# ESSENTIAL DIMENSION OF FINITE $p$-GROUPS

## NIKITA A. KARPENKO AND ALEXANDER S. MERKURJEV

ABSTRACT. We prove that the essential dimension and $p$-dimension of a $p$-group $G$ over a field $F$ containing a primitive $p$-th root of unity is equal to the least dimension of a faithful representation of $G$ over $F$.

The notion of the essential dimension $\mathrm{ed}(G)$ of a finite group $G$ over a field $F$ was introduced in [5]. The integer $\mathrm{ed}(G)$ is equal to the smallest number of algebraically independent parameters required to define a Galois $G$-algebra over any field extension of $F$. If $V$ is a faithful linear representation of $G$ over $F$ then $\mathrm{ed}(G) \leq \dim(V)$ (cf. [2, Prop. 4.15]). The essential dimension of $G$ can be smaller than $\dim(V)$ for every faithful representation $V$ of $G$ over $F$. For example, we have $\mathrm{ed}(\mathbb{Z}/3\mathbb{Z}) = 1$ over $\mathbb{Q}$ or any field $F$ of characteristic 3 (cf. [2, Cor. 7.5]) and $\mathrm{ed}(S_3) = 1$ over $\mathbb{C}$ (cf. [5, Th. 6.5]).

In this paper we prove that if $G$ is a $p$-group and $F$ is a field of characteristic different from $p$ containing $p$-th roots of unity, then $\mathrm{ed}(G)$ coincides with the least dimension of a faithful representation of $G$ over $F$ (cf. Theorem 4.1).

We also compute the essential $p$-dimension of a $p$-group $G$ introduced in [15]. We show that $\mathrm{ed}_p(G) = \mathrm{ed}(G)$ over a field $F$ containing $p$-th roots of unity.

## 1. PRELIMINARIES

In the paper the word "scheme" means a separated scheme of finite type over a field and "variety" an integral scheme.

1.1. **Severi-Brauer varieties.** (cf. [1]) Let $A$ be a central simple algebra of degree $n$ over a field $F$. The *Severi-Brauer variety* $P = \mathrm{SB}(A)$ of $A$ is the variety of right ideals in $A$ of dimension $n$. For a field extension $L/F$, the algebra $A$ is split over $L$ if and only if $P(L) \neq \emptyset$ if and only if $P_L \simeq \mathbb{P}_L^{n-1}$.

The change of field map $\deg : \mathrm{Pic}(P) \to \mathrm{Pic}(P_L) = \mathbb{Z}$ for a splitting field extension $L/F$ identifies $\mathrm{Pic}(P)$ with $e\mathbb{Z}$, where $e$ is the exponent (period) of

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$A$. In particular, $P$ has divisors of degree $e$. The algebra $A$ is split over $L$ if and only if $P_L$ has a prime divisor of degree 1 (a hyperplane).

1.2. **Groupoids and gerbes.** (cf. [4]) Let $\mathcal{X}$ be a groupoid over $F$ in the sense of [19]. We assume that for any field extension $L/F$, the isomorphism classes of objects in the category $\mathcal{X}(L)$ form a set which we denote by $\widehat{\mathcal{X}}(L)$. We can view $\widehat{\mathcal{X}}$ as a functor from the category *Fields*$/F$ of field extensions of $F$ to *Sets*.

**Example 1.2.1.** If $G$ is an algebraic group over $F$, then the groupoid $BG$ is defined as the category of $G$-torsors over a scheme over $F$. Hence the functor $\widehat{BG}$ takes a field extension $L/F$ to the set of all isomorphism classes of $G$-torsors over $L$.

Special examples of groupoids are *gerbes banded by a commutative group scheme $C$* over $F$. There is a bijection between the set of isomorphism classes of gerbes banded by $C$ and the Galois cohomology group $H^2(F, C)$ (cf. [7, Ch. 4] and [13, Ch. 4, §2]). The split gerbe $BC$ corresponds to the trivial element of $H^2(F, C)$.

**Example 1.2.2.** (Gerbes banded by $\mu_n$) Let $A$ be a central simple $F$-algebra and $n$ an integer with $[A] \in \mathrm{Br}_n(F) = H^2(F, \mu_n)$. Let $P$ be the Severi-Brauer variety of $A$ and $S$ a divisor on $P$ of degree $n$. Denote by $\mathcal{X}_A$ the gerbe banded by $\mu_n$ corresponding to $[A]$. For a field extension $L/F$, the set $\widehat{\mathcal{X}}_A(L)$ has the following explicit description (cf. [4]): $\widehat{\mathcal{X}}_A(L)$ is nonempty if and only if $P$ is split over $L$. In this case $\widehat{\mathcal{X}}_A(L)$ is the set of equivalence classes of the set

$$\big\{ f \in L(P)^\times : \quad \mathrm{div}(f) = nH - S_L, \quad \text{where } H \text{ is a hyperplane in } P_L \big\},$$

and two functions $f$ and $f'$ are equivalent if $f' = fh^n$ for some $h \in L(P)^\times$.

1.3. **Essential dimension.** Let $T : \textit{Fields}/F \to \textit{Sets}$ be a functor. For a field extension $L/F$ and an element $t \in T(L)$, the *essential dimension of $t$*, denoted $\mathrm{ed}(t)$, is the least $\mathrm{tr. \, deg}_F(L')$ over all subfields $L' \subset L$ over $F$ such that $t$ belongs to the image of the map $T(L') \to T(L)$. The *essential dimension* $\mathrm{ed}(T)$ *of the functor $T$* is the supremum of $\mathrm{ed}(t)$ over all $t \in T(L)$ and field extensions $L/F$.

Let $p$ be a prime integer and $t \in T(L)$. The *essential $p$-dimension of $t$*, denoted $\mathrm{ed}_p(t)$, is the least $\mathrm{tr. \, deg}_F(L'')$ over all subfields $L'' \subset L'$ over $F$, where $L'$ is a finite field extension of $L$ of degree prime to $p$ such that the image of $t$ in $T(L')$ belongs to the image of the map $T(L'') \to T(L')$. The *essential $p$-dimension* $\mathrm{ed}_p(T)$ *of the functor $T$* is the supremum of $\mathrm{ed}_p(t)$ over all $t \in T(L)$ and field extensions $L/F$. Clearly, $\mathrm{ed}(T) \geq \mathrm{ed}_p(T)$.

Let $G$ be an algebraic group over $F$. The *essential dimension* $\mathrm{ed}(G)$ *of $G$* (respectively the *essential $p$-dimension* $\mathrm{ed}(G)$) is the essential dimension (respectively the essential $p$-dimension) of the functor taking a field extension $L/F$ to the set of isomorphism classes of $G$-torsors over $\mathrm{Spec} \, L$.

If $G$ is a finite group, we view $G$ as a constant group over a field $F$. Every $G$-torsor over $\operatorname{Spec} L$ has the form $\operatorname{Spec} K$ where $K$ is a Galois $G$-algebra over $L$. Therefore, $\operatorname{ed}(G)$ is the essential dimension of the functor taking a field $L$ to the set of isomorphism classes of Galois $G$-algebras over $L$.

**Example 1.3.1.** Let $\mathcal{X}$ be a groupoid over $F$. The *essential dimension of $\mathcal{X}$*, denoted by $\operatorname{ed}(\mathcal{X})$, is the essential dimension $\operatorname{ed}(\widehat{\mathcal{X}})$ of the functor $\widehat{\mathcal{X}}$ defined in §1.2. The *essential $p$-dimension of $\operatorname{ed}_p(\mathcal{X})$* is defined similarly. In particular, $\operatorname{ed}(BG) = \operatorname{ed}(G)$ and $\operatorname{ed}_p(BG) = \operatorname{ed}_p(G)$ for an algebraic group $G$ over $F$.

1.4. **Canonical dimension.** (cf. [3], [11]) Let $F$ be a field and $\mathcal{C}$ a class of field extensions of $F$. A field $E \in \mathcal{C}$ is called *generic* if for any $L \in \mathcal{C}$ there is an $F$-place $E \rightsquigarrow L$.

The *canonical dimension* $\operatorname{cdim}(\mathcal{C})$ of the class $\mathcal{C}$ is the minimum of the $\operatorname{tr.deg}_F E$ over all generic fields $E \in \mathcal{C}$.

Let $p$ be a prime integer. A field $E$ in a class $\mathcal{C}$ is called *$p$-generic* if for any $L \in \mathcal{C}$ there is a finite field extension $L'$ of $L$ of degree prime to $p$ and an $F$-place $E \rightsquigarrow L'$. The *canonical $p$-dimension* $\operatorname{cdim}_p(\mathcal{C})$ of the class $\mathcal{C}$ is the least $\operatorname{tr.deg}_F E$ over all $p$-generic fields $E \in \mathcal{C}$. Obviously, $\operatorname{cdim}(\mathcal{C}) \geq \operatorname{cdim}_p(\mathcal{C})$.

Let $T : \mathsf{Fields}/F \to \mathsf{Sets}$ be a functor. Denote by $\mathcal{C}_T$ the class of *splitting fields of $T$*, i.e., the class of field extensions $L/F$ such that $T(L) \neq \emptyset$. The *canonical dimension ($p$-dimension) of $T$*, denoted $\operatorname{cdim}(T)$ (respectively $\operatorname{cdim}_p(T)$), is the canonical dimension ($p$-dimension) of the class $\mathcal{C}_T$.

If $X$ is a scheme over $F$, we write $\operatorname{cdim}(X)$ and $\operatorname{cdim}_p(X)$ for the canonical dimension and $p$-dimension of $X$ viewed as a functor $L \mapsto X(L) = \operatorname{Mor}_F(\operatorname{Spec} L, X)$.

**Example 1.4.1.** Let $\mathcal{X}$ be a groupoid over $F$. We define the *canonical dimension* $\operatorname{cdim}(\mathcal{X})$ and *$p$-dimension* $\operatorname{cdim}_p(\mathcal{X})$ of $\mathcal{X}$ as the canonical dimension and $p$-dimension of the functor $\widehat{\mathcal{X}}$.

**Example 1.4.2.** If $X$ is a regular and complete variety over $F$ viewed as a functor then $\operatorname{cdim}(X)$ is equal to the smallest dimension of a closed subvariety $Z \subset X$ such that there is a rational morphism $X \dashrightarrow Z$ (cf. [11, Cor. 4.6]). If $p$ is a prime integer then $\operatorname{cdim}_p(X)$ is equal to the smallest dimension of a closed subvariety $Z \subset X$ such that there are dominant rational morphisms $X' \dashrightarrow X$ of degree prime to $p$ and $X' \dashrightarrow Z$ for some variety $X'$ (cf. [11, Prop. 4.10]).

**Remark 1.4.3.** (A relation between essential and canonical dimension) Let $T : \mathsf{Fields}/F \to \mathsf{Sets}$ be a functor. We define the "contraction" functor $T^c : \mathsf{Fields}/F \to \mathsf{Sets}$ as follows. For a field extension $L/F$, we have $T^c(L) = \emptyset$ if $T(L)$ is empty and $T^c(L)$ is a one element set otherwise. If $X$ is a regular and complete variety over $F$ viewed as a functor then one can show that $\operatorname{ed}(X^c) = \operatorname{cdim}(X)$ and $\operatorname{ed}_p(X^c) = \operatorname{cdim}_p(X)$.

1.5. **Valuations.** Let $K/F$ be a regular field extension, i.e., for any field extension $L/F$, the ring $K \otimes_F L$ is a domain. We write $KL$ for the quotient field of $K \otimes_F L$.

Let $v$ be a valuation on $L$ over $F$ with residue field $R$. Let $O$ be the associated valuation ring and $M$ its maximal ideal. As $K \otimes_F R$ is a domain, the ideal $\widetilde{M} := K \otimes_F M$ in the ring $\widetilde{O} := K \otimes_F O$ is prime. The localization ring $\widetilde{O}_{\widetilde{M}}$ is a valuation ring in $KL$ with residue field $KR$. The corresponding valuation $\tilde{v}$ of $KL$ is called the *canonical extension of $v$ on $KL$*. Note that the groups of values of $v$ and $\tilde{v}$ coincide.

We shall need the following lemma.

**Lemma 1.1** (cf. [11, Lemma 3.2]). *Let $v$ be a discrete valuation (of rank 1) of a field $L$ with residue field $R$ and $L'/L$ a finite field extension of degree prime to $p$. Then $v$ extends to a discrete valuation of $L'$ with residue field $R'$ such that the ramification index and the degree $[R' : R]$ are prime to $p$.*

*Proof.* If $L'/L$ is separable and $v_1, \dots, v_k$ are all the extensions of $v$ on $L'$ then $[L' : L] = \sum e_i [R_i : R]$ where $e_i$ is the ramification index and $R_i$ is the residue field of $v_i$ (cf. [20, Ch. VI, Th. 20 and p. 63]). It follows that the integer $e_i [R_i : R]$ is prime to $p$ for some $i$.

If $L'/L$ is purely inseparable of degree $q$ then the valuation $v'$ of $L'$ defined by $v'(x) = v(x^q)$ satisfies the desired properties. The general case follows. □

## 2. Canonical dimension of a subgroup of $\mathrm{Br}(F)$

Let $F$ be an arbitrary field, $p$ a prime integer and $D$ a finite subgroup of $\mathrm{Br}_p(F)$ of dimension $r$ over $\mathbb{Z}/p\mathbb{Z}$. In this section we determine the canonical dimension $\mathrm{cdim}\, D$ and the canonical $p$-dimension $\mathrm{cdim}_p D$ of the class of common splitting fields of all elements of $D$. We say that a basis $\{a_1, a_2, \dots, a_r\}$ of $D$ is *minimal* if for any $i = 1, \dots, r$ and any element $d \in D$ outside of the subgroup generated by $a_1, \dots, a_{i-1}$, we have $\mathrm{ind}\, d \geq \mathrm{ind}\, a_i$.

One can construct a minimal basis of $D$ by induction as follows. Let $a_1$ be a nonzero element of $D$ of minimal index. If the elements $a_1, \dots, a_{i-1}$ are already chosen for some $i \leq r$, we take for the $a_i$ an element of $D$ of the minimal index among the elements outside of the subgroup generated by $a_1, \dots, a_{i-1}$.

In this section we prove the following

**Theorem 2.1.** *Let $F$ be an arbitrary field, $p$ a prime integer, $D \subset \mathrm{Br}_p(F)$ a subgroup of dimension $r$ and $\{a_1, a_2, \dots, a_r\}$ a minimal basis of $D$. Then*

$$\mathrm{cdim}_p(D) = \mathrm{cdim}(D) = \Big( \sum_{i=1}^{r} \mathrm{ind}\, a_i \Big) - r \ .$$

We prove Theorem 2.1 in several steps.

Let $\{a_1, a_2, \dots, a_r\}$ be a minimal basis of $D$. For every $i = 1, 2, \dots, r$, let $P_i$ be the Severi-Brauer variety of a central division $F$-algebra $A_i$ representing the element $a_i \in \mathrm{Br}_p F$. We write $P$ for the product $P_1 \times P_2 \times \cdots \times P_r$. We

have

$$\dim P = \sum_{i=1}^{r} \dim P_i = \Big( \sum_{i=1}^{r} \operatorname{ind} a_i \Big) - r.$$

Moreover, the classes of splitting fields of $P$ and $D$ coincide, hence $\operatorname{cdim}(D) = \operatorname{cdim}(P)$ and $\operatorname{cdim}_p(D) = \operatorname{cdim}_p(P)$. Thus, the statement of Theorem 2.1 is equivalent to the equality $\operatorname{cdim}_p(P) = \operatorname{cdim}(P) = \dim(P)$.

Let $r \geq 1$ and $0 \leq n_1 \leq n_2 \leq \cdots \leq n_r$ be integers and $K = K(n_1, \ldots, n_r)$ the subgroup of the polynomial ring $\mathbb{Z}[x]$ in $r$ variables $x = (x_1, \ldots, x_r)$ generated by the monomials $p^{e(j_1, \ldots, j_r)} x_1^{j_1} \ldots x_r^{j_r}$ for all $j_1, \ldots, j_r \geq 0$, where the exponent $e(j_1, \ldots, j_r)$ is 0 if all the $j_1, \ldots, j_r$ are divisible by $p$, otherwise $e(j_1, \ldots, j_r) = n_k$ with the maximum $k$ such that $j_k$ is not divisible by $p$. In fact, $K$ is a subring of $\mathbb{Z}[x]$.

**Remark 2.2.** Let $A_1, \ldots, A_r$ be central division algebras over some field such that for any non-negative integers $j_1, \ldots, j_r$, the index of the tensor product $A_1^{\otimes j_1} \otimes \cdots \otimes A_r^{\otimes j_r}$ is equal to $p^{e(j_1, \ldots, j_r)}$. The group $K$ can be interpreted as the colimit of the Grothendieck groups of the product over $i = 1, \ldots, r$ of the Severi-Brauer varieties of the matrix algebras $M_{l_i}(A_i)$ over all positive integers $l_1, \ldots, l_r$.

We set $h = (h_1, \ldots, h_r)$ with $h_i = 1 - x_i \in \mathbb{Z}[x]$.

**Proposition 2.3.** *Let $bh_1^{i_1} \ldots h_r^{i_r}$ be a monomial of the lowest total degree of a polynomial $f$ in the variables $h$ lying in $K$. Assume that the integer $b$ is not divisible by $p$. Then $p^{n_1} \,|\, i_1, \ldots, p^{n_r} \,|\, i_r$.*

*Proof.* We recast the proof for $r = 1$ given in [8, Lemma 2.1.2] to the case of arbitrary $r$.

We proceed by induction on $m = r + n_1 + \cdots + n_r$. The case $m = 1$ is trivial. If $m > 1$ and $n_1 = 0$, then $K = K(n_2, \ldots, n_r)[x_1]$ and we are done by induction applied to $K(n_2, \ldots, n_r)$. In what follows we assume that $n_1 \geq 1$.

Since $K(n_1, n_2, \ldots, n_r) \subset K(n_1 - 1, n_2, \ldots, n_r)$, by the induction hypothesis $p^{n_1 - 1} \,|\, i_1, \; p^{n_2} \,|\, i_2, \ldots, p^{n_r} \,|\, i_r$. It remains to show that $i_1$ is divisible by $p^{n_1}$.

Consider the additive operation $\varphi \colon \mathbb{Z}[x] \to \mathbb{Q}[x]$ which takes a polynomial $g \in \mathbb{Z}[x]$ to the polynomial $p^{-1} x_1 \cdot g'$, where $g'$ is the partial derivative of $g$ with respect to $x_1$. We have

$$\varphi(K) \subset K(n_1 - 1, n_2 - 1, \ldots, n_r - 1) \subset K(n_1 - 1)[x_2, \ldots, x_r]$$

and

$$\varphi(h_1^{j_1} h_2^{j_2} \cdots h_r^{j_r}) = -p^{-1} j_1 h_1^{j_1 - 1} h_2^{j_2} \cdots h_r^{j_r} + p^{-1} j_1 h_1^{j_1} h_2^{j_2} \cdots h_r^{j_r}.$$

Since $bh_1^{i_1} \cdots h_r^{i_r}$ is a monomial of the lowest total degree of the polynomial $f$, it follows that $-bp^{-1} i_1 h_1^{i_1 - 1} h_2^{i_2} \cdots h_r^{i_r}$ is a monomial of $\varphi(f)$ considered as a polynomial in $h$. As

$$\varphi(f) \in K(n_1 - 1)[x_2, \ldots, x_r] \,,$$

we see that $-bp^{-1}i_1 h_1^{i_1-1}$ is a monomial of a polynomial from $K(n_1-1)$. It follows that $p^{-1}i_1$ is an integer and by Lemma 2.4 below, this integer is divisible by $p^{n_1-1}$. Therefore $p^{n_1} \mid i_1$.                                                     $\square$

**Lemma 2.4.** *Let $g$ be a polynomial in $h_1$ lying in $K(m)$ for some $m \geq 0$. Let $bh_1^{i-1}$ be a monomial of $g$ such that $i$ is divisible by $p^m$. Then $b$ is divisible by $p^m$.*

*Proof.* We write $h$ for $h_1$ and $x$ for $x_1$. Note that $h^i \in K(m)$ since $i$ is divisible by $p^m$. Moreover, the quotient ring $K(m)/(h^i)$ is additively generated by $p^{e(j)}x^j$ with $j < i$. Indeed, the polynomial $x^i - (-h)^i = x^i - (x-1)^i$ is a linear combination with integer coefficients of $p^{e(j)}x^j$ with $j < i$. Consequently, for any $k \geq 0$, multiplying by $p^{e(k)}x^k$, we see that the polynomial $p^{e(i+k)}x^{i+k} = p^{e(k)}x^{i+k}$ modulo the ideal $(h^i)$ is a linear combination with integer coefficients of the $p^{e(j)}x^j$ with $j < i+k$.

Thus, $K(m)/(h^i)$ is additively generated by $p^{e(j)}(1-h)^j$ with $j < i$. Only the generator $p^{e(i-1)}(1-h)^{i-1} = p^m(1-h)^{i-1}$ has a nonzero $h^{i-1}$-coefficient and that coefficient is divisible by $p^m$.                                                     $\square$

Let $Y$ be a scheme over the field $F$. We write $\mathrm{CH}(Y)$ for the Chow group of $Y$ and set $\mathrm{Ch}(Y) = \mathrm{CH}(Y)/p\,\mathrm{CH}(Y)$. We define $\mathrm{Ch}(\overline{Y})$ as the colimit of $\mathrm{Ch}(Y_L)$ where $L$ runs over all field extensions of $F$. Thus for any field extension $L/F$, we have a canonical homomorphism $\mathrm{Ch}(Y_L) \to \mathrm{Ch}(\overline{Y})$. This homomorphism is an isomorphism if $Y = P$, the variety defined above, and $L$ is a splitting field of $P$.

We define $\overline{\mathrm{Ch}}(Y)$ to be the image of the homomorphism $\mathrm{Ch}(Y) \to \mathrm{Ch}(\overline{Y})$.

**Proposition 2.5.** *We have $\overline{\mathrm{Ch}}^j(P) = 0$ for any $j > 0$.*

*Proof.* Let $K_0(P)$ be the Grothendieck group of $P$. We write $K_0(\overline{P})$ for the colimit of $K_0(P_L)$ taken over all field extensions $L/F$. The group $K_0(\overline{P})$ is canonically isomorphic to $K_0(P_L)$ for any splitting field $L$ of $P$. Each of the groups $K_0(P)$ and $K_0(\overline{P})$ is endowed with the topological filtration. The subsequent factor groups $G^j K_0(P)$ and $G^j K_0(\overline{P})$ of these filtrations fit into the commutative square

$$
\begin{array}{ccc}
\mathrm{CH}^j(\overline{P}) & \longrightarrow & G^j K_0(\overline{P}) \\
\uparrow & & \uparrow \\
\mathrm{CH}^j(P) & \longrightarrow & G^j K_0(P)
\end{array}
$$

where the top map is an isomorphism. Therefore it suffices to show that the image of the homomorphism $G^j K_0(P) \to G^j K_0(\overline{P})$ is divisible by $p$ for any $j > 0$.

The ring $K_0(\overline{P})$ is identified with the quotient of the polynomial ring $\mathbb{Z}[h]$ by the ideal generated by $h_1^{\mathrm{ind}\,a_1}, \ldots, h_r^{\mathrm{ind}\,a_r}$. Under this identification, the element $h_i$ is the pull-back to $P$ of the class of a hyperplane in $P_i$ over a splitting field and the $j$-th term $K_0(\overline{P})^{(j)}$ of the filtration is generated by the classes

of monomials of degree at least $j$. The group $G^j K_0(\overline{P})$ is identified with the group of all homogeneous polynomials of degree $j$.

The group $K_0(P)$ is isomorphic to the direct sum of $K_0(B)$, where $B = A_1^{\otimes j_1} \otimes \cdots \otimes A_r^{\otimes j_r}$, over all $j_i$ with $0 \leq j_i < \operatorname{ind} a_i$ (cf. [14, §9]). The image of the natural map $K_0(B) \to K_0(B_L) = \mathbb{Z}$, where $L$ is a splitting field of $B$, is equal to $\operatorname{ind}(a_1^{j_1} \cdots a_r^{j_r})\mathbb{Z}$. The image of the homomorphism $K_0(P) \to K_0(\overline{P})$ (which is in fact an injection) is generated by

$$\operatorname{ind}(a_1^{j_1} \cdots a_r^{j_r})(1 - h_1)^{j_1} \cdots (1 - h_r)^{j_r}$$

over all $j_1, \ldots, j_r \geq 0$.

We embed $K_0(\overline{P})$ into the polynomial ring $\mathbb{Z}[x] = \mathbb{Z}[x_1, \ldots, x_r]$ as a subgroup by identifying a monomial $h_1^{j_1} \cdots h_r^{j_r}$ where $0 \leq j_i < \operatorname{ind} a_i$ with the polynomial $(1 - x_1)^{j_1} \cdots (1 - x_r)^{j_r}$. As the elements $a_1, \ldots, a_r$ form a minimal basis of $D$, the index $\operatorname{ind}(a_1^{j_1} \cdots a_r^{j_r})$ is a power of $p$ with the exponent at least $e(\log_p \operatorname{ind} a_1, \ldots, \log_p \operatorname{ind} a_r)$. Therefore,

$$K_0(P) \subset K(\log_p \operatorname{ind} a_1, \ldots, \log_p \operatorname{ind} a_r) \subset \mathbb{Z}[x].$$

An element of $K_0(P)^{(j)}$ with $j > 0$ is a polynomial $f$ in $h$ of degree at least $j$. The image of $f$ in $G^j K_0(\overline{P})$ is the $j$-th homogeneous part $f_j$ of $f$. As the degree of $f$ with respect to $h_i$ is less than $\operatorname{ind} a_i$, it follows from Proposition 2.3 that all the coefficients of $f_j$ are divisible by $p$. $\qquad \square$

Let $d = \dim P$ and $\alpha \in \operatorname{CH}^d(P \times P)$. The *first multiplicity* $\operatorname{mult}_1(\alpha)$ *of* $\alpha$ is the image of $\alpha$ under the push-forward map $\operatorname{CH}^d(P \times P) \to \operatorname{CH}^0(P) = \mathbb{Z}$ given by the first projection $P \times P \to P$ (cf. [10]). Similarly, we define the *second multiplicity* $\operatorname{mult}_2(\alpha)$.

**Corollary 2.6.** *For any element* $\alpha \in \operatorname{CH}^d(P \times P)$, *we have*

$$\operatorname{mult}_1(\alpha) \equiv \operatorname{mult}_2(\alpha) \quad modulo \; p.$$

*Proof.* We follow the proof of [9, Th. 2.1]. The homomorphism

$$f \colon \operatorname{CH}^d(P \times P) \to (\mathbb{Z}/p\mathbb{Z})^2,$$

taking an $\alpha \in \operatorname{CH}^d(P \times P)$ to $\big(\operatorname{mult}_1(\alpha), \operatorname{mult}_2(\alpha)\big)$ modulo $p$, factors through the group $\overline{\operatorname{Ch}}^d(P \times P)$. Since for any $i$, any projection $P_i \times P_i \to P_i$ is a projective bundle, the Chow group $\overline{\operatorname{Ch}}^d(P \times P)$ is a direct some of several copies of $\overline{\operatorname{Ch}}^i(P)$ for some $i$'s and the value $i = 0$ appears once. By Proposition 2.5, the dimension over $\mathbb{Z}/p\mathbb{Z}$ of the vector space $\overline{\operatorname{Ch}}^d(P \times P)$ is equal to 1 and consequently the dimension of the image of $f$ is at most 1. Since the image of the diagonal class under $f$ is $(1, 1)$, the image of $f$ is generated by $(1, 1)$. $\quad \square$

**Corollary 2.7.** *Any rational map* $P \dashrightarrow P$ *is dominant.*

*Proof.* Let $\alpha \in \operatorname{CH}^d(P \times P)$ be the class of the closure of the graph of a rational map $P \dashrightarrow P$. We have $\operatorname{mult}_1(\alpha) = 1$. Therefore, by Corollary 2.6, $\operatorname{mult}_2(\alpha) \neq 0$, and it follows that the rational map is dominant. $\quad \square$

**Corollary 2.8.** $\operatorname{cdim}_p P = \operatorname{cdim} P = \dim P$.

*Proof.* As $\operatorname{cdim}_p P \le \operatorname{cdim} P \le \dim P$, it suffices to show that $\operatorname{cdim}_p P = \dim P$. Let $Z \subset P$ be a closed subvariety and $f : P' \dashrightarrow P$ and $g : P' \dashrightarrow Z$ dominant rational morphisms such that $\deg f$ is prime to $p$. Let $\alpha$ be the class in $\operatorname{CH}^d(P \times P)$ of the closure in $P \times P$ of the image of $f \times g : P' \dashrightarrow P \times Z$. As $\operatorname{mult}_1(\alpha) = \deg f$ is prime to $p$, by Corollary 2.6, we have $\operatorname{mult}_2(\alpha) \ne 0$, i.e., $Z = P$. By Example 1.4.2, $\operatorname{cdim}_p P = \dim P$. $\qquad\square$

The corollary completes the proof of Theorem 2.1.

**Remark 2.9.** Theorem 2.1 can be generalized to the case of any finite subgroup $D \subset \operatorname{Br}(F)$ consisting of elements of $p$-primary orders. Let $\{a_1, a_2, \dots, a_r\}$ be elements of $D$ such that their images $\{a'_1, a'_2, \dots, a'_r\}$ in $D/D^p$ form a minimal basis, i.e., for any $i = 1, \dots r$ and any element $d \in D$ with the class in $D/D^p$ outside of the subgroup generated by $a'_1, \dots, a'_{i-1}$, the inequality $\operatorname{ind} d \ge \operatorname{ind} a_i$ holds. In particular, $\{a_1, a_2, \dots, a_r\}$ generate $D$. Then, as in Theorem 2.1, we have

$$\operatorname{cdim}_p(D) = \operatorname{cdim}(D) = \Big( \sum_{i=1}^r \operatorname{ind} a_i \Big) - r \ .$$

Indeed, the group $D$ and the variety $P = P_1 \times \cdots \times P_r$, where $P_i$ for every $i = 1, \dots, r$ is the Severi-Brauer variety of a central division algebra representing the element $a_i$, have the same splitting fields. Therefore, $\operatorname{cdim}(D) = \operatorname{cdim}(P)$ and $\operatorname{cdim}_p(D) = \operatorname{cdim}_p(P)$. Corollaries 2.6, 2.7 and 2.8 hold for $P$ since $K_0(P) \subset K(\log_p \operatorname{ind} a_1, \dots, \log_p \operatorname{ind} a_r)$.

**Remark 2.10.** One can compute the canonical $p$-dimension of an arbitrary finite subgroup of $D \subset \operatorname{Br}(F)$ as follows. Let $D'$ be the Sylow $p$-subgroup of $D$. Write $D = D' \oplus D''$ for a subgroup $D'' \subset D$ and let $L/F$ be a finite field extension of degree prime to $p$ such that $D''$ is split over $L$. Then $D_L = D'_L$ and $\operatorname{cdim}_p(D) = \operatorname{cdim}_p(D_L) = \operatorname{cdim}_p(D'_L) = \operatorname{cdim}_p(D') = \operatorname{cdim}(D')$.

3. ESSENTIAL AND CANONICAL DIMENSION OF GERBES BANDED BY $(\boldsymbol{\mu}_p)^s$

In this section we relate the essential and canonical ($p$-)dimensions of gerbes banded by $(\boldsymbol{\mu}_p)^s$ where $s \ge 0$. The following statement is a generalization of [4, Th. 7.1].

**Theorem 3.1.** *Let $p$ be a prime integer and $\mathcal{X}$ a gerbe banded by $(\boldsymbol{\mu}_p)^s$ over an arbitrary field $F$. Then*

$$\operatorname{ed}(\mathcal{X}) = \operatorname{ed}_p(\mathcal{X}) = \operatorname{cdim}_p(\mathcal{X}) + s = \operatorname{cdim}(\mathcal{X}) + s.$$

*Proof.* The gerbe $\mathcal{X}$ is given by an element in $H^2\big(F, (\boldsymbol{\mu}_p)^s\big) = \operatorname{Br}_p(F)^s$, i.e., by an $s$-tuple of central simple algebras $A_1, A_2, \dots, A_s$ with $[A_i] \in \operatorname{Br}_p(F)$. Let $P$ be the product of the Severi-Brauer varieties $P_i := \operatorname{SB}(A_i)$ and $D$ the subgroup of $\operatorname{Br}_p(F)$ generated by the $[A_i]$, $i = 1, \dots, s$. As the classes of splitting fields for $\mathcal{X}$, $D$ and $P$ coincide, we have

(1)     $\operatorname{cdim}(\mathcal{X}) = \operatorname{cdim}(P) = \operatorname{cdim}(D) = \operatorname{cdim}_p(D) = \operatorname{cdim}_p(P) = \operatorname{cdim}_p(\mathcal{X})$

by Theorem 2.1. We shall prove the inequalities $\operatorname{ed}_p(\mathcal{X}) \geq \operatorname{cdim}(P) + s \geq \operatorname{ed}(\mathcal{X})$.

Let $S_i$ be a divisor on $P_i$ of degree $p$. Let $L/F$ be a field extension and $f_i \in L(P_i)^\times$ with $\operatorname{div}(f_i) = pH_i - (S_i)_L$, where $H_i$ is a hyperplane in $(P_i)_L$ for $i = 1, \ldots, s$. We write $\langle f_i \rangle_{i=1}^s$ for the corresponding element in $\widehat{\mathcal{X}}(L)$ (cf. §1.2).

By Example 1.4.2, there is a closed subvariety $Z \subset P$ and a rational dominant morphism $P \dashrightarrow Z$ with $\dim(Z) = \operatorname{cdim}(P) = \operatorname{cdim}_p(P)$. We view $F(Z)$ as a subfield of $F(P)$. As $P(L) \neq \emptyset$ and $P$ is regular, there is an $F$-place $\gamma : F(P) \rightsquigarrow L$ (cf. [11, §4.1]). Since $Z$ is complete, the valuation ring of the restriction $\gamma|_{F(Z)} : F(Z) \rightsquigarrow L$ dominates a point in $Z$. It follows that $Z(L) \neq \emptyset$. Choose a point $y \in Z$ such that $F' := F(y) \subset L$.

Since $P(F') \neq \emptyset$, the $P_i$ are split over $F'$, hence $\operatorname{Pic}(P_i)_{F'} = \mathbb{Z}$ and there are functions $g_i \in F'(P_i)^\times$ with $\operatorname{div}(g_i) = pH_i' - (S_i)_{F'}$, where $H_i'$ is a hyperplane in $P_i$ for $i = 1, \ldots, s$. As $\operatorname{Pic}(P_i)_L = \mathbb{Z}$, there are functions $h_i \in L(P_i)^\times$ with $\operatorname{div}(h_i) = (H_i')_L - H_i$. We have

$$\operatorname{div}(g_i)_L = \operatorname{div}(f_i) + \operatorname{div}(h_i^p),$$

hence

$$a_i g_i = f_i h_i^p$$

for some $a_i \in L^\times$. It follows that $\langle f_i \rangle_{i=1}^s = \langle a_i g_i \rangle_{i=1}^s$ in $\mathcal{X}(L)$, therefore $\langle f_i \rangle_{i=1}^s$ is defined over the field $F'(a_1, a_2, \ldots, a_s)$. Hence

$$\operatorname{ed}\langle f_i \rangle_{i=1}^s \leq \operatorname{tr.deg}_F(F') + s \leq \dim(Z) + s = \operatorname{cdim}(P) + s,$$

and therefore $\operatorname{ed}(\mathcal{X}) \leq \operatorname{cdim}(P) + s$.

We shall prove the inequality $\operatorname{ed}_p(\mathcal{X}) \geq \operatorname{cdim}(P) + s$. As $P\big(F(Z)\big) \neq \emptyset$, there are functions $f_i \in F(Z)(P_i)^\times$ with $\operatorname{div}(f_i) = pH_i - (S_i)_{F(Z)}$, where $H_i$ is a hyperplane in $(P_i)_{F(Z)}$. Let $L := F(Z)(t_1, t_2, \ldots, t_s)$, where the $t_i$ are variables, and consider the point $\langle t_i f_i \rangle_{i=1}^s \in \widehat{\mathcal{X}}(L)$.

We claim that $\operatorname{ed}_p \langle t_i f_i \rangle_{i=1}^s \geq \operatorname{cdim}(P) + s$. Let $L'$ be a finite extension of $L$ of degree prime to $p$ and $L'' \subset L'$ a subfield such that the image of $\langle t_i f_i \rangle_{i=1}^s$ in $\widehat{\mathcal{X}}(L')$ is defined over $L''$, i.e., there are functions $g_i \in L''(P_i)^\times$ and $h_i \in L'(P_i)^\times$ with $t_i f_i = g_i h_i^p$. We shall show that $\operatorname{tr.deg}_F(L'') \geq \operatorname{cdim}(P) + s$.

Let $L_i := F(Z)(t_i, \ldots, t_s)$ and $v_i$ be the discrete valuation of $L_i$ corresponding to the variable $t_i$ for $i = 1, \ldots, s$. We construct a sequence of field extensions $L_i'/L_i$ of degree prime to $p$ and discrete valuations $v_i'$ of $L_i'$ for $i = 1, \ldots, s$ by induction on $i$ as follows. Set $L_1' = L'$. Suppose the fields $L_1', \ldots, L_i'$ and the valuations $v_1', \ldots, v_{i-1}'$ are constructed. By Lemma 1.1, there is a valuation $v_i'$ of $L_i'$ with residue field $L_{i+1}'$ extending the discrete valuation $v_i$ of $L_i'$ with the ramification index $e_i$ and the degree $[L_{i+1}' : L_{i+1}]$ prime to $p$.

The composition $v'$ of the discrete valuations $v_i'$ is a valuation of $L'$ with residue field of degree over $F(Z)$ prime to $p$. A choice of prime elements in all the $L_i'$ identifies the group of values of $v'$ with $\mathbb{Z}^s$. Moreover, for every

$i = 1, \ldots, s$, we have
$$v'(t_i) = e_i \varepsilon_i + \sum_{j>i} a_{ij} \varepsilon_j$$
where the $\varepsilon_i$'s denote the standard basis elements of $\mathbb{Z}^s$ and $a_{ij} \in \mathbb{Z}$.

Write $v''$ for the restriction of $v'$ on $L''$. Let $K = F(P)$. We extend canonically the valuations $v'$ and $v''$ to valuations $\tilde{v}'$ and $\tilde{v}''$ of $KL'$ and $KL''$ respectively (cf. §1.5). Note that $f_i \in K(Z)^\times$, $g_i \in (KL'')^\times$ and $h_i \in (KL')^\times$. We have
$$e_i \varepsilon_i + \sum_{j>i} a_{ij} \varepsilon_j = v'(t_i) = \tilde{v}'(t_i f_i) \equiv \tilde{v}''(g_i) \pmod{p}.$$
Since $e_i$ are prime to $p$, the elements $\tilde{v}''(g_i)$ generate a subgroup of $\mathbb{Z}^s$ of finite index. It follows that the value group of $\tilde{v}''$ is of rank $s$, hence $\mathrm{rank}(v'') = \mathrm{rank}(\tilde{v}'') = s$.

Let $R''$ and $R'$ be residue fields of $v''$ and $v'$ respectively. We have the inclusions $R'' \subset R' \supset F(Z)$ and $[R' : F(Z)]$ is prime to $p$. By [20, Ch. VI, Th. 3, Cor. 1],

(2)        $\mathrm{tr.deg}_F(L'') \geq \mathrm{tr.deg}_F(R'') + \mathrm{rank}(v'') = \mathrm{tr.deg}_F(R'') + s.$

As $P(L'') \neq \emptyset$, there is an $F$-place $F(P) \rightsquigarrow L''$. Composing it with the place $L'' \rightsquigarrow R''$ given by $v''$, we get an $F$-place $F(P) \rightsquigarrow R''$. As $P$ is complete, we have $P(R'') \neq \emptyset$, i.e., $R''$ is a splitting field of $P$.

We prove that $R''$ is a $p$-generic splitting field of $P$. Let $M$ be a splitting field of $P$. A regular system of parameters at the image of a morphism $\alpha : \mathrm{Spec}\, M \to P$ yields an $F$-place $F(P) \rightsquigarrow M$ that is a composition of places associated with discrete valuations (cf. [11, §1.4]). By [11, Lemma 3.2] applied to the restriction of $\alpha$ to $F(Z)$, there is a finite field extension $M'$ of $M$ and an $F$-place $R' \rightsquigarrow M'$. Restricting to $R''$ we get an $F$-place $R'' \rightsquigarrow M'$, i.e., $R''$ is a $p$-generic splitting field of $P$.

By the definition of the canonical $p$-dimension,
$$\mathrm{cdim}(P) = \mathrm{tr.deg}_F F(Z) = \mathrm{tr.deg}_F R' \geq \mathrm{tr.deg}_F(R'') \geq \mathrm{cdim}_p(P).$$
It follows that $\mathrm{tr.deg}_F(R'') = \mathrm{cdim}(P)$ by (1) and therefore, $\mathrm{tr.deg}_F(L'') \geq \mathrm{cdim}(P) + s$ by (2). The claim is proved.

It follows from the claim that $\mathrm{ed}_p(\mathcal{X}) \geq \mathrm{cdim}(P) + s$.     □

## 4. Main theorem

The main result of the paper is the following

**Theorem 4.1.** *Let $G$ be a $p$-group and $F$ a field of characteristic different from $p$ containing a primitive $p$-th root of unity. Then $\mathrm{ed}_p(G)$ over $F$ is equal to $\mathrm{ed}(G)$ over $F$ and coincides with the least dimension of a faithful representation of $G$ over $F$.*

The rest of the section is devoted to the proof of the theorem. As was mentioned in the introduction, we have $\mathrm{ed}_p(G) \leq \mathrm{ed}(G) \leq \dim(V)$ for any faithful

representation $V$ of $G$ over $F$. We shall construct a faithful representation $V$ of $G$ over $F$ with $\mathrm{ed}_p(G) \geq \dim(V)$.

Denote by $C$ the subgroup of all central elements of $G$ of exponent $p$ and set $H = G/C$, so we have an exact sequence

$$(3) \qquad\qquad\qquad 1 \to C \to G \to H \to 1.$$

Let $E \to \mathrm{Spec}\, F$ be an $H$-torsor and $\mathrm{Spec}\, F \to BH$ be the corresponding morphism. Set $\mathcal{X}^E := BG \times_{BH} \mathrm{Spec}\, F$. Then $\mathcal{X}^E$ is a gerbe over $F$ banded by $C$ and its class in $H^2(F, C)$ coincides with the image of the class of $E$ under the connecting map $H^1(F, H) \to H^2(F, C)$ (cf. [13, Ch. 4, §2]). An object of $\mathcal{X}^E$ over a field extension $L/F$ is a pair $(E', \alpha)$, where $E'$ is a $G$-torsor over $L$ and $\alpha : E'/C \xrightarrow{\sim} E_L$ is an isomorphism of $H$-torsors over $L$.

Alternatively, $\mathcal{X}^E = [E/G]$ with objects (over $L$) $G$-equivariant morphisms $E' \to E_L$, where $E'$ is a $G$-torsor over $L$ (cf. [19]).

A lower bound for $\mathrm{ed}(G)$ was established in [4, Prop. 2.20]. We give a similar bound for $\mathrm{ed}_p(G)$.

**Theorem 4.2.** *For any $H$-torsor $E$ over $F$, we have $\mathrm{ed}_p(G) \geq \mathrm{ed}_p(\mathcal{X}^E)$.*

*Proof.* Let $L/F$ be a field extension and $x = (E', \alpha)$ an object of $\mathcal{X}^E(L)$. Choose a field a field extension $L'/L$ of degree prime to $p$ and a subfield $L'' \subset L'$ over $F$ such that $\mathrm{tr.\,deg}(L'') = \mathrm{ed}_p(E')$ and there is a $G$-torsor $E''$ over $L''$ with $E''_{L'} \simeq E'_{L'}$.

Let $Z$ be the (zero-dimensional) scheme of isomorphisms $\mathrm{Iso}_{L''}(E''/C, E_{L''})$ of $H$-torsors over $L''$. The image of the morphism $\mathrm{Spec}\, L' \to Z$ over $L''$ representing the isomorphism $\alpha_{L'}$ is a one point set $\{z\}$ of $Z$. The field extension $L''(z)/L''$ is algebraic since $\dim Z = 0$.

The isomorphism $\alpha_{L'}$ descends to an isomorphism of the $H$-torsors $E''/C$ and $E$ over $L''(z)$. Hence the isomorphism class of $x_{L'}$ belongs to the image of the map $\widehat{\mathcal{X}}^E\big(L''(z)\big) \to \widehat{\mathcal{X}}^E(L')$. Therefore,

$$\mathrm{ed}_p(G) \geq \mathrm{ed}_p(E') = \mathrm{tr.\,deg}(L'') = \mathrm{tr.\,deg}(L''(z)) \geq \mathrm{ed}_p(x).$$

It follows that $\mathrm{ed}_p(G) \geq \mathrm{ed}_p(\mathcal{X}^E)$. $\qquad\qquad\square$

Let $C^* := \mathrm{Hom}(C, \mathbf{G}_\mathrm{m})$ denote the character group of $C$. An $H$-torsor $E$ over $F$ yields a homomorphism

$$\beta^E : C^* \to \mathrm{Br}(F)$$

taking a character $\chi : C \to \mathbf{G}_\mathrm{m}$ to the image of the class of $E$ under the composition

$$H^1(F, H) \xrightarrow{\partial} H^2(F, C) \xrightarrow{\chi_*} H^2(F, \mathbf{G}_\mathrm{m}) = \mathrm{Br}(F),$$

where $\partial$ is the connecting map for the exact sequence (3). Note that as $\mu_p \subset F^\times$, the intersection of $\mathrm{Ker}(\chi_*)$ over all characters $\chi \in C^*$ is trivial. It follows that the classes of splitting fields of the gerbe $\mathcal{X}^E$ and the subgroup $\mathrm{Im}(\beta^E)$ coincide. It follows that

$$(4) \qquad\qquad\qquad \mathrm{cdim}_p(\mathcal{X}^E) = \mathrm{cdim}_p\big(\mathrm{Im}(\beta^E)\big).$$

Let $\chi_1, \chi_2, \ldots, \chi_s$ be a basis of $C^*$ over $\mathbb{Z}/p\mathbb{Z}$ such that $\{\beta^E(\chi_1), \ldots, \beta^E(\chi_r)\}$ is a minimal basis of $\mathrm{Im}(\beta^E)$ for some $r$ and $\beta^E(\chi_i) = 1$ for $i > r$. By Theorem 2.1, we have

$$(5) \qquad \mathrm{cdim}_p\big(\mathrm{Im}(\beta^E)\big) = \Big( \sum_{i=1}^{r} \mathrm{ind}\, \beta^E(\chi_i) \Big) - r = \Big( \sum_{i=1}^{s} \mathrm{ind}\, \beta^E(\chi_i) \Big) - s.$$

In view of (4) and Theorems 3.1 and 4.2, we shall find an $H$-torsor $E$ (over a field extension of $F$) so that the integer in (5) is as large as possible. Let $U$ be a faithful representation of $H$ and $X$ an open subset of the affine space $\mathbb{A}(U)$ of $U$ where $H$ acts freely. Set $Y := X/H$. Let $E$ be the generic fiber of the $H$-torsor $\pi : X \to Y$. It is a "generic" $H$-torsor over the function field $L := F(Y)$.

Let $\chi : C \to \mathbf{G}_\mathrm{m}$ be a character and $\mathrm{Rep}^{(\chi)}(G)$ the category of all finite dimensional representations $\rho$ of $G$ such that $\rho(c)$ is multiplication by $\chi(c)$ for any $c \in C$. Fix a representations $\rho : G \to \mathbf{GL}(W)$ in $\mathrm{Rep}^{(\chi)}(G)$. The conjugation action of $G$ on $B := \mathrm{End}(W)$ factors through an $H$-action. By descent (cf. [13, Ch. 1, §2]), there is (a unique up to canonical isomorphism) Azumaya algebra $\mathcal{A}$ over $Y$ and an $H$-equivariant algebra isomorphism $\pi^*(\mathcal{A}) \simeq B_X := B \times X$. Let $A$ be the generic fiber of $\mathcal{A}$; it is a central simple algebra over $L = F(Y)$.

Consider the homomorphism $\beta^E : C^* \to \mathrm{Br}(L)$.

**Lemma 4.3.** *The class of $A$ in $\mathrm{Br}(L)$ coincides with $\beta^E(\chi)$.*

*Proof.* Consider the commutative diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & C & \longrightarrow & G & \longrightarrow & H & \longrightarrow & 1 \\
& & \chi \downarrow & & \rho \downarrow & & \alpha \downarrow & & \\
1 & \longrightarrow & \mathbf{G}_\mathrm{m} & \longrightarrow & \mathbf{GL}(W) & \longrightarrow & \mathbf{PGL}(W) & \longrightarrow & 1
\end{array}
$$

The image of the $H$-torsor $\pi : X \to Y$ under $\alpha$ is the $\mathbf{PGL}(W)$-torsor

$$E' := \mathbf{PGL}(W)_X/H \to Y$$

where $\mathbf{PGL}(W)_X := \mathbf{PGL}(W) \times X$ and $H$ acts on $\mathbf{PGL}(W)_X$ by $h(a, x) = (ah^{-1}, hx)$. The conjugation action of $\mathbf{PGL}(W)$ on $B$ gives rise to an isomorphism between $\mathbf{PGL}(W)_X$ and the $H$-torsor $\mathrm{Iso}_X\big(B_X, \mathrm{End}(W)_X\big)$ of isomorphisms between the (split) Azumaya $\mathcal{O}_X$-algebras $B_X$ and $\mathrm{End}(W)_X$. Note that this isomorphism is $H$-equivariant if $H$ acts by conjugation on $B_X$ and trivially on $\mathrm{End}(W)_X$. By descent,

$$E' \simeq \mathrm{Iso}_Y\big(\mathcal{A}, \mathrm{End}(W)_Y\big).$$

Therefore, the image of the class of the torsor $E' \to Y$ under the connecting map for the bottom row of the diagram coincides with the class of the Azumaya algebra $\mathcal{A}$. Restricting to the generic fiber yields $[A] = \beta^E(\chi)$.  $\square$

**Theorem 4.4.** *For any character $\chi \in C^*$, we have $\mathrm{ind}\, \beta^E(\chi) = \min \dim(V)$ over all representations $V$ in $\mathrm{Rep}^{(\chi)}(G)$.*

*Proof.* We follow the approach given in [12]. Let $H$ act on a scheme $Z$ over $F$. We also view $Z$ as a $G$-scheme. Denote by $\mathcal{M}(G, Z)$ the (abelian) category of left $G$-modules on $Z$ that are coherent $\mathcal{O}_Z$-modules (cf. [18, §1.2]). In particular, $\mathcal{M}(G, \operatorname{Spec} F) = \operatorname{Rep}(G)$, the category off all finite dimensional representations of $G$.

Note that $C$ acts trivially on $Z$. For a character $\chi : C \to \mathbf{G}_{\mathrm{m}}$, let $\mathcal{M}^{(\chi)}(G, Z)$ be the full subcategory of $\mathcal{M}(G, Z)$ consisting of $G$-modules on which $C$ acts via $\chi$. For example, $\mathcal{M}^{(\chi)}(G, \operatorname{Spec} F) = \operatorname{Rep}^{(\chi)}(G)$.

We write $K_0(G, Z)$ and $K_0^{(\chi)}(G, Z)$ for the Grothendieck groups of $\mathcal{M}(G, Z)$ and $\mathcal{M}^{(\chi)}(G, Z)$ respectively.

Every $M$ in $\mathcal{M}(G, Z)$ is a direct sum of unique submodules $M^{(\chi)}$ of $M$ in $\mathcal{M}^{(\chi)}(G, Z)$ over all characters $\chi$ of $C$. It follows that

$$K_0(G, Z) = \coprod K_0^{(\chi)}(G, Z).$$

Let $q$ be the order of $G$. By [17, Th. 24], every irreducible representation of $G$ is defined over the field $F(\mu_q)$. Since $F$ contains $p$-th roots of unity, the degree $[F(\mu_q) : F]$ is a power of $p$. Hence the dimension of any irreducible representation of $G$ over $F$ is a power of $p$. It follows by Lemma 4.3 that it suffices to show $\operatorname{ind}(A) = \gcd \dim(V)$ over all representations $V$ in $\operatorname{Rep}^{(\chi)}(G)$.

The image of the map $\dim : K_0(A) \to \mathbb{Z}$ given by the dimension over $L$ is equal to $\operatorname{ind}(A) \cdot \dim(W) \cdot \mathbb{Z}$. To finish the proof of the theorem is suffices to construct a surjective homomorphism

(6) $$K_0\big(\operatorname{Rep}^{(\chi)}(G)\big) \to K_0(A)$$

such that the composition $K_0\big(\operatorname{Rep}^{(\chi)}(G)\big) \to K_0(A) \xrightarrow{\dim} \mathbb{Z}$ is given by the dimension times $\dim(W)$.

First of all we have

(7) $$K_0\big(\operatorname{Rep}^{(\chi)}(G)\big) \simeq K_0^{(\chi)}(G, \operatorname{Spec} F).$$

Recall that $X$ an open subset of $\mathbb{A}(U)$ where $H$ acts freely. By homotopy invariance in the equivariant $K$-theory [18, Cor. 4.2],

$$K_0(G, \operatorname{Spec} F) \simeq K_0\big(G, \mathbb{A}(U)\big).$$

It follows that

(8) $$K_0^{(\chi)}(G, \operatorname{Spec} F) \simeq K_0^{(\chi)}\big(G, \mathbb{A}(U)\big).$$

By localization [18, Th. 2.7], the restriction homomorphism

(9) $$K_0^{(\chi)}\big(G, \mathbb{A}(U)\big) \to K_0^{(\chi)}(G, X).$$

is surjective.

Denote by $\mathcal{M}^{(1)}(G, X, B_X)$ the category of left $G$-modules $M$ on $X$ that are coherent $\mathcal{O}_X$-modules and right $B_X$-modules such that $C$ acts trivially on $M$ and the $G$-action on $M$ and the conjugation $G$-action on $B_X$ agree. The corresponding Grothendieck group is denoted by $K_0^{(1)}(G, X, B_X)$. For any

object $L$ in $\mathcal{M}^{(\chi)}(G, X)$, the group $C$ acts trivially on $L \otimes_F W^*$ and $B$ acts on the right on $L \otimes_F W^*$. We have Morita equivalence

$$\mathcal{M}^{(\chi)}(G, X) \xrightarrow{\sim} \mathcal{M}^{(1)}(G, X, B_X)$$

given by $L \mapsto L \otimes_F W^*$ (with the inverse functor $M \mapsto M \otimes_B W$). Hence

$$(10) \qquad\qquad K_0^{(\chi)}(G, X) \simeq K_0^{(1)}(G, X, B_X).$$

Now, as $C$ acts trivially on $X$ and $B_X$, the category $\mathcal{M}^{(1)}(G, X, B_X)$ is equivalent to the category $\mathcal{M}(H, X, B_X)$ of left $H$-modules $M$ on $X$ that are coherent $\mathcal{O}_X$-modules and right $B_X$-modules such that the $G$-action on $M$ and the conjugation $G$-action on $B_X$ agree. Hence

$$(11) \qquad\qquad K_0^{(1)}(G, X, B_X) \simeq K_0(H, X, B_X).$$

Recall that $Y = X/H$. By descent, the category $\mathcal{M}(H, X, B_X)$ is equivalent to the category $\mathcal{M}(Y, \mathcal{A})$ of coherent $\mathcal{O}_Y$-modules that are right $\mathcal{A}$-modules. Hence

$$(12) \qquad\qquad K_0(H, X, B_X) \simeq K_0(Y, \mathcal{A}).$$

The restriction to the generic point of $Y$ gives a surjective homomorphism

$$(13) \qquad\qquad K_0(Y, \mathcal{A}) \to K_0(A).$$

The homomorphism (6) is the composition of (7), (8), (9), (10), (11), (12) and (13). It takes the class of a representation $V$ to the class in $K_0(A)$ of the generic fiber of the vector bundle $\big((V \otimes W^*) \times X\big)/H$ over $Y$ of rank $\dim(V) \cdot \dim(W)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

**Remark 4.5.** The theorem holds with min replaced by the gcd (with the same proof) in a more general context when the sequence (3) is an arbitrary exact sequence of algebraic groups with $C$ a central diagonalizable subgroup of $G$.

**Example 4.6** (cf. [6], [4, §14], [16, Th. 7.3.8]). Let $p$ be a prime integer, $F$ be a field of characteristic different from $p$ and $C_m$ the cyclic group $\mathbb{Z}/p^m\mathbb{Z}$. Let $K = F(t_1, \ldots, t_{p^m})$ and $C_m$ act on the variables $t_1, \ldots, t_{p^m}$ by cyclic permutations. Then $K$ is a Galois $C_m$-algebra over $K^{C_m}$. Assume that $F$ contains a primitive root of unity $\xi_{p^k}$ for some $k$. The image of the class of $K$ under the connecting map $H^1(F, C_m) \to H^2(F, C_k) \simeq \mathrm{Br}_{p^k}(F)$ for the exact sequence

$$1 \to C_k \to C_n \to C_m \to 1,$$

where $n = k + m$, is the class of the cyclic algebra $A = \big(K/K^{C_m}, \xi_{p^k}\big)$. The group $C_n$ acts $F$-linearly on $F(\xi_{p^n})$ by multiplication by roots of unity making the $F$-space $F(\xi_{p^n})$ a faithful representation of $C_n$ of the smallest dimension. By Theorem 4.4 and Remark 4.5, we have

$$\mathrm{ind}(A) = \big[F(\xi_{p^n}) : F\big].$$

We can now complete the proof of Theorem 4.1. By Theorem 4.4, there are representations $V_i$ in $\mathrm{Rep}^{(\chi_i)}(G)$ such that $\mathrm{ind}\,\beta^E(\chi_i) = \dim(V_i)$, $i = 1, \ldots, s$. Let $V$ be the direct sum of all the $V_i$. By Theorem 4.2 (applied to the group $G$ over $L$ and the generic torsor $E$), Theorem 3.1, (4) and (5), we have

$$\mathrm{ed}_p(G) \geq \mathrm{ed}_p(G_L) \geq \mathrm{ed}_p(\mathcal{X}^E) = \mathrm{cdim}_p(\mathcal{X}^E) + s = \mathrm{cdim}_p\big(\mathrm{Im}(\beta^E)\big) + s$$

$$= \sum_{i=1}^{s} \mathrm{ind}\,\beta^E(\chi_i) = \sum_{i=1}^{s} \dim(V_i) = \dim(V).$$

Since $\chi_1, \chi_2, \ldots, \chi_s$ generate $C^*$, the restriction of $V$ on $C$ is faithful. As every nontrivial normal subgroup of $G$ intersects $C$ nontrivially, the $G$-representation $V$ is faithful. We have constructed a faithful representation $V$ of $G$ over $F$ with $\mathrm{ed}_p(G) \geq \dim(V)$. The theorem is proved.

**Remark 4.7.** The proof of Theorem 4.1 shows how to compute the essential dimension of $G$ over $F$. For every character $\chi \in C^*$ choose a representation $V_\chi \in \mathrm{Rep}^{(\chi)}(G)$ of the smallest dimension. It appears as an irreducible component of the smallest dimension of the induced representation $\mathrm{Ind}_C^G(\chi)$. We construct a basis $\chi_1, \ldots, \chi_s$ of $C^*$ by induction as follows. Let $\chi_1$ be a nonzero character with the smallest $\dim(V_{\chi_1})$. If the characters $\chi_1, \ldots, \chi_{i-1}$ are already constructed for some $i \leq s$, then we take for $\chi_i$ a character with minimal $\dim(V_{\chi_i})$ among all the characters outside of the subgroup generated by $\chi_1, \ldots, \chi_{i-1}$. Then $V$ is a faithful representation of the least dimension and $\mathrm{ed}(G) = \sum_{i=1}^{s} \dim(V_{\chi_i})$.

**Remark 4.8.** We can compute the essential $p$-dimension of an arbitrary finite group $G$ over a field $F$ of characteristic different from $p$. (We don't assume that $F$ contains $p$-th roots of unity.) Let $G'$ a Sylow $p$-subgroup of $G$. One can prove that $\mathrm{ed}_p(G) = \mathrm{ed}_p(G')$ and $\mathrm{ed}_p(G')$ does not change under field extensions of degree prime to $p$. In particular $\mathrm{ed}_p(G') = \mathrm{ed}_p(G'_{F'})$ where $F' = F(\mu_p)$. It follows from Theorem 4.1 that $\mathrm{ed}_p(G)$ coincides with the least dimension of a faithful representation of $G'$ over $F'$.

## 5. An application

**Theorem 5.1.** *Let $G_1$ and $G_2$ be two $p$-groups and $F$ a field of characteristic different from $p$ containing a primitive $p$-th root of unity. Then*

$$\mathrm{ed}(G_1 \times G_2) = \mathrm{ed}(G_1) + \mathrm{ed}(G_2).$$

*Proof.* The index $j$ in the proof takes the values 1 and 2. If $V_j$ is a faithful representation of $G_j$ then $V_1 \oplus V_2$ is a faithful representation of $G_1 \times G_2$. Hence $\mathrm{ed}(G_1 \times G_2) \leq \mathrm{ed}(G_1) + \mathrm{ed}(G_2)$ (cf. [5, Lemma 4.1(b)]).

Denote by $C_j$ the subgroup of all central elements of $G_j$ of exponent $p$. Set $C = C_1 \times C_2$. We identify $C^*$ with $C_1^* \oplus C_2^*$.

For every character $\chi \in C^*$ choose a representation $\rho_\chi : G_1 \times G_2 \to \mathbf{GL}(V_\chi)$ in $\mathrm{Rep}^{(\chi)}(G_1 \times G_2)$ of the smallest dimension. We construct a basis $\{\chi_1, \chi_2, \ldots, \chi_s\}$ of $C^*$ following Remark 4.7. We claim that all the $\chi_i$ can be

chosen in one of the $C_j^*$. Indeed, suppose the characters $\chi_1, \ldots, \chi_{i-1}$ are already constructed, and let $\chi_i$ be a character with minimal $\dim(V_{\chi_i})$ among the characters outside of the subgroup generated by $\chi_1, \ldots, \chi_{i-1}$. Let $\chi_i = \chi_i^{(1)} + \chi_i^{(2)}$ with $\chi_i^{(j)} \in C_j^*$. Denote by $\varepsilon_1$ and $\varepsilon_2$ the endomorphisms of $G_1 \times G_2$ taking $(g_1, g_2)$ to $(g_1, 1)$ and $(1, g_2)$ respectively. The restriction of the representation $\rho_{\chi_i} \circ \varepsilon_j$ on $C$ is given by the character $\chi_i^{(j)}$. We replace $\chi_i$ by $\chi_i^{(j)}$ with $j$ such that $\chi_i^{(j)}$ does not belong to the subgroup generated by $\chi_1, \ldots, \chi_{i-1}$. The claim is proved.

Let $W_j$ be the direct sum of all the $V_{\chi_i}$ with $\chi_i \in C_j^*$. Then the restriction of $W_j$ on $C_j$ is faithful, hence so is the restriction of $W_j$ on $G_j$. It follows that $\operatorname{ed}(G_j) \leq \dim(W_j)$. As $W_1 \oplus W_2 = V$, we have

$$\operatorname{ed}(G_1) + \operatorname{ed}(G_2) \leq \dim(W_1) + \dim(W_2) = \dim(V) = \operatorname{ed}(G_1 \times G_2). \qquad \square$$

**Corollary 5.2.** *Let $F$ be a field as in Theorem 5.1. Then*

$$\operatorname{ed}\big(\mathbb{Z}/p^{n_1}\mathbb{Z} \times \mathbb{Z}/p^{n_2}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{n_s}\mathbb{Z}\big) = \sum_{i=1}^{s} \big[F(\xi_{p^{n_i}}) : F\big].$$

*Proof.* By Theorem 5.1, it suffices to consider the case $s = 1$. This case has been done in [6]. It is also covered by Theorem 4.1 as the natural representation of the group $\mathbb{Z}/p^n\mathbb{Z}$ in the $F$-space $F(\xi_{p^n})$ is faithful irreducible of the smallest dimension (cf. Remark 4.6). $\qquad \square$

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Institut de Mathématiques de Jussieu, Université Pierre et Marie Curie (Paris 6), 4 place Jussieu, F-75252 Paris CEDEX 05, FRANCE
*E-mail address*: karpenko@math.jussieu.fr

Department of Mathematics, University of California, Los Angeles, CA 90095-1555, USA
*E-mail address*: merkurev@math.ucla.edu