chern classes in $\operatorname{CH}(Y_m)$ of the tautological vector bundle $'\mathcal{T}$ on Y_m.

We have $Y_m = G/P$ for certain parabolic subgroup $P \subset G := \operatorname{Sp}(2n)$. Namely, P is the standard maximal parabolic subgroup given by the m-th vertex of the Dynkin diagram of $\operatorname{Sp}(2n)$. Therefore $C \otimes k$ is identified with the coefficient k version

$$\operatorname{Ch}(G/P) := \operatorname{CH}(G/P) \otimes k$$

of the Chow ring.

Note that the group G is *special* meaning that every G-torsor over every extension field of F is trivial. In other terms, the torsion index of G is 1. It follows from [1, Proposition 20.5] that every parabolic subgroup of G is also special.

We are going to operate with the *Chow rings of the classifying spaces* of algebraic groups introduced in [14]. The Weyl group W of P acts on the integral Chow ring $\operatorname{CH}(\mathsf{B}T)$ of the classifying space $\mathsf{B}T$ of the standard split maximal torus $T \subset P$. Note that $\operatorname{CH}(\mathsf{B}T)$ is a ring of polynomials over $\mathbb{Z}$ in $n = \dim T$ variables. Namely, it is the symmetric $\mathbb{Z}$-algebra of the character group of T.

The ring of W-invariants $\operatorname{CH}(\mathsf{B}T)^W$ coincides with $\operatorname{CH}(\mathsf{B}P)$ (see [4, Proposition 6]) and therefore maps surjectively onto $\operatorname{CH}(G/P)$ (see, e.g., [8, Lemma 2.1]):

$$\operatorname{CH}(\mathsf{B}T)^W \twoheadrightarrow \operatorname{CH}(G/P). \tag{2.3}$$

Let $B \subset P \subset G$ be the standard Borel subgroup. The pull-back homomorphism $\operatorname{CH}(G/P) \to \operatorname{CH}(G/B)$ with respect to the projection $G/B \to G/P$ is injective (see, e.g., [3, Proof of Lemma 2.2]). The composition

$$\operatorname{CH}(\mathsf{B}T)^W \to \operatorname{CH}(G/P) \to \operatorname{CH}(G/B)$$

is the restriction of the characteristic map $\operatorname{CH}(\mathsf{B}T) \to \operatorname{CH}(G/B)$, studied in [2]. Since G has no torsion primes, it follows by [2, Corollaire 2] (together with [2, §8]) that the kernel of the characteristic map is the ideal generated by the image of

$$\mathrm{CH}^{>0}(BG) := \bigoplus_{i>0} \mathrm{CH}^i(BG)$$

under the (injective) ring homomorphism $\mathrm{CH}(BG) \to \mathrm{CH}(BT)^W$ from the Chow ring of the classifying space of G. By [2, Théorème], the graded $\mathrm{CH}(BT)^W$-algebra $\mathrm{CH}(BT)$ is a free graded module over $\mathrm{CH}(BT)^W$. In particular, the embedding of rings $\mathrm{CH}(BT)^W \hookrightarrow \mathrm{CH}(BT)$ is a split monomorphism of $\mathrm{CH}(BT)^W$-modules. It follows that the kernel of (2.3) is also generated by the image of $\mathrm{CH}^{>0}(BG)$.

The ring $\mathrm{CH}(BT)$ is identified with the polynomial ring

$$\mathbb{Z}[x_1, \dots, x_m, y_1, \dots, y_l]$$

in the n variables $x_1, \dots, x_m, y_1, \dots, y_l$, where $l := n - m$. The ring $\mathrm{Ch}(BT)$ is therefore identified with the polynomial ring

$$\widetilde{R} := k[x_1, \dots, x_m, y_1, \dots, y_l].$$

We view $\widetilde{R}$ as a graded ring with respect to the standard grading $\deg x_i = \deg y_j = 1$. Write u_i for the i-th elementary symmetric polynomial in $x_1, \dots, x_m$ ($i = 1, \dots, m$) and v_j for the j-th elementary symmetric polynomial in $y_1^2, \dots, y_l^2$ ($j = 1, \dots, l$). Let R be the subring $k[u_1, \dots, u_m, v_1, \dots, v_l]$ of $\widetilde{R}$. The ring R is also a polynomial ring over k in $n = m + l$ variables. Since the Weyl group W acts on $\mathrm{CH}(BT)$ by interchanging the variables $x_1, \dots, x_m$, interchanging the variables $y_1, \dots, y_l$, and changing signs of the latter variables, we have $R = \mathrm{CH}(BT)^W \otimes k$. Note that in general R does not coincide with $\mathrm{Ch}(BT)^W$, c.f. [2, Théorème (c)].

Write w_i for the i-th elementary symmetric polynomial in the squares

$$x_1^2, \dots, x_m^2, y_1^2, \dots, y_l^2$$

of our n variables ($i = 1, \dots, n$). Let $I \subset R$ be the ideal generated by $w_1, w_2, \dots, w_n$. The computation of $\mathrm{CH}(BG)$ given in [14, §15] (see also [10, Example 5.2]) tells us that I is the kernel of the surjective ring homomorphism (2.3)

$$R = \mathrm{CH}(BT)^W \otimes k \twoheadrightarrow \mathrm{Ch}(G/P) = C \otimes k. \tag{2.4}$$

To have a more complete understanding of the objects involved, let us mention that the quotient of $\tilde{R}$ by the ideal $I\tilde{R}$, generated by $w_1, \dots, w_n$, is the Chow ring $\mathrm{Ch}(G/B)$.

Consider the monomials $u := u_m = x_1 \cdots x_m \in R$ and $w := u_m v_l = u y_1^2 \cdots y_l^2 \in R$. Note that $w_n = uw$. Besides, the image of u under (2.4) is $c_m('\mathcal{T})$ whereas the image of w is $(-1)^n c_{2n-m}(-'\mathcal{T})$. In order to justify the statement on the image of w, let us mention that whereas the images of $x_1, \dots, x_m$ are the roots of the vector bundle $'\mathcal{T}$, the roots of the containing $'\mathcal{T}$ trivial vector bundle, given by the $2n$-dimensional vector space V_b of definition of our alternating bilinear form b, are $\pm x_1, \dots, \pm x_m, \pm y_1, \dots, \pm y_l$. Therefore $c_{2n-m}(-'\mathcal{T}) = c_{2n-m}(V_b/'\mathcal{T})$ is computed as claimed.

Proposition 2.2 translates as follows:

(2.5) Let $f \in R$ be a polynomial such that $uf \in I$. Then $f \in wR + I$.

To prove (2.5), since I is a homogeneous ideal, we may assume that the polynomial f is also homogeneous. By assumption,

$$uf = w_1f_1 + w_2f_2 + \ldots + w_nf_n$$

for some $f_1, f_2, \ldots, f_n \in R$, or equivalently,

$$uf_0 + w_1f_1 + w_2f_2 + \ldots + w_{n-1}f_{n-1} = 0, \tag{2.6}$$

where $f_0 = wf_n - f$.

From now on k is assumed to be a field. Let $M \subset R$ be the ideal generated by $u_1, \ldots, u_m, v_1, \ldots, v_l$. Clearly, M is a maximal ideal of R containing I.

We claim that M is the radical $\sqrt{I}$ of I. For a brief moment, let's view $u_1, \ldots, u_m$ and $v_1, \ldots, v_l$ as integral polynomials and let $p \in \mathbb{Z}[x_1, \ldots, x_m, y_1, \ldots, y_l]$ be one of them. The coefficients of the polynomial $f(t) := \prod(t - \sigma p)$ in a variable t, where σ runs over the Weyl group W_G of G, are W_G-symmetric polynomials with trivial constant terms. Since $\mathbb{Z}[x_1, \ldots, x_m, y_1, \ldots, y_l]^{W_G} = \mathbb{Z}[w_1, \ldots, w_n]$, the coefficients of $f(t)$, viewed now as a polynomial over R, belong to I. Since $p \in R$ is a root of the polynomial $f(t) \in R[t]$, we have $p^{|W_G|} \in I$, hence $p \in \sqrt{I}$.

Let $S = R_M$ be the localization of R at the maximal ideal M. Then S is a regular local ring of dimension n. Consider the sequence of polynomials in R:

$$w_0, w_1, w_2, \ldots, w_{n-1} \tag{2.7}$$

with $w_0 := u$. Since u divides w_n, the ideal in R generated by these polynomials contains I and hence its radical is equal to $M = \sqrt{I}$. This means that the sequence (2.7) is a system of parameters in S. Since S is a regular local ring, by [12, Theorem 31], the sequence (2.7) is regular. It follows from [12, Theorem 43], that the sequence (which is a part of the Koszul complex)

$$\Lambda^2(S^n) \xrightarrow{d'} S^n \xrightarrow{d} S, \tag{2.8}$$

where $d(e_i) = w_i$ and $d'(e_i \wedge e_j) = w_je_i - w_ie_j$ (here $e_0, e_1, \ldots e_{n-1}$ is the standard basis for S^n) is exact.

By (2.6), the n-tuple $(f_0, f_1, \ldots, f_{n-1})$ is in the kernel of d. The exactness of (2.8) yields elements $g_i \in S$ such that

$$f_0 = w_1g_1 + w_2g_2 + \ldots + w_{n-1}g_{n-1} \in IS.$$

It follows that $f = wf_n - f_0 \in wS + IS$. As $S = R_M$, there is a polynomial $h \in R$ with a nonzero constant term, satisfying $fh \in wR + I$. Since the polynomial f is homogeneous, it is also contained in $wR + I$. □

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