HYPERBOLICITY OF HERMITIAN FORMS OVER BIQUATERNION ALGEBRAS

NIKITA A. KARPENKO

Abstract. We show that a non-hyperbolic hermitian form over a biquaternion algebra over a field of characteristic $\neq 2$ remains non-hyperbolic over a generic splitting field of the algebra.

Contents

1. Introduction

Throughout this note (besides of §3 and §4) F is a field of characteristic $\neq 2$. The basic reference for the staff related to involutions on central simple algebras is [12]. The *degree* $\deg A$ of a (finite dimensional) central simple F-algebra A is the integer $\sqrt{\dim_F A}$; the *index* $\operatorname{ind} A$ of A is the degree of a central division algebra Brauer-equivalent to A.

Conjecture 1.1. *Let A be a central simple F-algebra endowed with an orthogonal involution σ. If σ becomes hyperbolic over the function field of the Severi-Brauer variety of A, then σ is hyperbolic (over F).*

In a stronger version of Conjecture 1.1, each of two words "hyperbolic" is replaced by "isotropic", cf. [10, Conjecture 5.2].

Here is the complete list of indices $\operatorname{ind} A$ and coindices $\operatorname{coind} A = \deg A / \operatorname{ind} A$ of A for which Conjecture 1.1 is known (over an arbitrary field of characteristic $\neq 2$), given in the chronological order:

- $\operatorname{ind} A = 1$ — trivial;

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- $\operatorname{coind} A = 1$ (the stronger version) — [10, Theorem 5.3];
- $\operatorname{ind} A = 2$ (the stronger version) — [15, Corollary 3.4];
- $\operatorname{coind} A$ odd — [6, appendix by Zainoulline] and independently [7, Theorem 3.3];
- $\operatorname{ind} A = 4$ and $\operatorname{coind} A = 2$ — [18, Proposition 3].

Let us note that Conjecture 1.1 for any given (A, σ) with $\operatorname{coind} A = 2$ implies the stronger version version of Conjecture 1.1 for this (A, σ): indeed, by [7, Theorem 3.3], if $\operatorname{coind} A = 2$ and σ becomes isotropic over the function field of the Severi-Brauer variety, then σ becomes hyperbolic over this function field and the weaker version applies. In particular, the stronger version of Conjecture 1.1 is known for A with $\operatorname{ind} A = 4$ and $\operatorname{coind} A = 2$ (the last case in the above list, also discussed below).

The main result of this note is as follows:

Theorem 1.2 (**Main Theorem**). *Conjecture 1.1 holds for any A of index 4.*

Sivatski's proof of the case with $\deg A = 8$ and $\operatorname{ind} A = 4$, mentioned above, is based on the following theorem, due to Laghribi:

Theorem 1.3 ([13, Théorème 4]). *Let φ be a quadratic form of dimension 8 and of trivial discriminant. Assume that the index of the Clifford algebra C of φ is 4. If φ becomes isotropic over the function field $F(X_1)$ of the Severi-Brauer variety X_1 of C, then φ is isotropic (over F).*

The following alternate proof of Theorem 1.3, given by Vishik, is a prototype of our proof of Main Theorem (Theorem 1.2). Let Y be the projective quadric of φ and let X_2 be the Albert quadric of the biquaternion division algebra Brauer-equivalent to C. Assume that $\varphi_{F(X_1)}$ is isotropic. Then for any field extension E/F, the Witt index of φ_E is at least 2 if and only if $X_2(E) \neq \emptyset$. By [19, Theorem 4.15] and since the Chow motive $M(X_2)$ of X_2 is indecomposable, it follows that the motive $M(X_2)(1)$ is a summand of the motive of Y. The complement summand of $M(Y)$ is then given by a *Rost projector* on Y in the sense of Definition 5.1. Since $\dim Y + 1$ is not a power of 2, it follows that Y is isotropic (cf. [5, Corollary 80.11]).

In §4 we produce a replacement of [19, Theorem 4.15] (used right above to split off the summand $M(X_2)(1)$ from the motive of Y) valid for more general (as projective quadrics) algebraic varieties. In §5 we reproduce some recent results due to Rost concerning the modulo 2 Rost correspondences and Rost projectors on more general (as projective quadrics) varieties. In §6 we apply some standard motivic decompositions of projective homogeneous varieties to certain varieties related to an index 4 central simple algebra with an isotropic orthogonal involution. Finally, in §7 we prove Main Theorem (Theorem 1.2).

2. Notation

Unlike [5], we understand under a *variety* a separated scheme of finite type over a field.

Let D be a central simple F-algebra. The F-dimension of any right ideal in D is divisible by $\deg D$; the quotient is the *reduced dimension* of the ideal. For any integer i, we write $X(i; D)$ for the generalized Severi-Brauer variety of the right ideals in D of reduced dimension i. In particular, $X(0; D) = \operatorname{Spec} F = X(\deg D; D)$ and $X(i, D) = \emptyset$ for $i < 0$ and for $i > \deg D$.

More generally, let V be a right D-module. The F-dimension of V is then divisible by $\deg D$ and the quotient $\operatorname{rdim} V = \dim_F V/\deg D$ is called the *reduced dimension* of V. For any integer i, we write $X(i;V)$ for the variety of the D-submodules in V of reduced dimension i (non-empty iff $0 \le i \le \operatorname{rdim} V$). For a finite sequence of integers $i_1,\dots,i_r$, we write $X(i_1 \subset \dots \subset i_r; V)$ for the variety of flags of the D-submodules in V of reduced dimensions $i_1,\dots,i_r$ (non-empty iff $0 \le i_1 \le \dots \le i_r \le \operatorname{rdim} V$).

Now we additionally assume that D is endowed with an orthogonal involution τ. Then we write $X(i;(D,\tau))$ for the variety of the totally isotropic right ideals in D of reduced dimension i (non-empty iff $0 \le i \le \deg D/2$).

If moreover V is endowed with a hermitian form h, we write $X(i;(V,h))$ for the variety of the totally isotropic D-submodules in V of reduced dimension i and we write

$$X(i_1 \subset \dots \subset i_r; (V,h))$$

for the variety of flags of the totally isotropic D-submodules in V of reduced dimensions $i_1,\dots,i_r$.

Let us observe that all the varieties introduced in this section are projective homogeneous.

3. Krull-Schmidt principle

The characteristic of the base field F is arbitrary in this section.

Our basic reference for Chow groups and Chow motives (including notation) is [5]. We fix an associative unital commutative ring Λ (we will take $\Lambda = \mathbb{F}_2$ in the application) and for a variety X we write $\mathrm{CH}(X;\Lambda)$ for its Chow group with coefficients in Λ. Our category of motives is the category $\mathrm{CM}(F,\Lambda)$ of *graded Chow motives with coefficients in* Λ, [5, definition of §64]. By a *sum* of motives we always mean the *direct* sum.

We shall often assume that our coefficient ring Λ is finite. This simplifies significantly the situation (and is sufficient for our application). For instance, for a finite Λ, the endomorphism rings of finite sums of Tate motives are also finite and the following easy statement applies:

Lemma 3.1. *An appropriate power of any element of any* finite *associative (not necessarily commutative) ring is idempotent.*

Proof. Since the ring is finite, for any its element x we have $x^a = x^{a+b}$ for some $a \ge 1$ and $b \ge 1$. It follows that x^{ab} is an idempotent. □

Let X be a smooth complete variety over F. We call X *split*, if its motive $M(X) \in \mathrm{CM}(F,\Lambda)$ is a finite sum of Tate motives. We call X *geometrically split*, if it splits over a field extension of F. We say that X satisfies the *nilpotence principle*, if for any field extension E/F the kernel of the change of field homomorphism $\mathrm{End}(M(X)) \to \mathrm{End}(M(X)_E)$ consists of nilpotents. Any projective homogeneous variety is geometrically split and satisfies the nilpotence principle, [3, Theorem 8.2].

Corollary 3.2. *Assume that the coefficient ring Λ is finite. Let X be a geometrically split variety satisfying the nilpotence principle. Then an appropriate power of any endomorphism of the motive of X is a projector.*

Proof. Let $\bar{F}/F$ be a field extension such that the motive $M(X)_{\bar{F}}$ is a finite sum of Tate motives. Let f be an endomorphism of $M(X)$. Since our coefficient ring Λ is finite, it follows that the ring $\operatorname{End}(M(X)_{\bar{F}})$ is finite. Therefore a power of $f_{\bar{F}}$ is idempotent by Lemma 3.1, and we may assume that $f_{\bar{F}}$ is idempotent. Since X satisfies the nilpotence principle, the element $\varepsilon := f^2 - f$ is nilpotent. Let n be a positive integer such that $\varepsilon^n = 0 = n\varepsilon$. Then $(f+\varepsilon)^{n^n} = f^{n^n}$ because the binomial coefficients $\binom{n^n}{i}$ for $i < n$ are divisible by n. Therefore f^{n^n} is a projector. □

Lemma 3.3 (cf. [4, Theorem 28]). *Assume that the coefficient ring Λ is finite. Let X be a geometrically split variety satisfying the nilpotence principle and let $p \in \operatorname{End}(M(X))$ be a projector. Then the motive (X, p) decomposes into a finite sum of indecomposable motives.*

Proof. If (X, p) does not decompose this way, we get an infinite sequence

$$p_0 = p,\ p_1,\ p_2,\ \ldots\ \in \operatorname{End}(M(X))$$

of pairwise distinct projectors such that $p_i \circ p_j = p_j = p_j \circ p_i$ for any $i < j$.

Let $\bar{F}/F$ be a splitting field of X. Since the ring $\operatorname{End}(M(X)_{\bar{F}})$ is finite, we have $(p_i)_{\bar{F}} = (p_j)_{\bar{F}}$ for some $i < j$. The difference $p_i - p_j$ is nilpotent and idempotent, therefore $p_i = p_j$. □

A (non necessarily commutative) ring is called *local*, if the sum of any two non-invertible elements differs from 1 in the ring. Since the sum of two nilpotents is never 1, we have

Lemma 3.4. *A ring, where each non-invertible element is nilpotent, is local. In particular, by Corollary 3.2, so is the ring $\operatorname{End}(M(X))$, if Λ is finite and X is a geometrically split variety satisfying the nilpotence principle and such that the motive $M(X)$ is indecomposable.* □

Theorem 3.5 ([1, Theorem 3.6 of Chapter I]). *If the endomorphism ring of any indecomposable object of a pseudo-abelian category is local, then any object of this category has at most one decomposition in a finite direct sum of indecomposable objects.*

We say that the *Krull-Schmidt principle* holds for a given pseudo-abelian category, if every object of the category has one and unique decomposition in a finite direct sum of indecomposable objects. In the sequel, we are constantly using the following statement which is an immediate consequence of Lemmas 3.3 and 3.4 and the above theorem:

Corollary 3.6 (cf. [4, Corollary 35]). *Assume that the coefficient ring Λ is finite. The Krull-Schmidt principle holds for the pseudo-abelian Tate subcategory in $\operatorname{CM}(F, \Lambda)$ generated by the motives of the geometrically cellular F-varieties satisfying the nilpotence principle.* □

Remark 3.7. Replacing the Chow groups $\operatorname{CH}(-;\Lambda)$ by the *reduced* Chow groups $\overline{\operatorname{CH}}(-;\Lambda)$ (cf. [5, §72]) in the definition of the category $\operatorname{CM}(F, \Lambda)$, we get a "simplified" motivic category $\overline{\operatorname{CM}}(F, \Lambda)$ (which is still sufficient for the main purpose of this paper). Working within this category, we do not need the nilpotence principle any more. In particular, the Krull-Schmidt principle holds (with a simpler proof) for the pseudo-abelian Tate subcategory in $\overline{\operatorname{CM}}(F, \Lambda)$ generated by the motives of the geometrically cellular F-varieties.

4. Splitting off a motivic summand

The characteristic of the base field F is still arbitrary in this section.

In this section we assume that the coefficient ring Λ is connected. We shall often assume that Λ is finite.

Before climbing to the main result of this section (which is Proposition 4.5), let us do some warm up.

Definition 4.1. A motive $M \in \mathrm{CM}(F, \Lambda)$ is called *outer*, if $\mathrm{CH}^0(M; \Lambda) \neq 0$. A summand M of the motive of a smooth complete variety is called *outer*, if this summand is an outer motive.

For instance, the motive of any smooth complete non-empty variety is outer and an outer summand of itself.

Given a correspondence $\alpha \in \mathrm{CH}_{\dim X}(X \times Y; \Lambda)$ between some smooth complete irreducible varieties X and Y, we write $\operatorname{mult} \alpha \in \Lambda$ for the *multiplicity* of α, [5, definition of §75]. Multiplicity of a composition of two correspondences is the product of multiplicities of the composed correspondences (cf. [10, Corollary 1.7]). In particular, multiplicity of a projector is idempotent and therefore $\in \{0, 1\}$ because the coefficient ring Λ is connected.

Lemma 4.2. *Let X be a smooth complete irreducible variety. The motive (X, p) given by a projector $p \in \mathrm{CH}_{\dim X}(X \times X; \Lambda)$ is outer iff $\operatorname{mult} p = 1$.*

Proof. The group $\mathrm{CH}^0((X, p); \Lambda)$ (defined as $\operatorname{Hom}\big((X, p), \Lambda\big)$) is the image of the endomorphism of $\mathrm{CH}^0(X; \Lambda) = \Lambda \cdot [X]$ given by the multiplication by $\operatorname{mult} p$. $\square$

Lemma 4.3. *Assume that a motive M decomposes into a sum of Tate motives. Then M is outer iff the Tate motive Λ is present in the decomposition.*

Proof. For any $i \in \mathbb{Z}$ we have: $\mathrm{CH}^0(\Lambda(i); \Lambda) \neq 0$ iff $i = 0$. $\square$

The following statement generalizes (the finite coefficient version of) [19, Corollary 3.9]:

Lemma 4.4. *Assume that the coefficient ring Λ is finite. Let X and Y be smooth complete irreducible varieties such that there exist multiplicity 1 correspondences*

$$\alpha \in \mathrm{CH}_{\dim X}(X \times Y; \Lambda) \quad \textit{and} \quad \beta \in \mathrm{CH}_{\dim Y}(Y \times X; \Lambda).$$

Assume that X is geometrically split and satisfies the nilpotence principle. Then there is an outer summand of $M(X)$ isomorphic to an outer summand of $M(Y)$. Moreover, for any outer summand M_X of $M(X)$ and any outer summand M_Y of $M(Y)$, there is an outer summand of M_X isomorphic to an outer summand of M_Y.

Proof. By Corollary 3.2, the composition $p := (\beta \circ \alpha)^{\circ n}$ for some $n \geq 1$ is a projector. Therefore $q := (\alpha \circ \beta)^{\circ 2n}$ is also a projector and the summand (X, p) of $M(X)$ is isomorphic to the summand (Y, q) of $M(Y)$: mutually inverse isomorphisms are, say,

$$\alpha \circ (\beta \circ \alpha)^{\circ(2n)} : (X, p) \to (Y, q) \;\text{ and }\; \beta \circ (\alpha \circ \beta)^{\circ(4n-1)} : (Y, q) \to (X, p)\ .$$

Since $\operatorname{mult} p = (\operatorname{mult} \beta \cdot \operatorname{mult} \alpha)^n = 1$ and similarly $\operatorname{mult} q = 1$, the summand (X, p) of $M(X)$ and the summand (Y, q) of $M(Y)$ are outer by Lemma 4.2.

We have proved the first statement of Lemma 4.4. As to the second statement, let

$$p' \in \mathrm{CH}_{\dim X}(X \times X; \Lambda) \quad \text{and} \quad q' \in \mathrm{CH}_{\dim Y}(Y \times Y; \Lambda)$$

be projectors such that $M_X = (X, p')$ and $M_Y = (Y, q')$. Replacing α and β by $q' \circ \alpha \circ p'$ and $p' \circ \beta \circ q'$, we get isomorphic outer motives (X, p) and (Y, q) which are summands of M_X and M_Y. □

Here comes the needed replacement of [19, Theorem 4.15]:

Proposition 4.5. *Assume that the coefficient ring Λ is finite. Let X be a geometrically split, geometrically irreducible variety satisfying the nilpotence principle and let M be a motive. Assume that there exists a field extension E/F such that*

1. *the E-motive $M(X)_E \in \mathrm{CM}(E, \Lambda)$ of the E-variety X_E is indecomposable (this implies that the F-motive $M(X) \in \mathrm{CM}(F, \Lambda)$ is indecomposable);*
2. *the field extension $E(X)/F(X)$ is purely transcendental;*
3. *the motive $M(X)_E$ is a summand of the motive M_E.*

Then the motive $M(X)$ is a summand of the motive M.

Proof. We may assume that $M = (Y, p, n)$ for some irreducible smooth complete F-variety Y, a projector $p \in \mathrm{CH}_{\dim Y}(Y \times Y; \Lambda)$, and an integer n.

By the assumption (3), we have morphisms of motives $f : M(X)_E \to M_E$ and $g : M_E \to M(X)_E$ with $g \circ f = \mathrm{id}_{M(X)_E}$.

Let $\xi : \operatorname{Spec} F(X) \to X$ be the generic point of the (irreducible) variety X. For any F-scheme Z, we write ξ_Z for the morphism $\xi_Z = (\xi \times \mathrm{id}_Z) : Z_{F(X)} = \operatorname{Spec}_{F(X)} \times Z \to X \times Z$. Note that for any $\alpha \in \mathrm{CH}(X \times Z)$, the image $\xi_Z^*(\alpha) \in \mathrm{CH}(Z_{F(X)})$ of α under the pull-back homomorphism $\xi_Z^* : \mathrm{CH}(X \times Z, \Lambda) \to \mathrm{CH}(Z_{F(X)}, \Lambda)$ coincides with the composition of correspondences $\alpha \circ [\xi]$, [5, Proposition 62.4(2)], where $[\xi] \in \mathrm{CH}_0(X_{F(X)}, \Lambda)$ is the class of the point ξ:

$$(*) \qquad \xi_Z^*(\alpha) = \alpha \circ [\xi].$$

In the commutative square

$$\begin{array}{ccc} \mathrm{CH}(X_E \times Y_E; \Lambda) & \xrightarrow{\xi^*_{Y_E}} & \mathrm{CH}(Y_{E(X)}; \Lambda) \\ {\scriptstyle \mathrm{res}_{E/F}} \uparrow & & {\scriptstyle \mathrm{res}_{E(X)/F(X)}} \uparrow \\ \mathrm{CH}(X \times Y; \Lambda) & \xrightarrow{\xi^*_{Y}} & \mathrm{CH}(Y_{F(X)}; \Lambda) \end{array}$$

the change of field homomorphism $\mathrm{res}_{E(X)/F(X)}$ is surjective[1] because of the assumption (2) by the homotopy invariance of Chow groups [5, Theorem 57.13] and by the localization property of Chow groups [5, Proposition 57.11]. Moreover, the pull-back homomorphism ξ^*_Y is surjective by [5, Proposition 57.11]. It follows that there exists an element $f' \in \mathrm{CH}(X \times Y; \Lambda)$ such that $\xi^*_{Y_E}(f'_E) = \xi^*_{Y_E}(f)$.

The multiplicity $\mathrm{mult}(g \circ f)$ of the correspondence given by the composition $g \circ f = \mathrm{id}_{M(X)_E}$ is equal to 1. The multiplicity $\mathrm{mult}(g \circ f'_E)$ coincides with $\mathrm{mult}(g \circ f)$. Indeed,

[1]In fact, $\mathrm{res}_{E(X)/F(X)}$ is even an isomorphism, but we do not need its injectivity (which can be obtained with a help of specialization).

$\operatorname{mult}(g \circ f) = \deg \xi^*_{X_E}(g \circ f)$ by [5, Lemma 75.1], where $\deg : \mathrm{CH}(X_{E(X)}) \to \Lambda$ is the degree homomorphism. Furthermore, $\xi^*_{X_E}(g \circ f) = (g \circ f) \circ [\xi_E]$ by $(*)$. Finally, $(g \circ f) \circ [\xi_E] = g \circ (f \circ [\xi_E])$ and $f \circ [\xi_E] = \xi^*_{Y_E}(f) = \xi^*_{Y_E}(f'_E)$ by construction of f'.

Since the multiplicity is multiplicative, the multiplicity of $(g \circ f'_E)^{\circ n}$ is still 1 and therefore $(g \circ f'_E)^{\circ n} \neq 0$ for any $n \geq 1$. On the other hand, by Corollary 3.2, $(g \circ f'_E)^{\circ n}$ is a projector for some $n \geq 1$. By the assumption (1) (indecomposability of $M(X)_E$), it follows that $(g \circ f'_E)^{\circ n} = \mathrm{id}_{M(X)_E}$ for this n. We set $g' = (g \circ f'_E)^{\circ(n-1)} \circ g$. Then for the new morphisms $f' : M(X) \to M$ and $g' : M_E \to M(X_E)$ we still have $g' \circ f'_E = \mathrm{id}_{M(X)_E}$.

Now we apply the same procedure to the dual morphisms

$$(g')^t : M(X)_E \to (Y, p, n + \dim X - \dim Y)_E$$
$$\text{and} \quad (f'_E)^t : (Y, p, n + \dim X - \dim Y)_E \to M(X)_E.$$

This way we get morphisms $g'' : M \to M(X)$ and $f'' : M(X) \to M$ such that

$$(f''_E)^t \circ (g''_E)^t = \mathrm{id}_{M(X)^*_E}, \quad \text{that is,} \;\; g''_E \circ f''_E = \mathrm{id}_{M(X)_E}.$$

It follows by the nilpotence principle that $(g'' \circ f'')^{\circ n} = \mathrm{id}_{M(X)}$ for some integer $n \geq 1$. Therefore $M(X)$ is a summand of M. $\square$

Remark 4.6. Replacing $\mathrm{CM}(F, \Lambda)$ by $\overline{\mathrm{CM}}(F, \Lambda)$ in Proposition 4.5, we get a weaker version of Proposition 4.5 which is still sufficient for our application. The nilpotence principle is no more needed in the proof of the weaker version. Because of that, there is no more need to assume that X satisfies the nilpotence principle.

5. Rost correspondences

In this section, X stands for a smooth complete absolutely irreducible variety of dimension $d > 0$.

The coefficient ring Λ of the motivic category is $\mathbb{F}_2$ in this section. We write $\mathrm{Ch}(-)$ for the Chow group $\mathrm{CH}(-; \mathbb{F}_2)$ with coefficients in $\mathbb{F}_2$. We write $\deg_{X/F}$ for the degree homomorphism $\mathrm{Ch}_0(X) \to \mathbb{F}_2$.

Definition 5.1. An element $\rho \in \mathrm{Ch}_d(X \times X)$ is called a *Rost correspondence* (on X), if $\rho_{F(X)} = \chi_1 \times [X_{F(X)}] + [X_{F(X)}] \times \chi_2$ for some 0-cycle classes $\chi_1, \chi_2 \in \mathrm{Ch}_0(X_{F(X)})$ of degree 1. A *Rost projector* is a Rost correspondence which is a projector.

Remark 5.2. Our definition of a Rost correspondence differs from the definition of a *special correspondence* in [16]. Our definition is weaker in the sense that a special correspondence on X (which is an element of the *integral* Chow group $\mathrm{CH}_d(X \times X)$) considered modulo 2 is a Rost correspondence but not any Rost correspondence is obtained this way. This difference gives a reason to reproduce below some results of [16].

Remark 5.3. Clearly, the set of all Rost correspondences on X is stable under transposition and composition. In particular, if ρ is a Rost correspondence, then its both symmetrizations $\rho^t \circ \rho$ and $\rho \circ \rho^t$ are (symmetric) Rost correspondences. Writing $\rho_{F(X)}$ as in Definition 5.1, we have $(\rho^t \circ \rho)_{F(X)} = \chi_1 \times [X_{F(X)}] + [X_{F(X)}] \times \chi_1$ (and $(\rho \circ \rho^t)_{F(X)} = \chi_2 \times [X_{F(X)}] + [X_{F(X)}] \times \chi_2$).

Lemma 5.4. *Assume that the variety X is projective homogeneous. Let $\rho \in \mathrm{Ch}_d(X \times X)$ be a projector. If there exists a field extension E/F such that $\rho_E = \chi_1 \times [X_E] + [X_E] \times \chi_2$ for some 0-cycle classes $\chi_1, \chi_2 \in \mathrm{Ch}_0(X_E)$ of degree 1, then ρ is a Rost projector.*

Proof. According to [3, Theorem 7.5], there exist some integer $n \geq 0$ and for $i = 1, \dots, n$ some integers $r_i > 0$ and some projective homogeneous varieties X_i satisfying $\dim X_i + r_i < d$ such that for $M = \bigoplus_{i=1}^n M(X_i)(r_i)$ the motive $M(X)_{F(X)}$ decomposes as $\mathbb{F}_2 \oplus M \oplus \mathbb{F}_2(d)$. Since there is no non-zero morphism between different summands of this decomposition, the ring $\mathrm{End}\, M(X)$ decomposes in the product of rings

$$\mathrm{End}\, \mathbb{F}_2 \times \mathrm{End}\, M \times \mathrm{End}\, \mathbb{F}_2(d) = \mathbb{F}_2 \times \mathrm{End}\, M \times \mathbb{F}_2.$$

Let $\chi \in \mathrm{Ch}_0(X_{F(X)})$ be a 0-cycle class of degree 1. We set

$$\begin{aligned}\rho' = \chi \times [X_{F(X)}] + [X_{F(X)}] \times \chi \in \mathbb{F}_2 \times \mathbb{F}_2 \\ \subset \mathbb{F}_2 \times \mathrm{End}\, M \times \mathbb{F}_2 = \mathrm{End}\, M(X)_{F(X)} = \mathrm{Ch}_d(X_{F(X)} \times X_{F(X)})\end{aligned}$$

and we show that $\rho_{F(X)} = \rho'$. The difference $\varepsilon = \rho_{F(X)} - \rho'$ vanishes over $E(X)$. Therefore ε is a nilpotent element of $\mathrm{End}\, M$. Choosing a positive integer m with $\varepsilon^m = 0$, we get

$$\rho_{F(X)} = \rho_{F(X)}^m = (\rho' + \varepsilon)^m = (\rho')^m + \varepsilon^m = (\rho')^m = \rho'. \qquad \square$$

Lemma 5.5. *Let $\rho \in \mathrm{Ch}_d(X \times X)$ be a projector. The motive (X, ρ) is isomorphic to $\mathbb{F}_2 \oplus \mathbb{F}_2(d)$ iff $\rho = \chi_1 \times [X] + [X] \times \chi_2$ for some some 0-cycle classes $\chi_1, \chi_2 \in \mathrm{Ch}_0(X)$ of degree 1.*

Proof. A morphism $\mathbb{F}_2 \oplus \mathbb{F}_2(d) \to (X, \rho)$ is given by some

$$f \in \mathrm{Hom}\big(\mathbb{F}_2, M(X)\big) = \mathrm{Ch}_0(X) \ \text{ and } \ f' \in \mathrm{Hom}\big(\mathbb{F}_2(d), M(X)\big) = \mathrm{Ch}_d(X).$$

A morphism in the inverse direction is given by some

$$g \in \mathrm{Hom}(M(X), \mathbb{F}_2) = \mathrm{Ch}^0(X) \ \text{ and } \ g' \in \mathrm{Hom}(M(X), \mathbb{F}_2(d)) = \mathrm{Ch}^d(X).$$

The two morphisms are mutually inverse isomorphisms iff $\rho = f \times g + f' \times g'$ and $\deg_{X/F}(fg) = 1 = \deg_{X/F}(f'g')$. The degree condition means that $f' = [X] = g$ and $\deg_{X/F}(f) = 1 = \deg_{X/F}(g')$. $\square$

Corollary 5.6. *If X is projective homogeneous and ρ is a projector on X such that*

$$(X, \rho)_E \simeq \mathbb{F}_2 \oplus \mathbb{F}_2(d)$$

for some field extension E/F, then ρ is a Rost projector. $\square$

A smooth complete variety is called *anisotropic*, if the degree of its any closed point is even.

Lemma 5.7 ([16, Lemma 9.2], cf. [17, proof of Lemma 6.2])**.** *Assume that X is anisotropic and possesses a Rost correspondence ρ. Then for any integer $i \neq d$ and any elements $\alpha \in \mathrm{Ch}_i(X)$ and $\beta \in \mathrm{Ch}^i(X_{F(X)})$, the image of the product $\alpha_{F(X)} \cdot \beta \in \mathrm{Ch}_0(X_{F(X)})$ under the degree homomorphism $\deg_{X_{F(X)}/F(X)} : \mathrm{Ch}_0(X_{F(X)}) \to \mathbb{F}_2$ is 0.*

Proof. Let $\gamma \in \mathrm{Ch}^i(X \times X)$ be a preimage of β under the surjection

$$\xi_X^* : \mathrm{Ch}^i(X \times X) \to \mathrm{Ch}^i(\operatorname{Spec} F(X) \times X)$$

(where ξ_X^* is as defined in the proof of Proposition 4.5). We consider the 0-cycle class

$$\delta = \rho \cdot ([X] \times \alpha) \cdot \gamma \in \mathrm{Ch}_0(X \times X).$$

Since X is anisotropic, so is $X \times X$, and it follows that $\deg_{(X\times X)/F} \delta = 0$. Therefore it suffices to show that $\deg_{(X\times X)/F} \delta = \deg_{X_{F(X)}/F(X)}(\alpha_{F(X)} \cdot \beta)$.

We have $\deg_{(X\times X)/F} \delta = \deg_{(X\times X)_{F(X)}/F(X)}(\delta_{F(X)})$ and

$$\delta_{F(X)} = (\chi_1 \times [X_{F(X)}] + [X_{F(X)}] \times \chi_2) \cdot ([X_{F(X)}] \times \alpha_{F(X)}) \cdot \gamma_{F(X)} = \\ ([\chi_1] \times [X_{F(X)}]) \cdot ([X_{F(X)}] \times \alpha_{F(X)}) \cdot \gamma_{F(X)}$$

(because $i \neq d$) where $\chi_1, \chi_2 \in \mathrm{Ch}_0(X_{F(X)})$ are as in Definition 5.1. For the first projection $pr_1 : X_{F(X)} \times X_{F(X)} \to X_{F(X)}$ we have

$$\deg_{(X\times X)_{F(X)}/F(X)} \delta_{F(X)} = \deg_{X_{F(X)}/F(X)} (pr_1)_*(\delta_{F(X)})$$

and by the projection formula

$$(pr_1)_*(\delta_{F(X)}) = [\chi_1] \cdot (pr_1)_*\big(([X_{F(X)}] \times \alpha_{F(X)}) \cdot \gamma_{F(X)}\big).$$

Finally,

$$(pr_1)_*\big(([X_{F(X)}] \times \alpha_{F(X)}) \cdot \gamma_{F(X)}\big) = \operatorname{mult}\big(([X_{F(X)}] \times \alpha_{F(X)}) \cdot \gamma_{F(X)}\big) \cdot [X_{F(X)}]$$

and

$$\operatorname{mult}\big(([X_{F(X)}] \times \alpha_{F(X)}) \cdot \gamma_{F(X)}\big) = \operatorname{mult}\big(([X] \times \alpha) \cdot \gamma\big).$$

Since $\operatorname{mult} \chi = \deg_{X_{F(X)}/F(X)} \xi_X^*(\chi)$ for any element $\chi \in \mathrm{Ch}_d(X \times X)$ by [5, Lemma 75.1], it follows that

$$\operatorname{mult}\big(([X] \times \alpha) \cdot \gamma\big) = \deg(\alpha_{F(X)} \cdot \beta). \qquad \square$$

For anisotropic X, we consider the homomorphism $\deg/2 : \mathrm{Ch}_0(X) \to \mathbb{F}_2$ induced by the homomorphism $\mathrm{CH}_0(X) \to \mathbb{Z}$, $\alpha \mapsto \deg(\alpha)/2$.

Corollary 5.8. *Assume that X is anisotropic and possesses a Rost correspondence. Then for any integer $i \neq d$ and any elements $\alpha \in \mathrm{Ch}_i(X)$ and $\beta \in \mathrm{Ch}^i(X)$ with $\beta_{F(X)} = 0$ one has* $(\deg/2)(\alpha \cdot \beta) = 0$.

Proof. Let $\beta' \in \mathrm{CH}^i(X)$ be an integral representative of β. Since $\beta_{F(X)} = 0$, we have $\beta'_{F(X)} = 2\beta''$ for some $\beta'' \in \mathrm{CH}^i(X_{F(X)})$. Therefore

$$(\deg/2)(\alpha \cdot \beta) = \deg_{X_{F(X)}/F(X)}\big(\alpha_{F(X)} \cdot (\beta'' \mod 2)\big) = 0$$

by Lemma 5.7. $\square$

Corollary 5.9. *Assume that X is anisotropic and possesses a Rost correspondence ρ. For any integer $i \notin \{0, d\}$ and any $\alpha \in \mathrm{Ch}_i(X)$ and $\beta \in \mathrm{Ch}^i(X)$ one has*

$$(\deg/2)\big((\alpha \times \beta) \cdot \rho\big) = 0.$$

Proof. Let $\alpha' \in \mathrm{CH}_i(X)$ and $\beta' \in \mathrm{CH}^i(X)$ be integral representatives of α and β. Let $\rho' \in \mathrm{CH}_d(X \times X)$ be an integral representative of ρ. It suffices to show that the degree of the 0-cycle class $(\alpha' \times \beta') \cdot \rho' \in \mathrm{CH}_0(X \times X)$ is divisible by 4.

Let χ_1 and χ_2 be as in Definition 5.1. Let $\chi_1', \chi_2' \in \mathrm{CH}_0(X_{F(X)})$ be integral representatives of χ_1 and χ_2. Then $\rho'_{F(X)} = \chi_1' \times [X_{F(X)}] + [X_{F(X)}] \times \chi_2' + 2\gamma$ for some $\gamma \in \mathrm{CH}_d(X_{F(X)} \times X_{F(X)})$. Therefore (since $i \notin \{0, d\}$)

$$(\alpha'_{F(X)} \times \beta'_{F(X)}) \cdot \rho'_{F(X)} = 2(\alpha'_{F(X)} \times \beta'_{F(X)}) \cdot \gamma.$$

Applying the projection pr_1 onto the first factor and the projection formula, we get twice the element $\alpha'_{F(X)} \cdot (pr_1)_*\big(([X_{F(X)}] \times \beta'_{F(X)}) \cdot \gamma\big)$ whose degree is even by Lemma 5.7 (here we use once again the condition that $i \neq d$). $\square$

Lemma 5.10. *Assume that X is anisotropic and possesses a Rost correspondence ρ. Then* $(\deg/2)(\rho^2) = 1$.

Proof. Let χ_1 and χ_2 be as in Definition 5.1. Let $\chi_1', \chi_2' \in \mathrm{CH}_0(X_E)$ be integral representatives of χ_1 and χ_2. The degrees of χ_1' and χ_2' are odd. Therefore, the degree of the cycle class

$$(\chi_1' \times [X_{F(X)}] + [X_{F(X)}] \times \chi_2')^2 = 2(\chi_1' \times \chi_2') \in \mathrm{CH}_0(X_{F(X)} \times X_{F(X)})$$

is not divisible by 4.

Let $\rho' \in \mathrm{CH}_d(X \times X)$ be an integral representative of ρ. Since $\rho'_{F(X)}$ is $\chi_1' \times [X_{F(X)}] + [X_{F(X)}] \times \chi_2'$ modulo 2, $(\rho'_{F(X)})^2$ is $(\chi_1' \times [X_{F(X)}] + [X_{F(X)}] \times \chi_2')^2$ modulo 4. Therefore $(\deg/2)(\rho^2) = 1$. $\square$

Theorem 5.11 ([16, Theorem 9.1], see also [17, proof of Lemma 6.2])**.** *Let X be an anisotropic smooth complete absolutely irreducible variety of dimension $d > 0$ over a field F of characteristic $\neq 2$ possessing a Rost correspondence. Then the degree of the highest Chern class $c_d(-T_X)$, where T_X is the tangent bundle on X, is not divisible by* 4.

Proof. In this proof, we write $c_\bullet(-T_X)$ for the total Chern class $\in \mathrm{Ch}(X)$ in the Chow group with coefficient in $\mathbb{F}_2$. It suffices to show that $(\deg/2)(c_d(-T_X)) = 1$.

Let $\mathrm{Sq}_\bullet^X : \mathrm{Ch}(X) \to \mathrm{Ch}(X)$ be the modulo 2 homological Steenrod operation, [5, §59]. We have a commutative diagram

$$\begin{tikzcd}
 & \mathrm{Ch}_d(X \times X) \arrow[dl, "(pr_1)_*"'] \arrow[dd, "\mathrm{Sq}_d^{X \times X}"] & \\
\mathrm{Ch}_d(X) \arrow[dd, "\mathrm{Sq}_d^X"'] & & \\
 & \mathrm{Ch}_0(X \times X) \arrow[dl, "(pr_1)_*"'] \arrow[dr, "(pr_2)_*"] \arrow[dd, "\deg/2"] & \\
\mathrm{Ch}_0(X) \arrow[dr, "\deg/2"'] & & \mathrm{Ch}_0(X) \arrow[dl, "\deg/2"] \\
 & \mathbb{F}_2 &
\end{tikzcd}$$

Since $(pr_1)_*(\rho) = [X]$ and $\mathrm{Sq}_d^X([X]) = c_d(-T_X)$ [5, formula (60.1)], it suffices to show that

$$(\deg/2)\big(\mathrm{Sq}_d^{X\times X}(\rho)\big) = 1.$$

We have $\mathrm{Sq}_\bullet^{X\times X} = c_\bullet(-T_{X\times X}) \cdot \mathrm{Sq}^\bullet_{X\times X}$, where $\mathrm{Sq}^\bullet$ is the cohomological Steenrod operation, [5, §61]. Therefore

$$\mathrm{Sq}_d^{X\times X}(\rho) = \sum_{i=0}^{d} c_{d-i}(-T_{X\times X}) \cdot \mathrm{Sq}^i_{X\times X}(\rho).$$

The summand with $i = d$ is $\mathrm{Sq}^d_{X\times X}(\rho) = \rho^2$ by [5, Theorem 61.13]. By Lemma 5.10, its image under $\deg/2$ is 1.

Since $c_\bullet(-T_{X\times X}) = c_\bullet(-T_X) \times c_\bullet(-T_X)$ and $\mathrm{Sq}^0 = \mathrm{id}$, the summand with $i = 0$ is

$$\left(\sum_{j=0}^{d} c_j(-T_X) \times c_{d-j}(-T_X)\right) \cdot \rho.$$

Its image under $\deg/2$ is 0 because

$$\begin{aligned}(\deg/2)\Big(\big(c_0(-T_X) \times c_d(-T_X)\big) \cdot \rho\Big) &= (\deg/2)(c_d(-T_X)) \\ &= (\deg/2)\Big(\big(c_d(-T_X) \times c_0(-T_X)\big) \cdot \rho\Big)\end{aligned}$$

while for $j \notin \{0, d\}$, we have $(\deg/2)\Big(\big(c_j(-T_X) \times c_{d-j}(-T_X)\big) \cdot \rho\Big) = 0$ by Corollary 5.9.

Finally, for any i with $0 < i < d$ the ith summand is the sum

$$\sum_{j=0}^{d-i} \big(c_j(-T_X) \times c_{d-i-j}(-T_X)\big) \cdot \mathrm{Sq}^i_{X\times X}(\rho).$$

We shall show that for any j the image of the jth summand under $\deg/2$ is 0. Note that the image under $\deg/2$ coincides with the image under the composition $(\deg/2) \circ (pr_1)_*$ and also under the composition $(\deg/2) \circ (pr_2)_*$ (look at the above commutative diagram). By the projection formula we have

$$\begin{aligned}(pr_1)_*\Big(\big(c_j(-T_X) \times c_{d-i-j}(-T_X)\big) \cdot \mathrm{Sq}^i_{X\times X}(\rho)\Big) = \\ c_j(-T_X) \cdot (pr_1)_*\Big(\big([X] \times c_{d-i-j}(-T_X)\big) \cdot \mathrm{Sq}^i_{X\times X}(\rho)\Big)\end{aligned}$$

and the image under $\deg/2$ is 0 for positive j by Corollary 5.8 applied to $\alpha = c_j(-T_X)$ and $\beta = (pr_1)_*\Big(\big([X] \times c_{d-i-j}(-T_X)\big) \cdot \mathrm{Sq}^i_{X\times X}(\rho)\Big)$. Corollary 5.8 can be indeed applied, because since $\rho_{F(X)} = \chi_1 \times [X_{F(X)}] + [X_{F(X)}] \times \chi_2$ and $i > 0$, we have $\mathrm{Sq}^i_{(X\times X)_{F(X)}}(\rho)_{F(X)} = 0$ and therefore $\beta_{F(X)} = 0$.

For $j = 0$ we use the projection formula for pr_2 and Corollary 5.8 with $\alpha = c_{d-i}(-T_X)$ and $\beta = (pr_2)_*\big(\mathrm{Sq}^i_{X\times X}(\rho)\big)$. □

Remark 5.12. The reason of the characteristic exclusion in Theorem 5.11 is that its proof makes use of Steenrod operations on Chow groups with coefficients in $\mathbb{F}_2$ which are not available in characteristic 2.

We would like to mention

Lemma 5.13 ([16, Lemma 9.10]). *Let X be an anisotropic smooth complete equidimensional variety over a field of arbitrary characteristic. If $\dim X + 1$ is not a power of 2, then the degree of the integral 0-cycle class $c_{\dim X}(-T_X) \in \mathrm{CH}_0(X)$ is divisible by 4.*

6. Motivic decompositions of some isotropic varieties

The coefficient ring Λ is arbitrary in the first half of this section and $\Lambda = \mathbb{F}_2$ in the second half (starting from Lemma 6.3). Throughout this section, D is a biquaternion division algebra, X_1 is the Severi-Brauer variety $X(1; D)$ of D, and X_2 is the generalized Severi-Brauer variety $X(2; D)$.

We say that motives M and N are *quasi-isomorphic* and write $M \approx N$, if there exist decompositions $M \simeq M_1 \oplus \cdots \oplus M_m$ and $N \simeq N_1 \oplus \cdots \oplus N_n$ such that

$$M_1(i_1) \oplus \cdots \oplus M_m(i_m) \simeq N_1(j_1) \oplus \cdots \oplus N_n(j_n)$$

for some (twist) integers $i_1, \ldots, i_m$ and $j_1, \ldots, j_n$.

Lemma 6.1. *Tensor product of any finite number of copies of $M(X_1)$ by any finite number of copies of $M(X_2)$ is quasi-isomorphic to a sum of finite number of copies of $M(X_1)$ and $M(X_2)$. (In other words, the Tate subcategory of $\mathrm{CM}(F, \Lambda)$ generated by $M(X_1)$ and $M(X_2)$ is a tensor subcategory, cf.* [4, Theorem 24].*)*

Proof. The product $X_1 \times X_1$ is a projective bundle over X_1. Therefore $M(X_1) \otimes M(X_1)$ is quasi-isomorphic to a sum of several copies of $M(X_1)$, [5, Theorem 63.10].

The product $X_2 \times X_1$ is a grassmann bundle over X_1. Therefore $M(X_2) \otimes M(X_1)$ is also quasi-isomorphic to a sum of several copies of $M(X_1)$.

Finally, by [4], the motive of the product $X_2 \times X_2$ quasi-decomposes into two copies of $M(X_2)$ and one copy of the motive of the flag variety $X(1 \subset 2 \subset 3; D)$. The projections

$$X(1 \subset 2 \subset 3; D) \to X(1 \subset 2; D) \to X(1; D) = X_1$$

are grassmann bundles. Consequently, $M(X(1 \subset 2 \subset 3; D))$ is quasi-isomorphic to a sum of several copies of $M(X_1)$. □

For the rest of this section, we fix an orthogonal involution on the biquaternion algebra D.

Lemma 6.2. *Let n be a positive integer. Let h be a hyperbolic hermitian form on the right D-module D^{2n} and let Y be the variety $X(4n; (D^{2n}, h))$ (of the* maximal *totally isotropic submodules). Then the motive $M(Y)$ is isomorphic to a finite sum of several twisted copies of the motives $M(X_1)$, $M(X_2)$, and Λ including one non-twisted copy of Λ.*

Proof. By [9, Следствие 15.9] (cf. [3]), the motive of the variety Y is quasi-isomorphic to the motive of the "total" variety

$$X(*; D^n) = \coprod_{i \in \mathbb{Z}} X(i; D^n) = \coprod_{i=0}^{4n} X(i; D^n)$$

of D-submodules in D^n. Furthermore, $M(X(*; D^n)) \approx M(X(*; D))^{\otimes n}$ by [9, Следствие 10.10] (cf. [3]). We finish by Lemma 6.1. The non-twisted copy of Λ is obtained as $\Lambda = M(X(0; D^n)) = M(X(0; D))^{\otimes n}$. □

As before, we write $\mathrm{Ch}(-)$ for the Chow group $\mathrm{CH}(-;\mathbb{F}_2)$ with coefficients in $\mathbb{F}_2$. We recall that a smooth complete variety is called *anisotropic*, if the degree of its any closed point is even (the empty variety is anisotropic).

Lemma 6.3. *Let Z be an anisotropic F-variety with a projector $p \in \mathrm{Ch}_{\dim Z}(Z \times Z)$ such that the motive $(Z,p)_L \in \mathrm{CM}(L,\mathbb{F}_2)$ for a field extension L/F is isomorphic to a finite sum of Tate motives. Then the number of the Tate summands is even. In particular, the motive in $\mathrm{CM}(F,\mathbb{F}_2)$ of any anisotropic F-variety does not contain a Tate summand.*

Proof. Mutually inverse isomorphisms between $(Z,p)_L$ and a sum of, say, n Tate summands, are given by two sequences of homogeneous elements $a_1,\dots,a_n$ and $b_1,\dots,b_n$ in $\mathrm{Ch}(Z_L)$ with $p_L = a_1 \times b_1 + \dots + a_n \times b_n$ and such that for any $i,j = 1,\dots,n$ the degree $\deg(a_i b_j)$ is 0 for $i \neq j$ and $1 \in \mathbb{F}_2$ for $i = j$. The pull-back of p via the diagonal morphism of Z is therefore a 0-cycle class on Z of degree n (modulo 2). □

Lemma 6.4. *Let n be a positive integer. Let h' be a hermitian form on the right D-module D^n such that h'_L is anisotropic for any finite odd degree field extension L/F. Let h be the hermitian form on the right D-module D^{n+2} which is the orthogonal sum of h' and a hyperbolic D-plane. Let Y' be the variety of totally isotropic submodules of D^{n+2} of reduced dimension 4 (= $\mathrm{ind}\, D$). Then the complete motivic decomposition of $M(Y') \in \mathrm{CM}(F,\mathbb{F}_2)$ (cf. Corollary 3.6) contains one summand $\mathbb{F}_2$, one summand $\mathbb{F}_2(\dim Y')$, and does not contain any other Tate motive.*

Proof. According to [9, Следствие 15.9], $M(Y')$ is quasi-isomorphic to the sum of the motives of the products

$$X(i \subset j; D) \times X(j-i; (D^n, h'))$$

where i,j run over all integers (the product is non-empty iff $0 \le i \le j \le 4$). The choices $i = j = 0$ and $i = j = 4$ give the summands $\mathbb{F}_2$ and $\mathbb{F}_2(\dim Y')$. The variety obtained by any other choice of i,j is anisotropic (the variety with $i = 0, j = 4$ is anisotropic by the assumption involving the odd degree field extensions), and we are done by Lemma 6.3. □

Lemma 6.5. *The motives $M(X_1)$ and $M(X_2)$ in the category $\mathrm{CM}(F,\mathbb{F}_2)$ are indecomposable.*

Proof. Since the varieties X_1 and X_2 satisfy the nilpotence principle, it suffices to show that the reduced Chow groups $\overline{\mathrm{Ch}}_{\dim X_1}(X_1 \times X_1)$ and $\overline{\mathrm{Ch}}_{\dim X_2}(X_2 \times X_2)$ do not contain non-trivial projectors. The statement on X_1 is a consequence of [8, Предложение 2.1.1].

The variety X_2 is isomorphic to the Albert quadric of D, [5, §16.A]. By [19, §7.3], the *splitting pattern* ([5, definition of §25]) of the Albert form of a biquaternion division algebra is $(0,1,3)$. It follows by [5, Theorem 73.26] that in the case of decomposable $M(X_2)$, one of the summands is given by a Rost projector on X_2. As already mentioned, there can't be a Rost projector on X_2 because X_2 is anisotropic and $\dim X_2 + 1 = 5$ is not a power of 2. Here one may refer to [5, Corollary 80.11], or to a more general Theorem 5.11, or to a simpler argument for this specific X_2 which uses [11, Lemma 6.5] and computation of $K_0(X_2)$. □

7. Proof of Main Theorem

We fix a central simple algebra A of index 4 with a non-hyperbolic orthogonal involution σ. We assume that σ becomes hyperbolic over the function field of the Severi-Brauer variety of A and we are looking for a contradiction.

According to [7, Theorem 3.3], $\operatorname{coind} A = 2n$ for some integer $n \geq 1$. We assume that Main Theorem (Theorem 1.2) is already proven for all algebras (over all fields) of index 4 and of coindex $< 2n$.

Let D be a biquaternion division algebra Brauer-equivalent to A. Let X_1 be the Severi-Brauer variety of D. Let us fix an orthogonal involution on D and an isomorphism of F-algebras $A \simeq \operatorname{End}_D(D^{2n})$. Let h be a hermitian form on the right D-module D^{2n} such that σ is adjoint to h. Then $h_{F(X_1)}$ is hyperbolic. Since the anisotropic kernel of h also becomes hyperbolic over $F(X_1)$, our induction hypothesis ensures that h is anisotropic. Moreover, h_L is hyperbolic for any field extension L/F such that h_L is isotropic. It follows by [2, Proposition 1.2] that h_L is anisotropic for any finite odd degree field extension L/F.

Let Y be the variety of totally isotropic submodules in D^{2n} of reduced dimension $4n$. (The variety Y is a twisted form of the variety of maximal totally isotropic subspaces of a quadratic form studied in [5, Chapter XVI].) It is isomorphic to the variety of totally isotropic right ideals in A of reduced dimension $4n$. Since σ is hyperbolic over $F(X_1)$ and the field F is algebraically closed in $F(X_1)$ (because the variety X_1 is geometrically integral), the discriminant of σ is trivial. Therefore the variety Y has two connected components Y_+ and Y_- corresponding to the components C_+ and C_- (cf. [5, Theorem 8.10]) of the Clifford algebra $C(A, \sigma)$.

Since $\sigma_{F(X_1)}$ is hyperbolic, $Y(F(X_1)) \neq \emptyset$. Since the varieties Y_+ and Y_- become isomorphic over $F(X_1)$, each of them has an $F(X_1)$-point.

The central simple algebras C_+ and C_- are related with A by the formula [12, (9.14)]:

$$[C_+] + [C_-] = [A] \in \operatorname{Br}(F).$$

Since $[C_+]_{F(X_1)} = [C_-]_{F(X_1)} = 0 \in \operatorname{Br}(F(X_1))$, we have $[C_+], [C_-] \in \{0, [A]\}$ and it follows that $[C_+] = [A]$, $[C_-] = 0$ up to exchange of the indices $+, -$.

By the index reduction formula for the varieties Y_+ and Y_- of [14, page 594], we have (up to exchange of the indices $+, -$): $\operatorname{ind} D_{F(Y_+)} = 4$, $\operatorname{ind} D_{F(Y_-)} = 1$. We replace Y by the component of Y whose function field does not reduce the index of D.

The coefficient ring Λ is $\mathbb{F}_2$ in this section. As before, we set $X_2 = X(2, D)$.

Lemma 7.1. *The motive of Y decomposes as $R_1 \oplus R_2$, where R_1 is quasi-isomorphic to a finite sum of several copies of the motives $M(X_1)$ and $M(X_2)$, and where $(R_2)_{F(Y)}$ is isomorphic to a finite sum of Tate motives including one exemplar of $\mathbb{F}_2$.*

Proof. According to Lemma 6.2, the motive $M(Y)_{F(Y)}$ is isomorphic to a sum of several twisted copies of $M(X_1)$, $M(X_2)$, and $\mathbb{F}_2$ including a copy of $\mathbb{F}_2$. If for some $i = 1, 2$ there is at least one copy of $M(X_i)$ (with a twist $j \in \mathbb{Z}$) in the decomposition, let us apply Proposition 4.5 taking as X the variety X_i, taking as M the motive $M(Y)(-j)$, and taking as E the function field $F(Y)$.

The condition (3) of Proposition 4.5 is fulfilled.

Since D_E is still a division algebra, the condition (1) is fulfilled by Lemma 6.5. Since $\operatorname{ind} D_{F(X)} < 4$, the hermitian form $h_{F(X)}$ is hyperbolic by [15]; therefore the variety $Y_{F(X)}$ is rational and the condition (2) is fulfilled.

It follows that $M(X_i)$ is a summand of $M(Y)(-j)$. Let now M be the complement summand of $M(Y)(-j)$. By Corollary 3.6, the complete decomposition of $M_{F(Y)}$ is the complete decomposition of $M(Y)(-j)_{F(Y)}$ with one copy of $M(X_i)$ erased. If $M_{F(Y)}$ contains one more copy of a twist of $M(X_i)$ (for some $i = 1, 2$), we once again apply Proposition 4.5 to the variety X_i and an appropriate twist of M. Doing this until we can, we get the desired decomposition in the end. □

Now let us consider a minimal right D-submodule $V \subset D^{2n}$ such that V becomes isotropic over a finite odd degree field extension of $F(Y)$. We set $v = \dim_D V$. Clearly, $2 \leq v \leq n+1$. [2] For $n > 1$, let Y' be the variety of totally isotropic submodules in V of reduced dimension 4 (that is, of "D-dimension" 1). For $n = 1$ we set $Y' = Y$.

The variety Y' has dimension $\dim Y' = 16v - 26$ which is even (and positive). Moreover, the variety Y' is anisotropic (because the hermitian form h is anisotropic and remains anisotropic over any finite odd degree field extension of the base field). Surprisingly, we can however prove the following

Lemma 7.2. *There is a Rost projector (Definition 5.1) on Y'.*

Proof. By the construction of Y', there exists a correspondence of odd multiplicity (that is, of multiplicity $1 \in \mathbb{F}_2$) $\alpha \in \mathrm{Ch}_{\dim Y}(Y \times Y')$. On the other hand, since $h_{F(Y')}$ is isotropic, $h_{F(Y')}$ is hyperbolic and therefore there exist a rational map $Y' \dashrightarrow Y$ and a multiplicity 1 correspondence $\beta \in \mathrm{Ch}_{\dim Y'}(Y' \times Y)$. Since the summand R_2 of $M(Y)$ given by Lemma 7.1 is outer (cf. Definition 4.1 and Lemma 4.3), by Lemma 4.4 there is an outer summand of $M(Y')$ isomorphic to a summand of R_2.

Let $\rho \in \mathrm{Ch}_{\dim Y'}(Y' \times Y')$ be the projector giving this summand. We claim that ρ is a Rost projector. We prove the claim by showing that the motive $(Y', \rho)_{\tilde{F}}$ is isomorphic to $\mathbb{F}_2 \oplus \mathbb{F}_2(\dim Y')$, cf. Corollary 5.6, where $\tilde{F}/F(Y)$ is a finite odd degree field extension such that V becomes isotropic over $\tilde{F}$.

Since $(R_2)_{F(Y)}$ is a finite sum of Tate motives, the motive $(Y', \rho)_{\tilde{F}}$ is also a finite sum of Tate motives. Since $(Y', \rho)_{\tilde{F}}$ is outer, the Tate motive $\mathbb{F}_2$ is included (Lemma 4.3). Now, by the minimal choice of V, the hermitian form $(h|_V)_{\tilde{F}}$ satisfies the condition on h in Lemma 6.4: $(h|_V)_{\tilde{F}}$ is an orthogonal sum of a hyperbolic $D_{\tilde{F}}$-plane and a hermitian form h' such that h'_L is anisotropic for any finite odd degree field extension $L/\tilde{F}$ of the base field $\tilde{F}$ (otherwise any D-hyperplane $V' \subset V$ would become isotropic over some odd degree extension of $F(Y)$). Therefore the complete motivic decomposition of $Y'_{\tilde{F}}$ (see Corollary 3.6) has one copy of $\mathbb{F}_2$, one copy of $\mathbb{F}_2(\dim Y')$, and no other Tate summands. By Corollary 3.6 and anisotropy of the variety Y' (see Lemma 6.3), it follows that

$$(Y', \rho)_{\tilde{F}} \simeq \mathbb{F}_2 \oplus \mathbb{F}_2(\dim Y'). \qquad \square$$

Lemma 7.2 contradicts to the general results of §5 (namely, to Theorem 5.11 and Lemma 5.13) thus proving Main Theorem (Theorem 1.2). We can avoid the use of Lemma 5.13 by showing that $\deg c_{\dim Y'}(-T_{Y'})$ is divisible by 16 for our variety Y'. Indeed, let K be

[2]One probably always have $v = n + 1$ here, but we do not need to know the precise value of v.

the field $F(t_1, \dots, t_{4v})$ of rational functions over F in $4v$ variables. Let us consider the (generic) diagonal quadratic form $\langle t_1, \dots, t_{4v}\rangle$ on the K-vector space K^{4v}. Let Y'' be the variety of 4-dimensional totally isotropic subspaces in K^{4v}. The degree of any closed point on Y'' is divisible by 2^4. In particular, the integer $\deg c_{\dim Y''}(-T_{Y''})$ is divisible by 2^4. Since over an algebraic closure $\bar{K}$ of K the varieties Y' and Y'' become isomorphic, we have

$$\deg c_{\dim Y'}(-T_{Y'}) = \deg c_{\dim Y''}(-T_{Y''}).$$

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UPMC Univ Paris 06, Institut de Mathématiques de Jussieu, F-75252 Paris, FRANCE

Web page: www.math.jussieu.fr/~karpenko
E-mail address: karpenko *at* math.jussieu.fr