# HERMITIAN FORMS OVER QUATERNION ALGEBRAS

NIKITA A. KARPENKO AND ALEXANDER S. MERKURJEV

Abstract. We study a hermitian form $h$ over a quaternion division algebra $Q$ over a field ($h$ is supposed to be alternating if the characteristic of the field is 2). For generic $h$ and $Q$, for any integer $i \in [1,\ n/2]$, where $n := \dim_Q h$, we show that the variety of $i$-dimensional (over $Q$) totally isotropic right subspaces of $h$ is 2-incompressible. The proof is based on a computation of the Chow ring for the classifying space of a certain parabolic subgroup in a split simple adjoint affine algebraic group of type $C_n$. As an application, we determine the smallest value of the $J$-invariant of a non-degenerate quadratic form divisible by a 2-fold Pfister form; we also determine the biggest values of the canonical dimensions of the orthogonal Grassmannians associated to such quadratic forms.

## Contents



*Date*: May, 2014.

*Key words and phrases.* Algebraic groups, classifying spaces, hermitian and quadratic forms, projective homogeneous varieties, Chow groups and motives. *Mathematical Subject Classification (2010):* 14L17; 14C25.

The first author acknowledges the support of the French Agence Nationale de la Recherche (ANR) under reference ANR-12-BL01-0005; his work has been also supported by the start-up grant of the University of Alberta.

The work of the second author has been supported by the NSF grant DMS #1160206.

## 1. Introduction

Let $p$ be a prime integer. The *canonical $p$-dimension* of a smooth complete variety $X$ is a non-negative integer $\mathrm{cd}_p(X)$ defined as the minimal dimension of a closed subvariety of $X$ which admits a closed point of degree prime-to-$p$ after scalar extension to the function field of $X$. In the case where $\mathrm{cd}_p(X) = \dim X$, the variety $X$ is said to be *$p$-incompressible*. The central motivation underlying these definitions is found in the investigation of splitting properties of torsors under linear algebraic groups, where the problem of determining the integer $\mathrm{cd}_p(X)$ for suitable $p$ and suitable projective homogeneous varieties of $p$-divisible index lies at the heart of many fundamental questions. In recent years, a systematic approach to the study of the canonical $p$-dimension of projective homogeneous varieties via the concept of Chow motives has emerged. Despite the substantial progress which has been made, there remain relatively few examples of varieties $X$ (of $p$-divisible index) for which the precise value of $\mathrm{cd}_p(X)$ has been determined.

In two recent articles [9, 10], the first author showed that, for each $1 \le i \le n/2$, the variety $X_i$ of $i$-dimensional totally isotropic subspaces associated to either

(1) a *generic* $n$-dimensional quadratic form over a field, or
(2) a *generic* $n$-dimensional hermitian form over a separable quadratic extension of a field

is 2-incompressible. The present paper represents a natural extension of this work. Its main result (Theorem 10.1) establishes, for each $1 \le i \le n/2$, the 2-incompressibility of the variety $X_i$ of $i$-dimensional totally isotropic subspace associated to

(3) a *generic* $n$-dimensional hermitian form over a *generic* quaternion algebra over a field.

As in [10], this result has an application to the theory of quadratic forms; namely, it determines the canonical 2-dimension of the orthogonal Grassmannians associated to any quadratic form $q$ defined as the tensor product of a *generic* 2-fold Pfister form and a *generic* $n$-dimensional symmetric bilinear form (Corollary 11.2); this, in turn, determines the $J$-invariant of $q$, and hence the Chow ring of the *maximal* orthogonal Grassmannian of $q$ modulo torsion and 2-divisible elements (Corollary 11.3).

For each $1 \le i \le n/2$, let $X_i$ denote the $F$-variety of $i$-dimensional totally isotropic subspaces associated to a generic $n$-dimensional hermitian form $h$ over a generic quaternion algebra $Q$ over a field $F$. The main result of the paper (Theorem 10.1) asserts that, for each $i$, a certain summand of the (mod-2) Chow motive of $X_i$ (known here as the "essential motive" of $X_i$) is indecomposable. By arguments which by now are well-known, this readily implies that the $X_i$ are 2-incompressible. Following arguments originally developed in [9] and [10], we reduce the proof of the theorem to the proof of a certain statement concerning generators of the ring $\overline{\mathrm{Ch}}(Y)$, where $Y$ denotes the variety of complete flags of totally isotropic subspaces in $h$ (this statement is proved in Proposition 6.1).

Now, via a specialization argument, the genericity assumptions on $h$ and $Q$ enable us to reduce the statement concerning $\overline{\mathrm{Ch}}(Y)$ to another concerning generators of the ring $\overline{\mathrm{Ch}}(E/P)$, where $P$ is a certain parabolic subgroup of the group $G = \mathbf{PGSp}_{2n}$ and $E$ is a "generic $G$-torsor" (this second statement is proved in Corollary 5.14). In order to prove this statement, it is sufficient to prove a general statement concerning generators of the mod-2 Chow ring of the classifying space of the parabolic subgroup $P$. This is done in Corollary 5.12.

We conclude the introduction with a couple of comments on our notation and terminology.

A *variety* in the paper is a separated scheme of finite type over a field. We write $\mathrm{Ch}(X)$ for its Chow group with coefficients in the field $\mathbb{F}_2 := \mathbb{Z}/2\mathbb{Z}$. So, $\mathrm{Ch}(X) = \mathrm{CH}(X)/2\,\mathrm{CH}(X)$, where $\mathrm{CH}(X)$ is the Chow group with integer coefficients. We are also working with the *reduced* Chow group $\overline{\mathrm{Ch}}(X)$ defined as the quotient of $\mathrm{Ch}(X)$ by the subgroup of the elements vanishing over an extension field of $F$. For smooth $X$, this subgroup is an ideal of the ring $\mathrm{Ch}(X)$ so that $\overline{\mathrm{Ch}}(X)$ is a ring as well.

We are using the Grothendieck Chow motives with coefficients in $\mathbb{F}_2$ and write $M(X)$ for the motive of a smooth complete variety $X$.

ACKNOWLEDGEMENTS. We thank the anonymous referee for numerous suggestions improving the exposition. We also thank Burt Totaro for useful advices. The first author gratefully acknowledges hospitality of the Fields Institute for Research in Mathematical Sciences (Toronto, Ontario), the Department of Mathematics of University of Toronto, and Thematic Program on Torsors, Nonassociative Algebras and Cohomological Invariants (January–June 2013).

## 2. The norm subgroup of a quaternion algebra

A *quaternion* algebra $Q$ over a variety $X$ is a rank 4 Azumaya algebra over $X$. Let $s : C \to X$ be the corresponding conic over $X$, i.e., the Severi-Brauer scheme of $Q$ (the scheme of right ideals in $Q$ of reduced rank 1).

The *norm subgroup* $N_Q$ of $Q$ is the image of the push-forward homomorphism

$$s_* : \mathrm{Ch}(C) \to \mathrm{Ch}(X).$$

We write $N_Q^i$ for the codimension-$i$ part of $N_Q$. By the projection formula, $N_Q$ is an ideal in $\mathrm{Ch}(X)$ if $X$ is smooth. The fiber $C_x$ of $s$ over a point $x \in X$ is the conic curve over the residue field $F(x)$ corresponding to the quaternion $F(x)$-algebra $Q_x$. If $Q_x$ is split, $C_x$ has a rational point (in fact $C_x \simeq \mathbb{P}^1$), hence $[x] \in N_Q$. If $Q_x$ is not split, then by Springer's Theorem [3, Corollary 18.5], the degree of every closed point on $C_x$ is even; therefore the subgroup of $\mathrm{CH}(X)$ generated by $s_*([y])$ for all $y$ in $C_x$ is equal to the subgroup generated by $2[x] = 0$ in this case. It follows that $N_Q$ is generated by the classes $[x]$ of the points $x \in X$ such that $Q_x$ is split.

**Lemma 2.1.** *Let $Q$ and $Q'$ be two quaternion algebras over a variety $X$. If $[Q] = [Q']$ in $\mathrm{Br}(X)$, then $N_Q = N_{Q'}$.*

*Proof.* For any $x \in X$, we have $[Q_x] = [Q'_x]$ in $\mathrm{Br}\,F(x)$ and hence $Q_x \simeq Q'_x$ (in general, two Brauer-equivalent central simple algebras of the same dimension are isomorphic, see [12, discussion after Definition 1.3]). It follows from the description of $N_Q$ and $N_{Q'}$ before the lemma that $N_Q = N_{Q'}$. □

Note that if $X$ is irreducible and smooth, $[Q] = [Q']$ in $\mathrm{Br}(X)$ if and only if the classes of the generic fibers of $Q$ and $Q'$ are equal in $\mathrm{Br}\,F(X)$, see [15, Corollary IV.2.6]).

We will need the following functorial property of $N_Q$:

**Lemma 2.2.** *Let $g : X' \to X$ be a morphism of smooth schemes and let $Q$ be a quaternion algebra over $X$. Let $Q'$ be the pull-back of $Q$ with respect to $g$. Then the inverse image homomorphism $g^* : \mathrm{Ch}(X) \to \mathrm{Ch}(X')$ takes $N_Q$ to $N_{Q'}$. In particular, $g^*$ yields a homomorphism*

$$\mathrm{Ch}(X)/N_Q \longrightarrow \mathrm{Ch}(X')/N_{Q'}.$$

*Proof.* Let $s : C \to X$ be the conic associated to $Q$. Then the conic curve $s' : C' \to X'$ associated with $Q'$ is the pull-back of $s$ and we have a commutative diagram (see [17, Proposition 12.5])

$$\begin{array}{ccc} \mathrm{Ch}(C) & \xrightarrow{h^*} & \mathrm{Ch}(C') \\ {\scriptstyle s'_*}\downarrow & & \downarrow{\scriptstyle s_*} \\ \mathrm{Ch}(X) & \xrightarrow{g^*} & \mathrm{Ch}(X'), \end{array}$$

where $h : C' \to C$ is the induced morphism. □

## 3. Projective $Q$-spaces

Let $Q$ be a quaternion division algebra over a field $F$. For any finite-dimensional right $Q$-vector space $V$, the *projective $Q$-space* $Q\mathbb{P}(V)$ is defined as the $F$-variety of 1-dimensional subspaces in $V$. The dimension of subspaces here is taken over $Q$; the dimension over $F$ is 4; the *reduced dimension* (defined as the dimension over $F$ divided by $\deg Q = 2$) is therefore 2. For any $n \geq 0$, we define $Q\mathbb{P}^n$ as $Q\mathbb{P}(Q^{n+1})$. In particular, $Q\mathbb{P}^0$ is the point $\mathbf{pt} = \operatorname{Spec} F$. For any $n \geq 0$, $Q\mathbb{P}^n$ is a smooth projective $F$-variety of dimension $4n$.

Of course, the variety $Q\mathbb{P}(V)$ (and therefore $Q\mathbb{P}^n$) can be defined for an arbitrary (non-split or split) quaternion algebra $Q$ as the variety of $Q$-submodules in $V$ (resp., in $Q^{n+1}$) of reduced dimension 2 (where $V$ is a right $Q$-module of even reduced dimension). Then $Q\mathbb{P}(V)_L = Q_L\mathbb{P}(V_L)$ and $Q\mathbb{P}^n_L = Q_L\mathbb{P}^n$ for any extension field $L/F$. Once $Q$ is split, an identification of $Q$ with the matrix algebra $M_2(F)$ identifies $Q\mathbb{P}^n$ with the Grassmannian of 2-planes in the $2(n+1)$-dimensional $F$-vector space $F^{2(n+1)}$. Our main case of interest, however, is the case where $Q$ is a division algebra.

The following statement is true for $Q$ non-split or split but only needs to be proved in the split case:

**Lemma 3.1.** *Let $a \in \mathrm{Ch}^4(Q\mathbb{P}^n)$ be the 4-th Chern class of the (rank 4) tautological vector bundle on $Q\mathbb{P}^n$. Then $\deg(a^n) = 1$.*

*Proof.* Replacing $F$ by a field extension splitting $Q$, we may assume that $Q$ itself is split. In this case there is an isomorphism of $Q\mathbb{P}^n$ with the Grassmannian $X$ of 2-planes in $F^{2(n+1)}$ such that the tautological bundle on $Q\mathbb{P}^n$ corresponds to a vector bundle on $X$ isomorphic to $\mathcal{T} \oplus \mathcal{T}$, where $\mathcal{T}$ is the tautological bundle on $X$. By the Whitney Sum Formula [3, Proposition 54.7], it follows that the element $a$ corresponds to $c_2^2(\mathcal{T})$. By [4, Proposition 14.6.5], $c_2^{2n}(\mathcal{T})$ is a generator of the group $\mathrm{Ch}_0(X)$. Therefore $\deg(a^n) = \deg(c_2^{2n}(\mathcal{T})) = 1$ (because $X$ possesses a closed point of degree 1). □

**Lemma 3.2.** *The* Ch*-motive $M(Q\mathbb{P}^n)$ decomposes in a direct sum of three summands: $M(\mathbf{pt})$, $\bigoplus_{i=1}^{2n} M(C)(i)$, and $M(Q\mathbb{P}^{n-1})(4)$.*

*Proof.* This is a particular case of [5, Theorem 10.9 and Corollary 10.19], where the characteristic of the base field is assumed to be different from 2. We refer to [1] for a characteristic-free treatment. □

**Corollary 3.3.** *The motive of* $Q\mathbb{P}^n$ *decomposes in a direct sum of two summands: the first one is* $\bigoplus_{i=0}^{n} M(\mathbf{pt})(4i)$ *and the second one is a direct sum of shifts of* $M(C)$. □

Since the product $C\times C$ is a projective bundle over $C$, the motive of $C\times C$ is isomorphic to the sum $M(C)\oplus M(C)(1)$ by [13, §7]. Therefore we get

**Corollary 3.4.** *For any integers* $n_1,\dots,n_r\geq 0$*, the motive of the product*

$$P := Q\mathbb{P}^{n_1}\times\cdots\times Q\mathbb{P}^{n_r}$$

*decomposes in a direct sum of* $(n_1+1)\dots(n_r+1)$ *shifts of* $M(\mathbf{pt})$ *and several shifts of* $M(C)$.

For an arbitrary smooth $F$-variety $X$, let us consider the (constant) Azumaya algebra $Q_X$ over $X$ given by the pull-back of $Q$. The norm subgroup $N_{Q_X}$ is defined in Section 2 as the image of the push-forward homomorphism $\mathrm{Ch}(C_X)\to\mathrm{Ch}(X)$ with respect to the projection $C_X := X\times C\to X$. (Note that $C_X$ is the conic over $X$ associated with the quaternion algebra $Q_X$.)

If $Q$ is split, the quotient $\mathrm{Ch}(X)/N_{Q_X}$ is 0. For a non-split $Q$ we, for instance, have $\mathrm{Ch}(\mathbf{pt})/N_Q=\mathbb{F}_2:=\mathbb{Z}/2\mathbb{Z}$ and $\mathrm{Ch}(C)/N_{Q_C}=0$. This proves the following

**Lemma 3.5.** *Assume that* $Q$ *is not split. Let* $X$ *be a smooth complete variety such that the motive of* $X$ *decomposes into a direct sum* $M\oplus M'$*, where* $M$ *is a direct sum of shifts of* $M(C)$ *and where* $M'$ *is a direct sum of shifts of* $M(\mathbf{pt})$*. Then the group* $N_{Q_X}$ *is isomorphic to the (homological as well as cohomological) Chow group of* $M$ *while the quotient* $\mathrm{Ch}(X)/N_{Q_X}$ *is isomorphic to the (homological as well as cohomological) Chow group of* $M'$. □

**Proposition 3.6.** *Assume that* $Q$ *is division algebra. For* $P$ *as in Corollary 3.4, let*

$$a_1,\dots,a_r\in\mathrm{Ch}(P)/N_{Q_P}$$

*be the elements given by the 4-th Chern classes of the tautological vector bundles of the factors of* $P$*. The* $\mathbb{F}_2$*-algebra* $\mathrm{Ch}(P)/N_{Q_P}$ *is generated by these elements subject to the relations* $a_1^{n_1+1}=0,\dots,a_r^{n_r+1}=0$.

*Proof.* Since by Springer's Theorem [3, Corollary 18.5] the degree of any closed point on $C$ is divisible by 2, the degree of any closed point on $P\times C$ is also divisible by 2 so that the degree homomorphism $\deg:\mathrm{Ch}(P)\to\mathbb{F}_2$ is defined on the quotient $\mathrm{Ch}(P)/N_{Q_P}$.

The monomials $a_1^{i_1}\dots a_r^{i_r}$ with $0\leq i_1\leq n_1$, …, $0\leq i_r\leq n_r$, are linearly independent. Indeed, if a linear combination $\alpha$ of the monomials with coefficients $\alpha_{i_1\dots i_r}\in\mathbb{F}_2$ is 0, then

$$0=\deg(\alpha\cdot a_1^{n_1-i_1}\dots a_r^{n_r-i_r})=\alpha_{i_1\dots i_r}$$

for every $i_1,\dots,i_r$.

It follows that the dimension (over $\mathbb{F}_2$) of the $\mathbb{F}_2$-subalgebra in $\mathrm{Ch}(P)/N_{Q_P}$ generated by $a_1,\dots,a_r$ is at least the product $(n_1+1)\dots(n_r+1)$. Since this product is equal to $\dim_{\mathbb{F}_2} \operatorname{Ch}(P)/N_{Q_P}$ (see Corollary 3.4 and Lemma 3.5), the $\mathbb{F}_2$-algebra $\operatorname{Ch}(P)/N_{Q_P}$ is generated by $a_1, \dots, a_r$.

The elements $a_1, \dots, a_r$ satisfy the indicated relations simply by dimension reason: $a_i$ is given by the pull-back of an element in $\operatorname{Ch}^4(Q\mathbb{P}^{n_i})$ and $\dim Q\mathbb{P}^{n_i} = 4n_i$. The resulting $\mathbb{F}_2$-algebra epimorphism

$$\mathbb{F}_2[t_1, \dots, t_r]/(t_1^{n_1+1}, \dots, t_r^{n_r+1}) \to \operatorname{Ch}(P)/N_{Q_P}, \ \ t_i \mapsto a_i$$

is an isomorphism by dimension reason once again. □

## 4. Smooth $Q$-fibrations

The following lemma is a distant descendant of [23, Statement 2.13]:

**Lemma 4.1.** *Let $f : X' \to X$ be a smooth morphism of smooth $F$-varieties, $Q$ a quaternion algebra on $X$, $Q' = f^*Q$ and $r$ an integer. Let $B \subset \operatorname{Ch}(X')$ be a homogeneous $\operatorname{Ch}(X)$-submodule containing $N_{Q'}$. Suppose that for any integer $i$ and any point $x \in X$ of codimension $i$ such that the restriction of $Q$ on $x$ is not split, the composition*

$$B^{r-i} \hookrightarrow \operatorname{Ch}^{r-i}(X') \to \operatorname{Ch}^{r-i}(X'_x) \to \operatorname{Ch}^{r-i}(X'_x)/N^{r-i}_{Q'_x}, \tag{4.2}$$

*where $Q'_x$ is the restriction of $Q'$ on the fiber $X'_x$ of $f$ over $x$, is surjective. Then $B^r = \operatorname{Ch}^r(X')$.*

*Proof.* First of all we notice that $\operatorname{Ch}(X'_x)/N_{Q'_x} = 0$ if the restriction of $Q$ on $x$ is split. Therefore the composition (4.2) is surjective for any point $x \in X$ of codimension $i$.

For any integer $i$, we write $\mathcal{F}^i \operatorname{Ch}^r(X')$ for the subgroup in $\operatorname{Ch}^r(X')$ generated by the classes of cycles on $X'$ whose images in $X$ have codimension $\geq i$; these are terms of a descending ring filtration on $\operatorname{Ch}^r(X')$. For any point $x \in X$ of codimension $i$, we have a homomorphism $\operatorname{Ch}^{r-i}(X'_x) \to \mathcal{F}^i \operatorname{Ch}^r(X')/\mathcal{F}^{i+1} \operatorname{Ch}^r(X')$ mapping the class of a point $x' \in X'_x$ to the the class modulo $\mathcal{F}^{i+1} \operatorname{Ch}^r(X')$ of the class of $x'$ considered as a point of $X'$. The composition

$$\operatorname{Ch}^{r-i}(X') \to \operatorname{Ch}^{r-i}(X'_x) \to \mathcal{F}^i \operatorname{Ch}^r(X')/\mathcal{F}^{i+1} \operatorname{Ch}^r(X')$$

is the multiplication by $[x] \in \operatorname{Ch}^i(X)$. The sum $\bigoplus_x \operatorname{Ch}^{r-i}(X'_x)$ over all points $x \in X$ of codimension $i$ surjects onto the quotient $\mathcal{F}^i \operatorname{Ch}^r(X')/\mathcal{F}^{i+1} \operatorname{Ch}^r(X')$.

Let $\alpha \in \operatorname{Ch}^r(X')$. Inducting on $i$, we will show that $\alpha \in B^r + \mathcal{F}^i \operatorname{Ch}^r(X')$ for any $i$. With a sufficiently large $i$ this will give the required statement.

The case of $i \leq 0$ being trivial, we assume that $\alpha \in B^r + \mathcal{F}^i \operatorname{Ch}^r(X')$ for some $i \geq 0$, and we show that $\alpha \in B^r + \mathcal{F}^{i+1} \operatorname{Ch}^r(X')$. We write $\alpha$ as a sum of an element of $B^r$ and some $\beta \in \mathcal{F}^i \operatorname{Ch}^r(X')$

The class of $\beta$ modulo $\mathcal{F}^{i+1} \operatorname{Ch}^r(X')$ decomposes into a sum of elements of two kinds. An element of the first kind is in the image of the composition

$$B^{r-i} \hookrightarrow \operatorname{Ch}^{r-i}(X') \to \operatorname{Ch}^{r-i}(X'_x) \to \mathcal{F}^i \operatorname{Ch}^r(X')/\mathcal{F}^{i+1} \operatorname{Ch}^r(X')$$

for some $x$, and therefore is represented by an element of $B^{r-i} \cdot [x] \subset B^r$.

An element of the second kind is in the image of the composition

$$N^{r-i}_{Q'_x} \hookrightarrow \operatorname{Ch}^{r-i}(X'_x) \to \mathcal{F}^i \operatorname{Ch}^r(X')/\mathcal{F}^{i+1} \operatorname{Ch}^r(X').$$

We claim that any element of this image is represented by an element of $N^r_{Q'}$. Since $N^r_{Q'} \subset B^r$, Lemma 4.1 is proved with this claim.

To prove the claim, we recall from §2 that $N^{r-i}_{Q'_x}$ is generated by $[x']$ with $x' \in X'_x$ of codimension $r-i$ such that the restriction of $Q'_x$ on $x'$ is split. The image of such a generator in $\mathcal{F}^i \operatorname{Ch}^r(X')/\mathcal{F}^{i+1}\operatorname{Ch}^r(X')$ is represented by the class $[x'] \in \operatorname{Ch}(X')$ of $x'$ viewed as a point of $X'$. Since the restriction of $Q'_x$ on $x' \in X'_x$ coincides with the restriction of $Q'$ on $x' \in X'$, the class $[x'] \in \operatorname{Ch}^r(X')$ is in $N^r_{Q'} \subset \operatorname{Ch}^r(X')$. □

**Example 4.3.** Let $V$ be a right $Q$-module of even reduced rank $2(n+1)$. The corresponding *projective $Q$-bundle* $Q\mathbb{P}(V)$ is then defined as the $X$-scheme of $Q$-submodules in $V$ of reduced rank 2. The structure morphism $Q\mathbb{P}(V) \to X$ is smooth and proper; its fiber over a point $x \in X$ is the projective $Q_x$-space $Q_x\mathbb{P}(V_x)$ defined in the previous section. Let $Q'$ the the pull-back of $Q$ to $Q\mathbb{P}(V)$. It follows by Propositions 3.6 and Lemma 4.1 that the $\operatorname{Ch}(X)/N_Q$-algebra $\operatorname{Ch}(Q\mathbb{P}(V))/N_{Q'}$ is generated by the 4-th Chern class $a$ of the tautological vector bundle on $Q\mathbb{P}(V)$. More precisely, the $\operatorname{Ch}(X)/N_Q$-module $\operatorname{Ch}(Q\mathbb{P}(V))/N_{Q'}$ is generated by the powers $a^i$ with $i = 0, \dots, n$.

## 5. Chow rings of some classifying spaces

5a. **Chow rings of classifying spaces.** Let $G$ be an algebraic group[1] over a field $F$. Write $BG$ for the *classifying space* of $G$ viewed as the stack of $G$-torsors over the category of $F$-varieties (see [24]).

In [21], Totaro defined the Chow ring $\operatorname{CH}(BG)$ as follows. Let $V$ be a generically free linear representation of $G$ over $F$ and $U \subset V$ a $G$-invariant open subscheme admitting a $G$-torsor $U \to U/G$ for a variety $U/G$. For any integer $i \geq 0$, the Chow group $\operatorname{CH}^i(U/G)$ does not depend (up to canonical isomorphism) on the choice of $V$ and $U$ provided that $\operatorname{codim}_V(V \setminus U) > i$. We write $\operatorname{CH}^i(BG)$ for $\operatorname{CH}^i(U/G)$ and refer to $U/G$ as to an *$i$-th approximation* of the classifying space $BG$ of $G$.

By [21, Theorem 1.3], the group $\operatorname{CH}^i(BG)$ is naturally identified with the set of functorial assignments $\alpha$ to every smooth variety $X$ over $F$ with a $G$-torsor $E$ over $X$ of an element $\alpha(E) \in \operatorname{CH}^i(X)$.

Let $E \to \mathbf{pt}$ be a $G$-torsor and ${}^EG := \operatorname{Aut}_G(E)$ the *twist* of $G$ by $E$ (see [19, §1 of Chapter V]). The correspondence $I \mapsto \operatorname{Iso}_G(I, E)$ gives rise to an equivalence between $BG$ and $B({}^EG)$. In particular, there is a natural ring isomorphism between $\operatorname{CH}(BG)$ and $\operatorname{CH}(B{}^EG)$.

**Example 5.1.** The ring $\operatorname{CH}(B\,\mathbf{GL}_n)$ is the polynomial ring on the Chern classes

$$c_1, c_2, \dots, c_n$$

of the tautological vector bundle on $B\,\mathbf{GL}_n$ (see [21]). A representation $\rho : G \to \mathbf{GL}_n$ yields the pull-back homomorphism $\rho^* : \operatorname{CH}(B\,\mathbf{GL}_n) \to \operatorname{CH}(BG)$. The elements $\rho^*(c_i)$ are the Chern classes of the pull-back of the tautological vector bundle under

$$B\rho : BG \to B\,\mathbf{GL}_n\,.$$

[1]By *algebraic group* we always mean *affine* algebraic group.

**Example 5.2.** Let $A$ be a central simple algebra of degree $n$ over $F$ and $G = \mathbf{GL}_1(A)$. For every $N > 0$, the group $G$ acts on the open subvariety $U_N$ of nondegenerate elements in the free $A$-module $A^N$. Then $BG$ is approximated by the Grassmannian varieties $A\mathbb{P}^{N-1} := U_N/G$ of $A$-submodules in $A^N$ of reduced dimension $n$. We write $BG = A\mathbb{P}^\infty$ (an infinite "projective space" over $A$). The left multiplication action of $A$ on itself yields a representation $\mathbf{GL}_1(A) \to \mathbf{GL}(A) = \mathbf{GL}_{n^2}$. The pull-back of the tautological vector bundle on $B\,\mathbf{GL}_{n^2}$ is the tautological vector bundle on $BG = A\mathbb{P}^\infty$.

**Example 5.3.** The projective linear group $\mathbf{PGL}_2$ is the automorphism group of the matrix algebra $M_2$. The twisted forms of $M_2$ are the quaternion algebras. It follows that every $\mathbf{PGL}_2$-torsor over a scheme $X$ is isomorphic to the torsor of isomorphisms between a unique (up to canonical isomorphism) quaternion algebra $Q$ over $X$ and the matrix algebra $M_2$. The algebra $Q$ carries a canonical (symplectic) involution $\tau$. The kernel $Q^0$ of the endomorphism $1 + \tau$ on $Q$ is a sub-bundle of $Q$ of rank 3. The natural homomorphism

$$\mathbf{PGL}_2 = \mathrm{Aut}(M_2) \to \mathrm{Aut}((M_2)^0) = \mathbf{O}_3^+$$

is an isomorphism between the group $\mathbf{PGL}_2$ of Dynkin type $\mathsf{A}_1$ and the split special orthogonal group $\mathbf{O}_3^+$ of type $\mathsf{C}_1$ (see [12, §15]). It is proved in [21, §16] (see also [16]) that if $\mathrm{char}(F) \neq 2$, then the ring $\mathrm{CH}(B\,\mathbf{PGL}_2) = \mathrm{CH}(B\,\mathbf{O}_3^+)$ is generated by the Chern classes $c_2(Q^0)$ and $c_3(Q^0)$ (i.e., the elements of the Chow group corresponding in view of the third paragraph of §5a to the assignments to every smooth variety $X$ over $F$ with the torsor over $X$ given by a quaternion algebra $Q$ over $X$ of the classes $c_2(Q^0)$ and $c_3(Q^0)$) with the relation $2c_3(Q^0) = 0$. If $\mathrm{char}(F) = 2$, the result (and the proof) still holds. Moreover, in this case, $Q^0$ contains the trivial line sub-bundle generated by 1, hence $c_3(Q^0) = 0$.

Let $V$ be a generically free representation of $G$ and let $U/G$ be an approximation of $BG$ as above. The morphism $U/G \to BG$ induced by the *versal* $G$-torsor $U \to U/G$ yields a ring homomorphism $\mathrm{CH}(BG) \to \mathrm{CH}(U/G)$.

**Lemma 5.4.** *The ring homomorphism* $\mathrm{CH}(BG) \to \mathrm{CH}(U/G)$ *is surjective.*

*Proof.* Suppose that an algebraic group $G$ acts on a variety $X$ over $F$. The *$G$-equivariant* Chow group $\mathrm{CH}^G(X)$ was defined in [2]. In particular, $\mathrm{CH}^G(\mathbf{pt}) = \mathrm{CH}(BG)$. The homomorphism $\mathrm{CH}(BG) \to \mathrm{CH}(U/G)$ coincides with the composition

$$\mathrm{CH}(BG) = \mathrm{CH}^G(\mathbf{pt}) \xrightarrow{\alpha} \mathrm{CH}^G(V) \xrightarrow{\beta} \mathrm{CH}^G(U) = \mathrm{CH}(U/G),$$

where the pull-back homomorphism $\alpha$ is an isomorphism by the homotopy invariance property and the restriction $\beta$ is surjective by localization. □

5b. **Semidirect products.** Let an algebraic group $K$ over $F$ act on another algebraic group $H$ by group automorphisms (so that we can form a semidirect product $H \rtimes K$). For a $K$-torsor $E$ over $\mathbf{pt}$ we can twist $H$ by $E$ (see [19, §1 of Chapter V]). The resulting group is denoted by ${}^E H$.

**Proposition 5.5.** *Let* $S = H \rtimes K$ *be a semidirect product of algebraic groups over* $F$. *Let* $U/S$ *and* $W/K$ *be* $i$*-th approximations of* $BS$ *and* $BK$ *respectively, and let* $E \to \mathrm{Spec}\, F$ *be the fiber of* $W \to W/K$ *over a point* $x \in (W/K)(F)$. *Then* $(U \times W)/S$ *is an* $i$*-th* *approximation of $BS$ and the fiber of the natural morphism $p : (U \times W)/S \to W/K$ over $x$ is an $i$-th approximation of $B({}^E H)$, where ${}^E H$ is the twist of $H$ by $E$.*

*Proof.* Write $E_{\mathrm{sep}} = K_{\mathrm{sep}} w$ for $w$ in $W_{\mathrm{sep}}$, where the subscript $_{\mathrm{sep}}$ means the scalar extension to $F_{\mathrm{sep}}$. Then the isomorphism

$$U_{\mathrm{sep}}/H_{\mathrm{sep}} \longrightarrow p^{-1}(x)_{\mathrm{sep}}, \quad H_{\mathrm{sep}} v \mapsto S_{\mathrm{sep}}(v, w)$$

over $F_{\mathrm{sep}}$ descents to an isomorphism between ${}^E U/{}^E H$ and $p^{-1}(x)$ over $F$. Note that the variety ${}^E U/{}^E H$ is an $i$-th approximation of $B({}^E H)$. □

5c. **The parabolic subgroup $P$.** Let $\widetilde{G} = \mathbf{Sp}_{2n}$ be the symplectic group of the alternating form on the $2n$-dimensional vector space $V$ with a symplectic basis $\{v_i, w_i\}$, $i = 1, 2, \dots, n$. Write $T$ for the maximal torus $T = (\mathbf{G}_m)^n$ acting on the basis vector by $tv_i = t_i v_i$ and $tw_i = t_i^{-1} w_i$. Write $\{e_i\}$ for the standard basis of the character group $T^* = \mathbb{Z}^n$. The simple roots of $\widetilde{G}$ are $\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \dots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = 2e_n$ (see [12, §24]).

Consider the subset $\Lambda = \{\alpha_1, \alpha_3, \dots\}$ of all odd simple roots. Let $\widetilde{S}$ be the (reductive) subgroup of $\widetilde{G}$ generated by the torus $T$ and the root subgroups $U_\alpha$ with $\alpha$ in the root system $\pm\Lambda \simeq A_1 + \dots + A_1$. The lattice $T^*$ splits accordingly into a direct sum of rank 2 lattices $\mathbb{Z}e_{2i-1} \oplus \mathbb{Z}e_{2i}$ for $i = 1, 2, \dots, m := [n/2]$ (and $\mathbb{Z}e_n$ if $n$ is odd). It follows that

$$\widetilde{S} \simeq \begin{cases} (\mathbf{GL}_2)^m, & \text{if } n \text{ is even } (n = 2m); \\ (\mathbf{GL}_2)^m \times \mathbf{SL}_2, & \text{if } n \text{ is odd } (n = 2m+1). \end{cases} \tag{5.6}$$

The group $\widetilde{S}$ is the Levi subgroup of the parabolic subgroup $\widetilde{P}$ of $\widetilde{G}$ corresponding to the set of simple roots $\Lambda$. The group $\widetilde{P}$ is the stabilizer of the flag

$$\{0\} = W_0 \subset W_1 \subset W_2 \subset \dots \subset W_m \subset V$$

of totally isotropic subspaces $W_i := \mathrm{span}(v_1, v_2, \dots, v_{2i})$. The projection of $\widetilde{S}$ onto the $i$-th component $\mathbf{GL}_2$ in (5.6), $i = 1, \dots, m$, is given by the action on the factor space $W_i/W_{i-1}$, i.e., coincides with $\widetilde{S} \to \mathbf{GL}(W_i/W_{i-1}) = \mathbf{GL}_2$.

The projective symplectic group $G = \mathbf{PGSp}_{2n}$ is the factor group of $\widetilde{G}$ by $\boldsymbol{\mu}_2$. Write $P$ for the parabolic subgroup $\widetilde{P}/\boldsymbol{\mu}_2$ in $G$. The Levi subgroup $S$ of $P$ is $\widetilde{S}/\boldsymbol{\mu}_2$ with the $\boldsymbol{\mu}_2$ embedded diagonally into the product of $\mathbf{GL}_2$ and $\mathbf{SL}_2$ with respect to the decomposition (5.6). By the above, we have

$$S = \begin{cases} (\mathbf{GL}_2)^m/\boldsymbol{\mu}_2, & \text{if } n \text{ is even;} \\ \big((\mathbf{GL}_2)^m \times \mathbf{SL}_2\big)/\boldsymbol{\mu}_2, & \text{if } n \text{ is odd.} \end{cases}$$

Let

$$H := \begin{cases} (\mathbf{GL}_2)^{m-1} \times \mathbf{G}_m, & \text{if } n \text{ is even;} \\ (\mathbf{GL}_2)^m, & \text{if } n \text{ is odd.} \end{cases}$$

We view $H$ as a subgroup of $S$ via the map

$$(s_1, s_2, \dots, s_{m-1}, \lambda^2) \mapsto (s_1\lambda, s_2\lambda, \dots, s_{m-1}\lambda, \lambda)\boldsymbol{\mu}_2$$

if $n$ is even and

$$(s_1, s_2, \dots, s_m) \mapsto (s_1, s_2, \dots, s_m, 1)\boldsymbol{\mu}_2$$

if $n$ is odd.

We also view the group $K := \mathbf{PGL}_2$ as a subgroup of $P$ embedded diagonally. Note that $S$ coincides with the the semidirect product $H \rtimes K$ and $K$ acts on $H$ by component-wise conjugation.

Consider the representations

$$\rho_i : S \to \mathbf{GL}_4$$

defined by

$$\rho_i((a_1, a_2, \dots a_m)\boldsymbol{\mu}_2) = a_i \otimes a_m, \quad i = 1, 2, \dots, m-1,$$

if $n$ is even and by

$$\rho_i((a_1, a_2, \dots a_{m+1})\boldsymbol{\mu}_2) = a_i \otimes a_{m+1}, \quad i = 1, 2, \dots, m,$$

is $n$ is odd.

A quaternion algebra $Q$ over $F$ can be viewed as a $K$-torsor. Twisting by $Q$ the composition of the embedding of the $i$-th component $\mathbf{GL}_2 \hookrightarrow H$ and $\rho_i$ restricted to $H$, we get a natural representation

$$\mathbf{GL}_1(Q) \to \mathbf{GL}_4 .$$

If $n$ is even, write $\tau : S \to \mathbf{G}_m$ for the homomorphism

$$\tau((a_1, a_2, \dots a_m)\boldsymbol{\mu}_2) = \det(a_m).$$

5d. **Quaternion Azumaya algebra associated to a $P$-torsor.** Let $G = \mathbf{PGSp}_{2n}$ be a split adjoint group $G$ of type $C_n$. There is a natural embedding of $G$ into $\mathbf{PGL}_{2n}$. Therefore, for every $G$-torsor $h : E \to X$ we have associated a $\mathbf{PGL}_{2n}$-torsor, i.e., an Azumaya algebra $A(h)$ over $X$ of degree $2n$.

Let $P$ be the parabolic subgroup of $G$ introduced in Section 5c, $S$ its Levi subgroup, and $P \to S$ the projection. We have the composition $\beta : P \to S = H \rtimes K \to K = \mathbf{PGL}_2$. Therefore, for every $P$-torsor $f : I \to X$ we have associated a $\mathbf{PGL}_2$-torsor, i.e., a quaternion Azumaya algebra $Q = Q(f)$ over $X$.

Every $P$-torsor $f : I \to X$ yields a $G$-torsor $\operatorname{res}_{G/P}(f) := (G \times I)/P \to X$.

**Lemma 5.7.** *For a $P$-torsor $f : I \to X$, we have $[A(\operatorname{res}_{G/P}(f))] = [Q(f)]$ in $\operatorname{Br}(X)$.*

*Proof.* Consider the group $G' = \mathbf{GSp}_{2n}$ of symplectic similitudes (see [12, §12]). The group $G$ is the factor group of $G'$ by the center $\mathbf{G}_m$ of scalar matrices. Let $P'$ the inverse image of $P$ in $G'$ and let $S'$ be the Levi subgroup of $P'$. By Section 5c, the diagram

$$\begin{array}{ccccccc}
\mathbf{G}_m & = & \mathbf{G}_m & = & \mathbf{G}_m & = & \mathbf{G}_m \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbf{GL}_2 & \xleftarrow{\beta'} & P' & \hookrightarrow & G' & \hookrightarrow & \mathbf{GL}_{2n} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbf{PGL}_2 & \xleftarrow{\beta} & P & \hookrightarrow & G & \hookrightarrow & \mathbf{PGL}_{2n},
\end{array}$$

where $\beta'$ is the composition $P' \to S' \to \mathbf{GL}(W_m/W_{m-1}) = \mathbf{GL}_2$ and $\beta$ is (as above) the composition $P \to S \to K = \mathbf{PGL}_2$, is commutative. It follows that the diagram

$$\begin{array}{ccccc} H^1_{\text{ét}}(X,P) & \xrightarrow[Q(-)]{} & H^1_{\text{ét}}(X,\mathbf{PGL}_2) & \hookrightarrow & \mathrm{Br}(X) \\ \downarrow{\scriptstyle \mathrm{res}_{G/P}} & & & & \| \\ H^1_{\text{ét}}(X,G) & \xrightarrow[A(-)]{} & H^1_{\text{ét}}(X,\mathbf{PGL}_{2n}) & \hookrightarrow & \mathrm{Br}(X) \end{array}$$

is also commutative, whence the result. □

5e. **Chow ring of $BP$.**

**Lemma 5.8.** *Let $P$ be a parabolic subgroup in a semisimple group $G$ defined over $F$, $R$ the unipotent radical of $P$ and $S$ a Levi subgroup of $P$. Then*

1. *the map $H^1(F,S) \to H^1(F,P)$ is a bijection;*
2. *the map $A^p(\operatorname{Spec} F, K_q) \to A^p(R, K_q)$ of $K$-cohomology groups is an isomorphism for all $p$ and $q$.*

*Proof.* (1) We have $P = R \rtimes S$, hence the map $s : H^1(F,P) \to H^1(F,S)$ is split surjective. By [12, Proposition 28.11], for any cocycle $\xi \in Z^1(F,S)$ there is a surjection from $H^1(F, {}^{\xi}R)$ to the fiber of $s$ over $\xi$. The group ${}^{\xi}R$ is the unipotent radical of the parabolic subgroup ${}^{\xi}P$ of the semisimple group ${}^{\xi}G$. Hence by [20, Proposition. 16.1.1], ${}^{\xi}R$ is split over $F$, therefore, $H^1(F, {}^{\xi}R) = 1$. It follows that $s$ is a bijection.

(2) As $R$ is split, there is a sequence of normal subgroups $1 = R_0 \subset R_1 \subset \cdots \subset R_n = R$ such that $R_{i+1}/R_i$ is isomorphism to the additive group $\mathbf{G}_a$ for all $i$. Hence every fiber of the natural morphism $R/R_i \to R/R_{i+1}$ is isomorphic to an affine line. By homotopy invariance, the map $A^p(R/R_{i+1}, K_q) \to A^p(R/R_i, K_q)$ is an isomorphism for all $p$ and $q$. □

**Proposition 5.9.** *Let $P$ be a parabolic subgroup in a semisimple group $G$ defined over $F$ and $S$ a Levi subgroup of $P$. Then the natural map $\mathrm{CH}(BS) \to \mathrm{CH}(BP)$ is an isomorphism.*

*Proof.* By Lemma 5.8(1), every fiber of (an approximation of) the natural morphism $BP \to BS$ over a point $\xi$ has a rational point and hence is isomorphic to ${}^{\xi}(P/S) \simeq {}^{\xi}R$, where $R$ is the unipotent radical of $P$. The result follows from Lemma 5.8(2) applied to the parabolic subgroup ${}^{\xi}P$ of the semisimple group ${}^{\xi}G$ and [3, Proposition 52.10]. □

Let $P$ be the parabolic subgroup in the split adjoint group of type $C_n$ introduced in Section 5c, $S$ the Levi subgroup of $P$, $K$ and $H$ as in Section 5c.

A point $x$ of $BK$ over a field $L$ is a $K$-torsor over $L$, i.e., a quaternion algebra $Q_x$ over $L$. Let $H_x$ be the twist of $H$ by $x$, i.e.,

$$H_x = \begin{cases} \mathbf{GL}_1(Q_x)^{m-1} \times \mathbf{G}_m, & \text{if } n \text{ is even;} \\ \mathbf{GL}_1(Q_x)^m, & \text{if } n \text{ is odd,} \end{cases}$$

Let us determine the space $BH_x$. By Example 5.2, $B\,\mathbf{GL}_1(Q_x) = Q_x\mathbb{P}^\infty$. It follows that

$$BH_x = \begin{cases} B\,\mathbf{GL}_1(Q_x)^{m-1} \times B\,\mathbf{G}_m = (Q_x\mathbb{P}^\infty)^{m-1} \times \mathbb{P}^\infty, & \text{if } n \text{ is even;} \\ B\,\mathbf{GL}_1(Q_x)^m = (Q_x\mathbb{P}^\infty)^m, & \text{if } n \text{ is odd.} \end{cases} \tag{5.10}$$

Let $Q'$ be the pull-back to $BS$ of the tautological quaternion algebra over $BK$. Let $J$ be the following subset of $\mathrm{Ch}(BS)$ (see Section 5c):

$$J = \begin{cases} \{\rho_1^*(c_4), \rho_2^*(c_4), \dots, \rho_{m-1}^*(c_4), \tau^*(c_1)\}, & \text{if } n \text{ is even;} \\ \{\rho_1^*(c_4), \rho_2^*(c_4), \dots, \rho_m^*(c_4)\}, & \text{if } n \text{ is odd.} \end{cases}$$

**Theorem 5.11.** *The* $\mathrm{Ch}(BK)$*-algebra* $\mathrm{Ch}(BS)$ *is generated by* $J \cup N_{Q'}$*.*

*Proof.* Let $r > 0$ be an integer and choose the $r$-th approximations

$$f : Y := (U \times W)/S \to W/K =: X$$

of the morphism $BS \to BK$ as in Proposition 5.5.

We apply Lemma 4.1 to the $\mathrm{Ch}(X)$-submodule $B$ of $\mathrm{Ch}(Y)$ generated by $J \cup N_{Q'}$. Let $x$ be a point in $X$ of codimension $i$ such that the restriction $Q_x$ of $Q$ on $x$ is not split. By Proposition 5.5, the fiber of $f$ over $x$ is an $r$-th approximation of $BH_x$ as in (5.10).

By Proposition 3.6 and the Projective Bundle Theorem [13, §7], the ring $\mathrm{Ch}(BH_x)/N_{Q_x}$ is generated by the first Chern class of the (line) tautological bundle on $\mathbb{P}^\infty$ and by the 4-th Chern classes of the (rank 4) tautological bundles on the $m-1$ factors $Q\mathbb{P}^\infty$ if $n$ is even. If $n$ is odd, the ring $\mathrm{Ch}(BH_x)/N_{Q_x}$ is generated by the 4-th Chern classes of the tautological bundles on the $m$ factors $Q\mathbb{P}^\infty$. By Example 5.2 and the end of Section 5c, these Chern classes are the restrictions of the Chern classes in $J$. □

By Example 5.3, the ring $\mathrm{CH}(BK)$ is generated by two elements: one in codimension 2 and the other one is 2-torsion in codimension 3.

Let $Q$ be the quaternion algebra over $BP$ associated to a $P$-torsor $U \to U/P$ approximating $BP$. By definition, $Q$ is the pull-back of $Q'$ under $BP \to BS$. By Proposition 5.9, the natural map $\mathrm{CH}(BS) \to \mathrm{CH}(BP)$ is an isomorphism. This isomorphism takes $N_{Q'}$ to $N_Q$ by Lemma 2.2. We have proved

**Corollary 5.12.** *The ring* $\mathrm{Ch}(BP)/N_Q$ *is generated by a set of elements of degree at most* 4 *(with all elements of degree* 3 *in the set being represented by* 2*-torsion elements in* $\mathrm{CH}^3(BP)$*).* □

Let $U \to U/P$ be a $P$-torsor approximating $BP$ with the associated quaternion algebra $Q(U)$. By Lemma 5.4, the natural ring homomorphism $\mathrm{CH}(BP) \to \mathrm{CH}(U/P)$ is surjective. The induced surjection $\mathrm{Ch}(BP) \to \mathrm{Ch}(U/P)$ takes $N_Q$ to $N_{Q(U)}$ (see Lemma 2.2). We have proved

**Corollary 5.13.** *Let* $U \to U/P$ *be a* $P$*-torsor with* $U/P$ *approximating* $BP$ *with the associated quaternion algebra* $Q(U)$*. Then the ring* $\mathrm{Ch}(U/P)/N_{Q(U)}$ *is generated by a set of elements of degree at most* 4 *(with all elements of degree* 3 *in the set being represented by* 2*-torsion elements in* $\mathrm{CH}^3(U/P)$*).* □

Consider a $G$-torsor $U \to U/G$ with $U/G$ approximating $BG$. The generic fiber $f : E \to \mathrm{Spec}(K)$, where $K = F(U/G)$ is the associated *generic* $G$-torsor. Let $L/K$ be a field extension. The fiber of $g : E \to E/P =: X$ over an $L$-point of $X$ is a $P$-torsor $h : I \to \mathrm{Spec}(L)$ over $L$. Clearly, $\mathrm{res}_{G/P}(h) = f_L$. Therefore, $[Q(h)] = [A(f)_L]$ in $\mathrm{Br}(L)$ by Lemma 5.7.

Now take $L = F(X)$. We have the following two quaternion Azumaya algebras over $X_L$. One is $Q(g_L)$ for the $P$-torsor $g_L : E_L \to X_L$. Another one is the constant algebra coming from the $L$-algebra $Q(h)$.

We claim that the pull-backs to $L(X) = L(X_L)$ of both algebras are Brauer equivalent (and hence isomorphic). We see that the two pull-backs are obtained from each other by the automorphism of the field $L(X)$ induced by the exchange automorphism of $X \times X$. Since the Brauer class of $Q(h)$ comes from the field $F$, the exchange automorphisms acts identically on $Q(h)$ proving the claim.

As a consequence of the claim, the norm subgroups $N$ in $\mathrm{Ch}(X_L)$ for both quaternion algebras are the same by Lemma 2.1.

**Corollary 5.14.** *Let $f : E \to \operatorname{Spec} K$ be a generic $G$-torsor and let $L$ be the function field of $X := E/P$. Then the image of the natural ring homomorphism $\mathrm{Ch}(X) \to \mathrm{Ch}(X_L)$ is generated by elements of codimensions 1, 2, 4 and the norm subgroup $N_Q \subset \mathrm{Ch}(X_L)$ of the constant quaternion algebra $Q$ on $X_L$ Brauer equivalent to $A(f)$ lifted to $X_L$.*

*Proof.* Recall that $E$ is the generic fiber of $U \to U/G$ for an appropriate $U$, hence $X$ is a localization of $U/P$. By the localization property for Chow groups, the pull-back ring homomorphism $\mathrm{Ch}(U/P) \to \mathrm{Ch}(X)$ is surjective. By Lemma 2.2 and the discussion before the corollary, it takes the standard norm group in $\mathrm{Ch}(U/P)$ to the norm group $N_Q$. The result follows from Corollary 5.13 and from the fact that the (integral) Chow group $\mathrm{CH}(X_L)$ is torsion free as the (integral) CH-motive of the projective homogeneous variety $X_L$ is a sum of shifts of several copies of the motives of the point and of a conic (see [5] and [1]). □

## 6. Generic flag variety

Let $k$ be a field, $n$ an integer $\geq 2$ and let $F$ be the field of rational functions over $k$ in $n+2$ variables $t, t', t_1, \dots, t_n$. Let $Q$ be the quaternion (division) $F$-algebra given by the elements $t, t'$ and let $h$ be the hermitian form $\langle t_1, \dots, t_n \rangle$ on the right $Q$-module $Q^n$. Note that in the characteristic 2 case, the hermitian form $h$ is *alternating* (as defined in [12, §4.A]). By [12, Theorem 4.2], this means that the adjoint to $h$ involution on the matrix algebra $M_n(Q)$ is symplectic (in any characteristic).

The pair $Q, h$ is generic in the following sense: there exist a smooth connected $k$-variety $X$ (namely, the affine space of dimension $n+2$), a quaternion algebra $\tilde{Q}$ over $X$ and a hermitian form $\tilde{h}$ on $\tilde{Q}^n$ such that:
(1) the pair $Q, h$ is $\tilde{Q}, \tilde{h}$ restricted to the generic point of $X$ and
(2) for any quaternion algebra $Q'$ over an extension field $F'/k$ with a non-degenerate hermitian form $h'$ on $Q'$ (alternating in characteristic 2), there exists an $F'$-point of $X$ such that $Q', h'$ is isomorphic to the restriction of $\tilde{Q}, \tilde{h}$ to the point.

Indeed, $h'$ can be diagonalized and the diagonal entries are elements of $Q$ which are symmetric (alternating in characteristic 2) with respect to the canonical symplectic involution on $Q$. It follows by [12, Proposition 2.6] that the diagonal entries are elements of $F'$.

A different construction of such a generic pair occurs in the proof of the following proposition. We are using the *reduced* Chow ring here defined in the introduction.

**Proposition 6.1.** *Let $Y$ be the $F$-variety of flags of totally isotropic subspaces in $Q^n$ of $Q$-dimensions $1, 2, \dots, [n/2]$ (i.e., of reduced dimensions $2, 4, \dots, 2[n/2]$). Then the ring $\overline{\mathrm{Ch}}(Y)/\overline{N}_{Q_Y}$, where $\overline{N}_{Q_Y}$ is the image of $N_{Q_Y}$ in $\overline{\mathrm{Ch}}(Y)$, is generated by codimension 1, 2, and 4.*

*Proof.* Let $P$ be the parabolic subgroup of the split simple adjoint affine algebraic group $G$ of type $C_n$ over the field $k$ considered ($P$ and $G$) in Section 5c. Let $E$ be a generic $G$-torsor over an extension field $K/k$, and let us consider the projective homogeneous $K$-variety $Y' := E/P$. The torsor $E$ is given by a central simple $K$-algebra $A$ of degree $2n$ with a symplectic involution $\sigma$. The variety $Y'$ is isomorphic to the variety of flags of $\sigma$-isotropic right ideals in $A$ of reduced dimensions $2, 4, 6, \dots, 2[n/2]$. There is a quaternion algebra $Q'$ over $Y'$ Brauer-equivalent to $A_{Y'}$.

Let $X$ be the generalized Severi-Brauer variety $\mathrm{SB}_2(A)$. By Example 4.3, the $\mathrm{Ch}(Y')$-algebra $\mathrm{Ch}(Y' \times_K X)/N_{Q'_{Y'\times X}}$ is generated by an element of codimension 4. The pull-back $\mathrm{Ch}(Y' \times_K X) \to \mathrm{Ch}(Y'_{K(X)})$ along the morphism given by the generic point of $X$ is surjective by [3, Corollary 57.11]. It follows by Corollary 5.14 that the ring $\overline{\mathrm{Ch}}(Y'_{K(X)})/\overline{N}_{Q'_{Y'_{K(X)}}}$ is generated by codimensions $1, 2, 4$. By a specialization argument similar to that used in [10, Proof of Corollary 4.8], the ring $\overline{\mathrm{Ch}}(Y'_{K(X)})/\overline{N}_{Q'_{Y'_{K(X)}}}$ is isomorphic to the ring $\overline{\mathrm{Ch}}(Y)/\overline{N}_{Q_Y}$. □

## 7. Projective $Q$-bundles for constant $Q$

In this section, $Q$ is a quaternion algebra over $F$, $X$ is a smooth $F$-variety, $V$ is a right $Q_X$-module of reduced dimension $2(n+1)$.

The following statement generalizes Lemma 3.1:

**Lemma 7.1.** *As usual, let $a$ be the 4-th Chern class of the tautological vector bundle on $Q\mathbb{P}(V)$. Let $\pi_X$ be the structure morphism $Q\mathbb{P}(V) \to X$. Then $\pi_{X*}(a^i) = 0$ for any $i = 0, \dots, n-1$ and $\pi_{X*}(a^n) = [X]$.*

*Proof.* For $i < n$, we have $\pi_{X*}(a^i) = 0$ by dimension reason. The remaining formula $\pi_{X*}(a^n) = [X]$ may be checked over an extension of $F$ so that we may assume $Q$ is split. In this case $Q\mathbb{P}(V)$ is identified with the Grassmannian of 2-planes in a rank $2(n+1)$ vector bundle over $X$ the way that $a = c_2^2$, where $c_2$ is the 2-nd Chern class of the tautological vector bundle on the Grassmannian. The desired formula becomes a particular case of Duality Theorem [4, 14.6.3]. □

**Proposition 7.2.** *For $V$ of even reduced rank $2(n+1)$, the module $\mathrm{Ch}(Q\mathbb{P}(V))/N_{Q_{Q\mathbb{P}(V)}}$ over the ring $\mathrm{Ch}(X)/N_{Q_X}$ is free with the basis $\{a^i\}_{i=0}^n$.*

*Proof.* We know already by Example 4.3 that the system $\{a^i\}_{i=0}^n$ generates the module. It remains to check that it is free.

Assuming that $\alpha := \sum_{i=0}^n \alpha_i a^i \in N_{Q_{Q\mathbb{P}(V)}}$ for some $\alpha_0, \dots, \alpha_n \in \mathrm{Ch}(X)$, we show that all $\alpha_i$ are in $N_{Q_X}$ using a descending induction on $i$. Let $i$ be the biggest index for which we didn't prove $\alpha_i \in N_{Q_X}$ yet. Calculating $\pi_{X*}(\alpha \cdot a^{n-i}) \in N_{Q_X}$ using Lemma 7.1, we get that $\alpha_i \in N_{Q_X}$. □

Now we are going to look at the reduced Chow ring $\overline{\mathrm{Ch}}(X)$ (defined in the introduction). Note that for any extension field $L/F$, the change of field homomorphism $\overline{\mathrm{Ch}}(X) \to \overline{\mathrm{Ch}}(X_L)$ is injective. As before, we write $\overline{N}_{Q_X}$ for the image of $N_{Q_X}$ in $\overline{\mathrm{Ch}}(X)$.

**Proposition 7.3.** *The element $a$ (more precisely, its class) in the $\overline{\mathrm{Ch}}(X)/\overline{N}_{Q_X}$-algebra $\overline{\mathrm{Ch}}(Q\mathbb{P}(V))/\overline{N}_{Q_{Q\mathbb{P}(V)}}$ satisfies the relation*

$$\sum_{i=0}^{n+1} c_{4i} a^{n+1-i} = 0,$$

*where $c_i := c_i(V)$ is the $i$-th Chern class of the vector bundle $V$.*

*Proof.* The tautological vector bundle $\mathcal{T}$ on $Y := Q\mathbb{P}(V)$ is a subbundle of the vector bundle $V_Y$. The exact sequence of vector bundles

$$0 \longrightarrow \mathcal{T} \longrightarrow V_Y \longrightarrow V_Y/\mathcal{T} \longrightarrow 0$$

gives by the Whitney Sum Formula [3, Proposition 54.7] the relation

$$c(\mathcal{T})c(V_Y/\mathcal{T}) = c(V)$$

of the total Chern classes. In particular,

$$\begin{aligned} a + a_3b_1 + a_2b_2 + a_1b_3 + b_4 &= c_4(V) \\ ab_4 + a_3b_5 + a_2b_6 + a_1b_7 + b_8 &= c_8(V) \\ &\vdots \\ ab_{4n-4} + a_3b_{4n-3} + a_2b_{4n-2} + a_1b_{4n-1} + b_{4n} &= c_{4n}(V) \\ ab_{4n} &= c_{4n+4}(V), \end{aligned} \tag{7.4}$$

where $a_i := c_i(\mathcal{T})$ and $b_i := c_i(V_Y/\mathcal{T})$.

We claim that the classes of $a_1, a_2, a_3$ in $\overline{\mathrm{Ch}}(Y)/\overline{N}_{Q_Y}$ are zero. Indeed, let $L/F$ be an extension field splitting $Q$. Choosing a simple left module $M$ over the split quaternion $L$-algebra $Q_L$ and defining $\mathcal{T}'$ as the tensor product of $\mathcal{T}_L$ and $M$ over $Q_L$, we get an isomorphism $\mathcal{T}_L \simeq \mathcal{T}' \oplus \mathcal{T}'$ of vector bundles over $Y_L$. It follows that the images of $a_1$ and $a_3$ in $\mathrm{Ch}(Y_L)$ are 0. Therefore $a_1$ and $a_3$ are zero already in $\overline{\mathrm{Ch}}(Y)$ – before factorization by $\overline{N}_{Q_Y}$.

It remains to deal with the image of $a_2$ in $\mathrm{Ch}(Y_L)$ which is equal to $c_1^2(\mathcal{T}')$. It suffices to prove the following

**Lemma 7.5.** *The element $c_1^2(\mathcal{T}')$ is in the image of the composition*

$$\mathrm{Ch}(Y \times C) \to \mathrm{Ch}(Y) \to \mathrm{Ch}(Y_L).$$

*Proof.* The following square commutes:

$$\begin{array}{ccc} \mathrm{Ch}(Y \times C) & \longrightarrow & \mathrm{Ch}(Y \times C)_L \\ \downarrow & & \downarrow \\ \mathrm{Ch}(Y) & \longrightarrow & \mathrm{Ch}(Y_L). \end{array}$$

Let $\mathcal{C}$ be the tautological vector bundle on $C$. This is a right $Q$-module, but we may view it as a left $Q$-module via the canonical involution on $Q$ in order to take the tensor product $\mathcal{F}$ of $\mathcal{T}_{Y\times C}$ and $\mathcal{C}_{Y\times C}$ over $Q_{Y\times C}$. Pulling-back the $(Y\times C)$-vector bundle $\mathcal{F}$ to $(Y\times C)_L = Y_L \times C_L$, we get a vector bundle isomorphic to the tensor product of $\mathcal{T}'$ and $\mathcal{C}'$, where $\mathcal{C}'$ is the (rank 1) vector bundle on $C_L$ defined as the tensor product over $Q_L$ of $\mathcal{C}$ and $M$.

Since $\mathcal{C}'$ is a line bundle and $\mathcal{T}'$ is a vector bundle of rank 2, we have

$$c_2(\mathcal{T}'\otimes\mathcal{C}') = c_2(\mathcal{T}')\times[C_L] + c_1(\mathcal{T}')\times c_1(\mathcal{C}') + [Y_L]\times c_1^2(\mathcal{C}')$$

(see [4, Remark 3.2.3]). Note that the last summand is 0 by dimension reason and that $\deg(\mathcal{C}')=1$. Therefore the image of $c_2(\mathcal{T}'\otimes\mathcal{C}')$ under the push-forward to $\mathrm{Ch}(Y_L)$ is $c_1(\mathcal{T}')$ showing that $c_1(\mathcal{T}')$ is in the image of the composition $\mathrm{Ch}(Y\times C)\to\mathrm{Ch}(Y)\to\mathrm{Ch}(Y_L)$. Therefore $c_1^2(\mathcal{T}')$ is also in the image of the composition. □

Turning back to the proof of Proposition 7.3 and passing from $\mathrm{Ch}(Y)$ to $\overline{\mathrm{Ch}}(Y)/\overline{N}_{Q_Y}$ in the relations (7.4), we get the following simpler relations

$$\begin{aligned} a+b_4 &= c_4(V)\\ ab_4+b_8 &= c_8(V)\\ &\vdots\\ ab_{4n-4}+b_{4n} &= c_{4n}(V)\\ ab_{4n} &= c_{4n+4}(V).\end{aligned}$$

Starting with the last relation and sequentially excluding $b_{4i}$ for $i=n,n-1,\dots$ with the help of the previous relations, we get the relation desired. □

**Corollary 7.6.** *The $\overline{\mathrm{Ch}}(X)/\overline{N}_{Q_X}$-algebra $\overline{\mathrm{Ch}}(Q\mathbb{P}(V))/\overline{N}_{Q_{Q\mathbb{P}(V)}}$ is generated by the element $a$ subject to one relation*

$$\sum_{i=0}^{n+1} c_{4i}a^{n+1-i} = 0,$$

*where $c_i := c_i(V)$ is the $i$-th Chern class of the vector bundle $V$.*

*Proof.* Let $A$ be an $\overline{\mathrm{Ch}}(X)/\overline{N}_{Q_X}$-algebra generated by one element $t$ subject to one relation

$$\sum_{i=0}^{n+1} c_{4i}t^{n+1-i} = 0.$$

The $\overline{\mathrm{Ch}}(X)/\overline{N}_{Q_X}$-algebra homomorphism $A\to\overline{\mathrm{Ch}}(Q\mathbb{P}(V))/\overline{N}_{Q_{Q\mathbb{P}(V)}}$, $t\mapsto a$ is well defined by Proposition 7.3 and surjective by Proposition 7.2. Since both algebras, considered as $\overline{\mathrm{Ch}}(X)/\overline{N}_{Q_X}$-modules, are free of rank $n+1$ (the right one is so by Proposition 7.2 once again), the epimorphism is an isomorphism. □

## 8. Generic maximal Grassmannian

We use the settings of the beginning of Section 6. Let $m := [n/2]$. We consider the $F$-variety $X$ of $m$-dimensional totally isotropic subspaces in $h$.

**Proposition 8.1.** *The components of positive codimension of the ring $\overline{\mathrm{Ch}}(X)/\overline{N}_{Q_X}$ are trivial.*

*Proof.* Let $I$ be the set of integers $[1,\ m] := \{1, 2, \dots, m\}$. For any subset $J \subset I$ we consider the variety $X_J$ of flags of totally isotropic subspaces in $h$ of $Q$-dimensions given by $J$.

By induction on $l \in I$, we prove the following statement: the ring $\overline{\mathrm{Ch}}(X_{[l,\ m]})/\overline{N}_{Q_{X_{[l,\ m]}}}$ is generated by codimensions $1, 2, 4$ and the Chern classes of the tautological rank $4l$ vector bundle $\mathcal{T}_l$ on $X_{[l,\ m]}$ (which is the pull-back to $X_{[l,\ m]}$ of the tautological vector bundle on $X_l$). This statement with $l = m$ gives the required statement of Proposition 8.1 due to the following

**Lemma 8.2.** *Let $F$ be a field, $Q$ a quaternion division $F$-algebra, $h$ a hermitian form on $Q^n$ which is hyperbolic for even $n$ and almost hyperbolic for odd $n$, $X$ the corresponding maximal Grassmannian. Then all the elements of codimensions $1, 2, 4$ in the ring $\mathrm{Ch}(X)/N_{Q_X}$ as well as the elements given by the Chern classes of positive codimensions of the tautological vector bundle on $X$ are $0$.*

*Proof.* It follows by [5, Theorem 15.8 and Corollary 15.14] and [1] that the motive of $X$ decomposes in a direct sum with one summand $M(\mathbf{pt})$, one summand $M(\mathbf{pt})(3)$, and every of the remaining summands being either $M(\mathbf{pt})(i)$ with $i > 4$ or a shift of $M(C)$. Therefore all elements of codimensions $1, 2, 4$ in the ring $\mathrm{Ch}(X)/N_{Q_X}$ are $0$, and it remains to prove the statement about the Chern classes of the tautological bundle.

The ring $\mathrm{Ch}(X)$ imbeds into $\mathrm{Ch}(\bar{X})$ (see, e.g., Corollary 9.2). The variety $\bar{X}$ is identified with the Grassmannian of $2m$-planes in a $2n$-dimensional vector space $V$ which are totally isotropic with respect to a fixed non-degenerate alternating form on $V$. The tautological bundle on $X$ gives rise to a vector bundle on $\bar{X}$ isomorphic to $\mathcal{T} \oplus \mathcal{T}$, where $\mathcal{T}$ is the tautological bundle on $\bar{X}$.

Let us consider the case of even $n$ first. We claim that in this case the Chern classes of the tautological bundle on $X$ are trivial already in $\mathrm{Ch}(\bar{X})$. Indeed, there is an exact sequence

$$0 \longrightarrow \mathcal{T} \longrightarrow V_{\bar{X}} \longrightarrow \mathcal{T}^* \longrightarrow 0$$

relating the bundle $\mathcal{T}$ with its dual $\mathcal{T}^*$ and the trivial vector bundle $V_{\bar{X}}$ (where the epimorphism $V_{\bar{X}} \to \mathcal{T}^*$ is induced by the alternating form). Since $c(\mathcal{T}) = c(\mathcal{T}^*) \in \mathrm{Ch}(\bar{X})$, it follows that $c(\mathcal{T} \oplus \mathcal{T}) = 1$. (We are repeatedly using the Whitney Sum Formula [3, Proposition 54.7] in this proof.)

In the case of odd $n$, there is an exact sequence

$$0 \longrightarrow \mathcal{T}^\perp \longrightarrow V_{\bar{X}} \longrightarrow \mathcal{T}^* \longrightarrow 0,$$

where $\mathcal{T}^\perp$ is the orthogonal complement of $\mathcal{T}$ in $V_{\bar{X}}$. The bundle $\mathcal{T}^\perp$ contains $\mathcal{T}$ as a subbundle, the quotient $\mathcal{T}^\perp/\mathcal{T}$ is of rank 2. We have

$$1 = c(\mathcal{T}^\perp)c(\mathcal{T}^*) = c(\mathcal{T})^2 c(\mathcal{T}^\perp/\mathcal{T}) \in \mathrm{Ch}(\bar{X}).$$

Multiplying by $c(\mathcal{T}^\perp/\mathcal{T})$, we get that $c(\mathcal{T}^\perp/\mathcal{T}) \in \mathrm{Ch}(X)$. Passing to the quotient by $N_{Q_X}$, we see that $c(\mathcal{T}^\perp/\mathcal{T}) = 1 \in \mathrm{Ch}(X)/N_{Q_X}$ (because $\mathcal{T}^\perp/\mathcal{T}$ is of rank 2 and the ring $\mathrm{Ch}(X)/N_{Q_X}$ has no non-zero elements in codimensions 1 and 2). Therefore $c(\mathcal{T})^2 = 1 \in \mathrm{Ch}(X)/N_{Q_X}$. □

We turn back to the inductive proof of the statement formulated in the beginning of the proof of Proposition 8.1. The induction base $l = 1$ follows from Proposition 6.1. Now, assuming that $l \geq 2$, let us do the passage from $l - 1$ to $l$.

The projection $X_{[l-1,\, m]} \to X_{[l,\, m]}$ is the projective $Q$-bundle given by the dual of the vector $Q$-bundle $\mathcal{T}_l$. Therefore, by Corollary 7.6, the $\overline{\mathrm{Ch}}(X_{[l,\, m]})/\overline{N}_{Q_{X_{[l,\, m]}}}$-algebra $\overline{\mathrm{Ch}}(X_{[l-1,\, m]})/\overline{N}_{Q_{X_{[l-1,\, m]}}}$ is generated by certain codimension 4 element $a$ subject to one relation $\sum_{i=0}^{l} c_{4i} a^{l-i} = 0$, where $c_i := c_i(\mathcal{T}_l)$. In particular, the $\overline{\mathrm{Ch}}(X_{[l,\, m]})/\overline{N}_{Q_{X_{[l,\, m]}}}$-module $\overline{\mathrm{Ch}}(X_{[l-1,\, m]})/\overline{N}_{Q_{X_{[l-1,\, m]}}}$ is free of rank $l$.

Now let $C \subset \overline{\mathrm{Ch}}(X_{[l,\, m]})/\overline{N}_{Q_{X_{[l,\, m]}}}$ be the subring generated by all $c_i$ together with the elements of codimensions $1, 2, 4$. The coefficients of the above relation are then in $C$. Therefore the subring of $\overline{\mathrm{Ch}}(X_{[l-1,\, m]})/\overline{N}_{Q_{X_{[l-1,\, m]}}}$ generated by $C$ and $a$ is also free (now as a $C$-module) of rank $l$. On the other hand, this subring coincides with the total ring by the induction hypothesis. Indeed, it contains all the elements of codimension $1, 2, 4$ in $\overline{\mathrm{Ch}}(X_{[l-1,\, m]})/\overline{N}_{Q_{X_{[l-1,\, m]}}}$ because any such element is either equal to $a$ or lies in the image of $\overline{\mathrm{Ch}}(X_{[l,\, m]})/\overline{N}_{Q_{X_{[l,\, m]}}}$. It also contains the Chern classes of the vector bundle $\mathcal{T}_{l-1}$ on $X_{[l-1,\, m]}$ because these Chern classes are polynomials in $c_i := c_i(\mathcal{T}_l)$ and $a := c_4(\mathcal{T}_l/\mathcal{T}_{l-1})$ (we recall that the 1-st, 2-nd and 3-d Chern classes of the quotient $\mathcal{T}_l/\mathcal{T}_{l-1}$ are trivial in $\overline{\mathrm{Ch}}(X_{[l-1,\, m]})/\overline{N}_{Q_{X_{[l-1,\, m]}}}$ as shown in the proof of Proposition 7.3).

It follows that $C = \overline{\mathrm{Ch}}(X_{[l,\, m]})/\overline{N}_{Q_{X_{[l,\, m]}}}$. □

## 9. Essential motives of $Q$-Grassmannians

We fix the following notation. Let $F$ be a field (of arbitrary characteristic). Let $Q$ be a quaternion division $F$-algebra and $C$ the corresponding conic. Let $n$ be an integer $\geq 0$. Let $V$ be a right vector space over $Q$ of dimension $n$. Let $h$ be a non-degenerate hermitian (with respect to the canonical involution of $Q$) form on $V$. If $\operatorname{char} F = 2$, we additionally assume that $h$ is alternating. For any integer $r$, let $X_r$ be the $F$-variety of totally isotropic subspaces in $V$ of $Q$-dimension $r$ (so that $X_0 = \operatorname{Spec} F$ and $X_r = \emptyset$ for $r$ outside of the interval $[0,\ n/2]$).

**Lemma 9.1** ([5, Theorem 15.8 and Corollary 15.14] and [1])**.** *Assume that the hermitian form $h$ is isotropic: $n$ is $\geq 2$ and $h \simeq \mathbb{H} \perp h'$, where $\mathbb{H}$ is the hyperbolic plane, $h'$ a hermitian form of dimension $n - 2$. For any integer $r$ one has*

$$M(X_r) \simeq M(X'_{r-1}) \oplus M(X'_r)(i) \oplus M(X'_{r-1})(j) \oplus M,$$

*where $X'_{r-1}$ and $X'_r$ are the varieties of $h'$, $i = (\dim X_r - \dim X'_r)/2$, $j = \dim X_r - \dim X'_{r-1}$, and $M$ is a sum of shifts of the motive of $C$.*

The following Corollary is also a consequence of a general result of [7] or of [1]:

**Corollary 9.2.** *If $h$ is* split *(meaning* hyperbolic *for even $n$ or* "almost hyperbolic" *for odd $n$), then $M(X_r)$ is a sum of shifts of $M(\mathbf{pt})$ and of $M(C)$.* □

**Corollary 9.3.** *There is a decomposition $M(X_r) \simeq M_r \oplus M$ such that the motive $M$ is a sum of shifts of $M(C)$ and for any field extension $L/F$ with split $h_L$ the motive $M_r$ is* split *(meaning* is a sum of Tate motives*).*

*Proof.* Apply [8, Proposition 4.1] inductively to $E := F(X_{[n/2]})$ and $S := C$. Note that the variety $S_E$ is still irreducible and has indecomposable motive because the quaternion $E$-algebra $Q \otimes_F E$ is non-split (see, e.g., [14]). Therefore Condition (1) of [8, Proposition 4.1] is satisfied.

Since $X_{[n/2]}(F(C)) \neq \emptyset$, the field extension $E(S)/F(S)$ is purely transcendental, that is, Condition (2) is satisfied as well.

Clearly, the hermitian form $h_E$ is split so that the motive of $X_r$ *over* $E$ is a sum of shifts of $M(\mathbf{pt})$ and of $M(C)$ (Corollary 9.2). The inductive application of [8, Proposition 4.1] shows that the sum $M$ of all copies of shifts of $M(C)$ present in the complete decomposition of $M(X_r)$ over $E$, can be extracted from $M(X_r)$ over $F$. The complementary summand $M_r$ of the motive of $X_r$ has the desired property. □

**Remark 9.4.** The reduced Chow group (homological or cohomological one) of the motive $M$ (as a subgroup of $\overline{\mathrm{Ch}}(X_r)$) is equal to $\overline{N}_{Q_{X_r}}$ (see Lemma 3.5; note that one can find $L$ as in Corollary 9.3 such that $C_L$ is not split). Therefore the reduced Chow group of $M_r$ is identified with the quotient $\overline{\mathrm{Ch}}(X_r)/\overline{N}_{Q_{X_r}}$. In the case where $h$ is hyperbolic or almost hyperbolic, the variety $X_r$ satisfies the condition of Lemma 3.5. Therefore the reduced Chow group in the above statements can be replaced by the usual Chow group. We refer to [3, §64] for the definition of homological and cohomological Chow group of a motive. The coincidence of descriptions of homological and cohomological Chow groups for the motives $M$ and $M_r$ is explained by their symmetry: $M \simeq M^*(\dim X_r)$ (and the same for $M_r$), where $M^*$ is the *dual* motive, [3, §65].

**Definition 9.5.** The motive $M_r$ (defined by $X_r$ uniquely up to an isomorphism) will be called the *essential motive* of $X_r$ (or the *essential part* of the motive of $X_r$).

It follows that the decomposition of the essential motive in the isotropic case has precisely the same shape as the decomposition of the motive of an isotropic orthogonal Grassmannian [9, Decomposition 2.6]:

**Corollary 9.6.** *Under the hypotheses of Lemma 9.1, one has*

$$M_r \simeq M'_{r-1} \oplus M'_r(i) \oplus M'_{r-1}(j),$$

*where $M'_{r-1}$ and $M'_r$ are the essential motives of $X'_{r-1}$ and $X'_r$, $i = (\dim X_r - \dim X'_r)/2$, $j = \dim X_r - \dim X'_{r-1}$.* □

According to the general result of [7], any summand of the complete motivic decomposition of the variety $X_r$ is a shift of the *upper motive* $U(X_s)$ for some $s \geq r$ or a shift of $M(C)$. Therefore we get

**Corollary 9.7.** *Any summand of the complete decomposition of the essential motive $M_r$ is a shift of the upper motive $U(X_s)$ for some $s \geq r$.* □

**Remark 9.8.** A motive is *split* if it is isomorphic to a finite direct sum of Tate motives. A motive is *geometrically split* if it becomes split over an extension of the base field.

*Dimension* $\dim P$ of a geometrically split motive $P$ is the maximum of the distance $|i-j|$ between $i$ and $j$ running over the integers such that the Tate motives $M(\mathbf{pt})(i)$ and $M(\mathbf{pt})(j)$ are direct summands of $P_L$, where $L/F$ is a field extension splitting $N$.

Since the quaternion $F$-algebra $Q$ remains non-split over the function field $L := F(X_{[n/2]})$ and the motive of $(X_r)_L$ contains the Tate summands $\mathbb{F}_2$ and $\mathbb{F}_2(\dim X_r)$, these Tate motives are summands of $(M_r)_L$. It follows that $\dim M_r = \dim X_r$.

## 10. Generic Grassmannians

In the statements below we use the notion of the *essential motive* $M_r$ of the variety $X_r$, introduced in the previous section. It turns out that in the generic case, this motive is indecomposable and the variety $X_r$ is 2-incompressible (see Introduction for definition of canonical 2-dimension and 2-incompressibility):

**Theorem 10.1.** *For $F$, $Q$, $n$, and $h$ as in the beginning of Section 6, for any $r = 0, 1, \dots, [n/2]$, the essential motive $M_r$ of the variety $X_r$ is indecomposable, the variety $X_r$ is 2-incompressible.*

*Proof.* We induct on $n$ in the proof of the first statement. The induction base is the trivial case of $n < 2$. Now we assume that $n \geq 2$.

We do a descending induction on $r$. The case of the maximal $r = [n/2]$ is an immediate consequence of Proposition 8.1. Indeed, one summand of the complete decomposition of the motive $M_r$ for such $r$ is the upper motive $U(X_r)$ of $X_r$. The remaining summands (if any) are positive shifts $U(X_r)(i)$ $(i > 0)$ of the upper motive, see Corollary 9.7. But if we have a summand $U(X_r)(i)$, then the reduced Chow group $\overline{\mathrm{Ch}}^i(M_r)$ is non-zero. However by Remark 9.4, $\overline{\mathrm{Ch}}(M_r)$ is isomorphic to $\overline{\mathrm{Ch}}(X_r)/\overline{N}_{Q_{X_r}}$ which is 0 in positive codimensions by Proposition 8.1.

Now we assume that $r < [n/2]$. Since the case of $r = 0$ is trivial, we may assume that $r \geq 1$ (and therefore $n \geq 4$).

Let $L := F(X_1)$. We have $h_L \simeq \mathbb{H} \perp h'$, where $h'$ is a hermitian form of dimension $n-2$ and $\mathbb{H}$ is the hyperbolic plane.

For any integer $s$, we write $X'_s$ for the variety $X_s$ of the hermitian form $h'$, and we write $M'_s$ for the essential motive of the variety $X'_s$. By Corollary 9.6, the motive $(M_r)_L$ decomposes in a sum of three summands:

$$(M_r)_L \simeq M'_{r-1} \oplus M'_r(i) \oplus M'_{r-1}(j), \tag{10.2}$$

where $i := (\dim X_r - \dim X'_r)/2$ and $j := \dim X_r - \dim X'_{r-1}$. Let us check that each of three summands of decomposition (10.2) is indecomposable.

Let $F'$ be the function field of the variety of totally isotropic subspaces of $Q$-dimension 1 of the hermitian form $\langle t_{n-1}, t_n \rangle$. Then $h_{F'} \simeq \mathbb{H} \perp \langle t_1, \dots, t_{n-2} \rangle$ so that we have a motivic decomposition similar to (10.2) where each of the three summands is indecomposable by the induction hypothesis. Since the field extension $F'(X_1)/F'$ is purely transcendental, the complete decomposition of $(M_r)_{F'(X_1)}$ has only three summands. Since $L = F(X_1) \subset F'(X_1)$, the complete decomposition of $(M_r)_L$ has at most three summands so that the summands of decomposition (10.2) are indecomposable.

It follows by [9, Proposition 2.4]) that if the motive $M_r$ is decomposable (over $F$), then it has a summand $P$ with $P_L \simeq M'_r(i) = U(X'_r)(i)$. Note that $U(X'_r) \simeq U((X_{r+1})_L)$. Again by [9, Proposition 2.4], $P \simeq U(X_{r+1})(i)$, showing that $U(X_{r+1})_L \simeq M'_r$. By the induction hypothesis, the motive $M_{r+1}$ is indecomposable, that is, $U(X_{r+1}) = M_{r+1}$. Therefore we have an isomorphism $(M_{r+1})_L \simeq M'_r$ and, in particular, $\dim X_{r+1} = \dim X'_r$ (see Remark 9.8). However $\dim X_{r+1} = (r+1)\big(4n - 6(r+1) + 1\big)$, $\dim X'_r = r\big(4(n-2) - 6r + 1\big)$, and the difference is $4n - 4r - 5 > 2n - 5 > 0$ (recall that $n \geq 4$ now).

To show that $X_r$ is 2-incompressible, we show that its canonical 2-dimension $\operatorname{cd}_2 X_r$ equals $\dim X_r$. By [6, Theorem 5.1], $\operatorname{cd}_2 X_r = \dim U(X_r)$. By the first part of Theorem 10.1, $U(X_r) = M_r$. Finally, $\dim M_r = \dim X_r$ (see Remark 9.8). □

## 11. Connection with quadratic forms

Let $F$, $Q$, $V$, and $h$ be as in the beginning of Section 9. For any $v \in V$ the value $h(v,v)$ is in $F$ and the map $q : V \to F$, $v \mapsto h(v,v)$ is a non-degenerate quadratic form on $V$ considered this time as a vector space over $F$. Note that the dimension of $q$, that is, the dimension of $V$ over $F$ is the dimension $n$ of $V$ over $Q$ multiplied by 4. Moreover, $q$ is isomorphic to the tensor product of the 2-fold Pfister quadratic form $\mathrm{Nrd}_Q$ given by the reduced norm of $Q$ by an $n$-dimensional non-degenerate symmetric bilinear form. Note that an arbitrary anisotropic 2-fold Pfister form over $F$ is isomorphic to $\mathrm{Nrd}_Q$ for a unique up to an isomorphism quaternion division $F$-algebra $Q$ ([3, Corollary 2.15]). Any non-degenerate quadratic form divisible by $\mathrm{Nrd}_Q$ arises the way described above from an appropriate hermitian form over $Q$ (unique up to an isomorphism). For the case of characteristic not 2, we may refer to [18, §1 of Chapter 10].

The Witt indexes $i(h)$ and $i(q)$ of $h$ and $q$ are related as follows (and this relationship implies the above uniqueness statement, c.f. [10, Corollary 9.2], known as Jacobson Theorem):

**Lemma 11.1.** *$i(q) = 4i(h)$.*

*Proof.* For any integer $r \geq 0$, the inequality $i(h) \geq r$ implies $i(q) \geq 4r$. Indeed, if $i(h) \geq r$, $V$ contains a totally $h$-isotropic $Q$-subspace $W$ of dimension $r$. This $W$ is also totally $q$-isotropic and has dimension $4r$ over $F$. Therefore $i(q) \geq 4r$.

To finish, we prove by induction on $r \geq 0$ that $i(q) \geq 4r - 3$ implies $i(h) \geq r$. This is trivial for $r = 0$. If $r > 0$ and $i(q) \geq 4r - 3$, then $q$ is isotropic. But any $q$-isotropic vector is also $h$ isotropic, therefore the $Q$-vector space $V$ decomposes in a direct sum of $h$-orthogonal subspaces $V = U \oplus V'$ such that $h|_U$ is a hyperbolic plane. The subspaces $U$ and $V'$ are also $q$-orthogonal and $q|_U$ is hyperbolic (of dimension 4). For $h' := h|_{V'}$ and $q' := q|_{V'}$ it follows that $i(h') = i(h) - 1$ and $i(q') = i(q) - 4$, and we are done by the induction hypothesis applied to $h'$ (of course, $q'$ is the quadratic form given by $h'$). □

For any integer $r$, let $X_r$ be the variety of totally $h$-isotropic $r$-dimensional $Q$-subspaces in $V$ and let $Y_r$ be the variety of totally $q$-isotropic $r$-dimensional $F$-subspaces in $V$. The variety $Y_{2n}$, where $n := \dim_Q V$, is not connected and has two isomorphic connected components; changing notation, we let $Y_{2n}$ be one of its connected component in this case.

**Corollary 11.2.** *For any $r$, the upper motives of the varieties $X_r$, $Y_{4r}$, $Y_{4r-1}$, $Y_{4r-2}$, and $Y_{4r-3}$ are isomorphic. In particular, these varieties have the same canonical 2-dimension.*

*This canonical dimension is maximal and equal to*

$$\dim X_r = r(4n - 6r + 1)$$

*in the case of* generic *(see §6) $Q$ and $h$.*

*Proof.* By Lemma 11.1, each of the three varieties possesses a rational map to each other. Therefore the upper motives are isomorphic by [11, Corollary 2.15]. For the first statement on canonical dimension see [6, Theorem 5.1]. The statement on the maximal canonical dimension follows from Theorem 10.1. □

Let us recall that according to the original definition [22, Definition 5.11(2)] due to A. Vishik of the $J$-invariant $J(q)$ of a non-degenerate quadratic form $q$ over $F$ of dimension $4n$, $J(q)$ is a certain subset of the set of integers $\{0, 1, \dots, 2n-1\}$. Note that in [3, §88], the name *$J$-invariant* and the notation $J(q)$ stand for the complement of the above subset (with the "excuse" that this choice simplifies several formulas involving the $J$-invariant). In the present paper we are using the original definition and notation.

Let $q$ be a quadratic form given by tensor product of an $n$-dimensional non-degenerate symmetric bilinear form by a 2-fold quadratic Pfister form $\pi$. Let us first assume that $n$ is odd. Then $q$ is hyperbolic if and only if $\pi$ is hyperbolic. It follows that for anisotropic $\pi$, the canonical 2-dimension of the maximal orthogonal Grassmannian associated to $q$ is 1. Therefore by [3, Theorem 90.3] the $J$-invariant of $q$ is $\{0, 2, 3, \dots, 2n-1\}$ (everything but 1) for odd $n$ and anisotropic $\pi$ (note that $0 \in J(q)$ beacuse $q$ is an even-dimension form of trivial discriminant).

For any even $n$, the Witt index of $q$ over any extension field of $F$ is divisible by 4. It follows by [3, Proposition 88.8] that the $J$-invariant of $q$ contains the set $J_0$ of the integers in the interval $[0,\ 2n-1]$ which are not congruent to 3 modulo 4. Theorem 10.1 with Corollary 11.2 make it possible (see Corollary 11.3 right below) to show that $J(q_0) = J_0$ for the quadratic form $q_0$ associated with the hermitian form $h$ on $Q^n$, where $h$ and the quaternion algebra $Q$ are as in the beginning of §6. The quadratic form $q_0$ is of the type we are interested in because it is isomorphic to the tensor product of an $n$-dimensional non-degenerate symmetric bilinear form by the 2-fold quadratic Pfister form $\mathrm{Nrd}_Q$. This means that for any even $n \geq 2$, $J_0$ is the smallest (in the sense of inclusion) value of the $J$-invariant of a quadratic form given by tensor product of an $n$-dimensional non-degenerate symmetric bilinear form by a 2-fold quadratic Pfister form.

**Corollary 11.3.** $J(q_0) = J_0$.

*Proof.* We recall that $q_0$ is the quadratic form associated with the hermitian form $h$ introduced in the beginning of Section 6. Let us calculate the $J$-invariant of $q = q_0$. We are using the above notation for the varieties associated to $h$ and to $q$.

By [3, Theorem 90.3], the canonical 2-dimension of $Y_{2n}$ is $\dim Y_{2n} = n(2n-1)$ minus the sum of the elements of $J(q)$. On the other hand, by Corollary 11.2 and Theorem 10.1, the canonical 2-dimension of $Y_{2n}$ is equal to the dimension of $X_{n/2}$ which is

$$\dim X_{n/2} = n(n+1)/2 = n(2n-1) - \sum_{J_0} j.$$

Therefore $J(q) = J_0$. □

Lemma 11.1 and Corollaries 11.2 and 11.3 are analogues of [10, Lemma 9.1 and Corollaries 9.3 and 9.4]. The reader may discover on his own the analogues of the remaining statements of [10, §9].

## References


[1] Chernousov, V., Gille, S., and Merkurjev, A. Motivic decomposition of isotropic projective homogeneous varieties. *Duke Math. J. 126*, 1 (2005), 137–159.

[2] Edidin, D., and Graham, W. Equivariant intersection theory. *Invent. Math. 131*, 3 (1998), 595–634.

[3] Elman, R., Karpenko, N., and Merkurjev, A. *The algebraic and geometric theory of quadratic forms*, vol. 56 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2008.

[4] Fulton, W. *Intersection theory*, second ed., vol. 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 1998.

[5] Karpenko, N. A. Cohomology of relative cellular spaces and of isotropic flag varieties. *Algebra i Analiz 12*, 1 (2000), 3–69.

[6] Karpenko, N. A. Canonical dimension. In *Proceedings of the International Congress of Mathematicians. Volume II* (New Delhi, 2010), Hindustan Book Agency, pp. 146–161.

[7] Karpenko, N. A. Upper motives of outer algebraic groups. In *Quadratic forms, linear algebraic groups, and cohomology*, vol. 18 of *Dev. Math.* Springer, New York, 2010, pp. 249–258.

[8] Karpenko, N. A. Hyperbolicity of unitary involutions. *Sci. China Math. 55*, 5 (2012), 937–945.

[9] Karpenko, N. A. Sufficiently generic orthogonal Grassmannians. *J. Algebra 372* (2012), 365–375.

[10] Karpenko, N. A. Unitary Grassmannians. *J. Pure Appl. Algebra 216*, 12 (2012), 2586–2600.

[11] Karpenko, N. A. Upper motives of algebraic groups and incompressibility of Severi-Brauer varieties. *J. Reine Angew. Math. 677* (2013), 179–198.

[12] Knus, M.-A., Merkurjev, A., Rost, M., and Tignol, J.-P. *The book of involutions*, vol. 44 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1998. With a preface in French by J. Tits.

[13] Manin, J. I. Correspondences, motifs and monoidal transformations. *Mat. Sb. (N.S.) 77 (119)* (1968), 475–507.

[14] Merkurjev, A. S., Panin, I. A., and Wadsworth, A. R. Index reduction formulas for twisted flag varieties. I. *$K$-Theory 10*, 6 (1996), 517–596.

[15] Milne, J. S. *Étale cohomology*, vol. 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1980.

[16] Molina Rojas, L. A., and Vistoli, A. On the Chow rings of classifying spaces for classical groups. *Rend. Sem. Mat. Univ. Padova 116* (2006), 271–298.

[17] Rost, M. Chow groups with coefficients. *Doc. Math. 1* (1996), No. 16, 319–393 (electronic).

[18] Scharlau, W. *Quadratic and Hermitian forms*, vol. 270 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1985.

[19] Serre, J.-P. *Cohomologie galoisienne*, fifth ed., vol. 5 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1994.

[20] Springer, T. A. *Linear algebraic groups*, second ed. Modern Birkhäuser Classics. Birkhäuser Boston Inc., Boston, MA, 2009.

[21] Totaro, B. The Chow ring of a classifying space. In *Algebraic $K$-theory (Seattle, WA, 1997)*, vol. 67 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI, 1999, pp. 249–281.

[22] Vishik, A. On the Chow groups of quadratic Grassmannians. *Doc. Math. 10* (2005), 111–130 (electronic).

[23] Vishik, A. Fields of $u$-invariant $2^r + 1$. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II*, vol. 270 of *Progr. Math.* Birkhäuser Boston Inc., Boston, MA, 2009, pp. 661–685.

[24] Vistoli, A. Grothendieck topologies, fibered categories and descent theory. In *Fundamental algebraic geometry*, vol. 123 of *Math. Surveys Monogr.* Amer. Math. Soc., Providence, RI, 2005, pp. 1–104.

Mathematical & Statistical Sciences, University of Alberta, Edmonton, CANADA
*E-mail address*: `karpenko` *at* `ualberta.ca`

Department of Mathematics, University of California, Los Angeles, CA, USA
*E-mail address*: `merkurev` *at* `math.ucla.edu`