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# Torsion in $\mathrm{CH}^2$ of Severi-Brauer varieties and indecomposability of generic algebras

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**Summary.** We compute degrees of algebraic cycles on certain Severi-Brauer varieties and apply it to show that:
- a generic division algebra of index $p^\alpha$ and exponent $p$ is not decomposable (in a tensor product of two algebras) for any prime $p$ and any $\alpha$ except the case when $p = 2$ and $2 \mid \alpha$;
- the 2-codimensional Chow group $\mathrm{CH}^2$ of the Severi-Brauer variety corresponding to the generic division algebra of index 8 and exponent 2 has a non-trivial torsion.

**Key words:** central simple algebras – Severi-Brauer varieties – algebraic cycles – Chow groups – Grothendieck group

*Subject Classifications*: Primary: 14C25, 16K20; Secondary: 16D15, 19E08.

## 0. Introduction

We are working with finite dimensional over a field central simple algebras and use the standard terminology related [12].

Let $n$ and $m$ be some positive integers with $m \mid n$. We construct a generic [1] algebra $D(n,m)$ of degree $n$ and exponent $m$ in the way proposed in [16]: take a division algebra $A$ of degree and exponent $n$ defined over a field $F$, consider the Severi-Brauer variety $Y$ [3] corresponding to $A^{\otimes m}$ and put $D(n,m) = A_{F(Y)}$ where $F(Y)$ is the function field of $Y$.

It follows from the construction immediately that $\exp D(n,m) \mid m$. It is the index reduction formula [16] which shows that in fact $\exp D(n,m) = m$.

[1] The meaning of the word "generic" in this context is not essential for the subsequent.

To discuss the indecomposability problem fix $n = p^\alpha$ and $m = p^\beta$ with a prime $p$. The question under consideration is whether any algebra of degree $p^\alpha$ and exponent $p^\beta$ can be decomposed in a tensor product of two non-trivial algebras ?

The index reduction formula shows that

$$\operatorname{ind} D(p^\alpha, p^\beta) = p^\alpha$$

and if $\beta > 1$ then moreover

$$\operatorname{ind} D(p^\alpha, p^\beta)^{\otimes p} = p^{\alpha-1} .$$

Whence in the case when $\beta > 1$ the algebra $D(p^\alpha, p^\beta)$ is indecomposable by a trivial reason and so, it remains the case of algebras of prime exponent only.

A classical result in this area is the theorem of Albert [1] which states that any algebra of degree 4 and exponent 2 decomposes. After that Amitsur, Rowen and Tignol constructed an indecomposable algebra (not the generic one) of exponent 2 having degree 8 [2]. A bit later Rowen did the same in characteristic 2 (the case of the characteristic 2 was excluded previously) [14]. Certain indecomposable algebras of degree $p^2$, exponent $p$ was found by Tignol [17]. Most recently Jacob gave indecomposable examples of degree $p^\alpha$ and exponent $p$ for any prime $p$ and any $\alpha$ except $p = \alpha = 2$ [5]. However the examples of Tignol and those of Jacob appear only in characteristic 0.

So, the prime characteristic case remained open.[2]

In this note we will show (see 3.1) that any algebra $D(p^\alpha, p)$ except the case $p = 2 \mid \alpha$ is indecomposable (the characteristic of the base field completely doesn't matter in the proofs). The way of doing it is to compute degrees of algebraic cycles on the Severi-Brauer variety [3] of $D(p^\alpha, p)$ (see 1.3) and show that any decomposition of the algebra would give a cycle of an impossible degree.

This computation of degrees of cycles can be also used to get a non-trivial torsion in the Chow group $\mathrm{CH}^*$ [4] of the variety. [4] More precisely, this torsion appears in the quotients of the topological filtration on the Grothendieck group $K_0'$ [13] and it is an essential difference from the torsion in Chow groups of Severi-Brauer varieties which was found by Merkurjev [10] because the Merkurjev's torsion vanished by passing to $K_0'$. In particular, it never appeared in codimension 2, so the question whether $\mathrm{CH}^2$ of a Severi-Brauer variety is always torsion-free was open.

We answer this question negatively by showing that $\mathrm{CH}^2$ in the case of $D(8,2)$ has a non-trivial torsion (see 4.5).

Here is the structure of the note.

In the section 1, we compute degrees of cycles for the algebra $D(n,m)$ with any $n$ and $m$.

In the section 2, we prove that the algebra $D(p^2, p)$ where $p \neq 2$ is indecomposable (see 2.1). This case is considered separately since almost no calculations are required here while the general case of $D(p^\alpha, p)$ done in the section 3 contains several (elementary) calculations with integers which are more difficult to check through.

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[2] In positive characteristic different from $p$ there was unpublished examples due to Saltman.

[3] What we mean by "degree of a cycle on a Severi-Brauer variety" is explained in section 1.

[4] In fact from the beginning, hunting torsion was the only aim of the author. The idea to apply this business to the indecomposability problem was proposed by P. Mammone.

In the section 4, we catch torsion in $\mathrm{CH}^2$. A person interested only in this question can omit the sections 2 and 3.

We denote the Severi-Brauer variety of an algebra $A$ by $\mathrm{SB}(A)$. Sometimes we write "index (exponent) of $\mathrm{SB}(A)$" instead of "index (exponent) of $A$". Recall that $\deg A = \dim \mathrm{SB}(A) + 1$.

## 1. Degrees of cycles

Let $X$ be a Severi-Brauer variety over a field $F$ and $E/F$ any extension which splits $X$ (i.e. $X_E$ is isomorphic to a projective space over $E$). For any $i$ with $0 \le i \le \dim X$, consider the homomorphism

$$\deg : \mathrm{CH}^i X \longrightarrow \mathbb{Z}$$

which value $\deg Z$ on a simple cycle $Z \subset X$ is by definition the degree of $Z_E$ considered as a subvariety of the projective space $X_E$ [4]. In other words, deg is the composition of the restriction map $\mathrm{CH}^i X \to \mathrm{CH}^i X_E$ and of the canonical isomorphism $\mathrm{CH}^i X_E \simeq \mathbb{Z}$.

First of all we formulate

**Proposition 1.1 ([6]).** *Let $n$ be the index of a Severi-Brauer variety $X$. Then*

$$\frac{n}{(i,n)} \in \deg \mathrm{CH}^i X$$

*for any $i = 0, 1, \ldots, \dim X$ (where $(i,n)$ stays for the greatest common divisor of $i$ and $n$).*

*If moreover the exponent of $X$ coincides with $n$ then each of the groups $\deg \mathrm{CH}^i X$ is generated by the element given.*

We note that the proof of the first statement uses transfers and is more or less trivial while in the proof of the second one the computation of K-theory of $X$ [13] is needed.

**Lemma 1.2.** *If $X$ is a Severi-Brauer variety of exponent $m$ then the group $\deg \mathrm{CH}^1 X$ is generated by $m$.*

*Proof.* We can refer for example to a classical computation [3] which states even more: $\mathrm{CH}^1 X = m\mathbb{Z}$.

**Proposition 1.3.** *Let $n$ and $m$ be some positive integers with $m \mid n$ and $D(n,m)$ the generic algebra defined in the introduction. The ring*

$$\deg \mathrm{CH}^* = \bigoplus_i \deg \mathrm{CH}^i$$

*of the corresponding Severi-Brauer variety is generated by*

$$m \in \deg \mathrm{CH}^1$$

*and by the elements*

$$\frac{n}{(i,n)} \in \deg \mathrm{CH}^i \ .$$

*Proof.* Take the division algebra $A$ which is used in the construction of $D(n,m)$ and put
$$X = \mathrm{SB}(A),\ Y = \mathrm{SB}(A^{\otimes m})\ .$$
The Severi-Brauer variety of $D(n,m)$ coincides with $X_{F(Y)}$.

Consider the pull-back
$$f : \mathrm{CH}^*(X \times Y) \longrightarrow \mathrm{CH}^* X_{F(Y)}$$
with respect to the morphism of varieties $X_{F(Y)} \to X \times Y$ obtained from $\operatorname{Spec} F(Y) \to Y$ by the base change. It is easy to show that $f$ is an epimorphism (see e.g. [9]). Since $X \times Y$ is a projective space bundle over $X$ (via the first projection $pr_X : X \times Y \to X$) [11] the ring $\mathrm{CH}^*(X \times Y)$ is generated by $\mathrm{CH}^1(X \times Y)$ and $pr_X^* \mathrm{CH}^* X$ [4]. Thus the ring $\mathrm{CH}^* X_{F(Y)}$ is generated by $\mathrm{CH}^1 X_{F(Y)}$ and $\mathrm{res}_{F(Y)/F} \mathrm{CH}^* X$.

Now we pass to degrees. According to (1.2) the group $\deg \mathrm{CH}^1 X_{F(Y)}$ is generated by $m = \exp D(n,m)$. According to (1.1) each of the groups $\deg \mathrm{CH}^i X$ is generated by $\frac{n}{(i,n)}$ and since a restriction preserves degrees of cycles we get the same generators in $\deg \mathrm{CH}^* X_{F(Y)}$.

## 2. Indecomposability of $D(p^2, p)$

First we describe a way how a decomposition of an algebra produces a cycle and compute its degree.

Let $D$ be an (finite-dimensional central simple) algebra over a field $F$ and suppose that $D$ admits a decomposition $D = D_1 \otimes_F D_2$. Tensor product of ideals gives rise to a closed imbedding of varieties [3]
$$\mathrm{SB}(D_1) \times \mathrm{SB}(D_2) \hookrightarrow \mathrm{SB}(D)\ .$$
This imbedding is a twisted form of a Segre imbedding: if we pass to a separable closure $F_s$ of $F$ we will get the Segre imbedding
$$\mathbb{P}^{n_1-1}_{F_s} \times \mathbb{P}^{n_2-1}_{F_s} \hookrightarrow \mathbb{P}^{n_1 n_2 - 1}_{F_s}$$
where $n_j = \deg D_j$. Hence degree of the cycle $\mathrm{SB}(D_1) \times \mathrm{SB}(D_2)$ coincides with that of $\mathbb{P}^{n_1-1} \times \mathbb{P}^{n_2-1}$ which is equal to the binomial coefficient [4]:
$$\binom{n_1 + n_2 - 2}{n_1 - 1}\ .$$

**Theorem 2.1.** *For any odd prime $p$ the generic algebra $D(p^2, p)$ of degree $p^2$ and exponent $p$ is indecomposable.*

*Proof.* Applying (1.3) to our particular situation we see that for each $i$ the group $\deg \mathrm{CH}^i$ of the Severi-Brauer variety corresponding to the algebra $D = D(p^2, p)$ is generated by
$$\begin{cases} 1 & \text{if } i = 0; \\ p & \text{if } i = 1 \text{ or } p \mid i; \\ p^2 & \text{otherwise.} \end{cases}$$

Suppose that $D = D_1 \otimes D_2$ with $\deg D_j = p$. Then according to the computations above the theorem we have a cycle on $\mathrm{SB}(D)$ which has dimension $2p-2$ (i.e. codimension $p^2 - 2p + 1$) and degree

$$\binom{2p-2}{p-1} .$$

Since the binomial coefficient is not divisible by $p^2$ it shows that

$$p \in \deg \mathrm{CH}^{p^2-2p+1} \mathrm{SB}(D) .$$

Since $p$ does not divide $p^2 - 2p + 1$ the inclusion can take place only if

$$p^2 - 2p + 1 \leq 1 ,$$

i.e. if $p = 2$.

*Remark 2.2.* If $A$ is a central simple algebra over a field $F$, $X = \mathrm{SB}(A)$ and $L/F$ a finite extension of degree prime to $\deg A$ then the transfer argument shows that

$$\deg \mathrm{CH}^* X = \deg \mathrm{CH}^* X_L .$$

Consequently, the generic algebra $D(p^2, p)$ remains indecomposable over any extension of its center having degree prime to $p$ (compare with [15]).

## 3. Indecomposability of $D(p^\alpha, p)$

In this section we show that the algebra $D(p^\alpha, p)$ is not decomposable by using exactly the same method as we did in the previous section for $\alpha = 2$. The only difference will be in a little bit more complicated calculations with integers which are met here.

**Theorem 3.1.** *For any prime number $p$ and any $\alpha$, the generic algebra $D(p^\alpha, p)$ of degree $p^\alpha$ and exponent $p$ is indecomposable except the case when $p = 2$ and $2 \mid \alpha$.*

*Proof.* Put $D = D(p^\alpha, p)$ and consider the Chow group $\mathrm{CH}^*$ of the variety $\mathrm{SB}(D)$. According to (1.3), the ring $\deg \mathrm{CH}^*$ is generated by

$$p \in \deg \mathrm{CH}^1 \quad \text{and} \quad p^{\alpha - v_p(i)} \in \deg \mathrm{CH}^i$$

where $i \geq 1$ and $v_p(i)$ is the multiplicity of $p$ in $i$. Hence each of the groups $\deg \mathrm{CH}^k$ is generated by a minimal number among

$$\left\{ p^{i+\alpha-v_p(k-i)} \right\}_{i=0}^{k-1} \text{ and } p^k .$$

Now suppose that $D = D_1 \otimes D_2$ where $\deg D_j = p^{\beta_j}$ and $\beta_1 \geq \beta_2 \geq 1$. Using this decomposition we obtain in the way described in the section 2 a cycle on $\mathrm{SB}(D)$ having dimension $p^{\beta_1} + p^{\beta_2} - 2$ (i.e. codimension $p^\alpha - p^{\beta_1} - p^{\beta_2} + 1$) and degree

$$\binom{p^{\beta_1} + p^{\beta_2} - 2}{p^{\beta_2} - 1} .$$

It is easy to calculate that

$$v_p \begin{pmatrix} p^{\beta_1} + p^{\beta_2} - 2 \\ p^{\beta_2} - 1 \end{pmatrix} = \beta_1$$

whence

$$p^{\beta_1} \in \deg \mathrm{CH}^{p^\alpha - p^{\beta_1} - p^{\beta_2} + 1}$$

what according to the previous paragraph implies that

$$\beta_1 \geq p^\alpha - p^{\beta_1} - p^{\beta_2} + 1 \tag{i}$$

or

$$\beta_1 \geq i + \alpha - v_p(p^\alpha - p^{\beta_1} - p^{\beta_2} + 1 - i) \tag{ii}$$

for some $i$ between 0 and $p^\alpha - p^{\beta_1} - p^{\beta_2}$.

It is easy to see that the inequality (i) takes place only if $p = 2$ and $\beta_1 = \beta_2 = 1$ (and hence $\alpha = \beta_1 + \beta_2 = 2$) what is one of the exceptional cases in the theorem. Consequently for some $i$ the inequality (ii) holds, i.e.

$$v_p\left(p^\alpha - p^{\beta_1} - p^{\beta_2} - (i-1)\right) \geq \beta_2 + i\ .$$

We see that $(i-1)$ is divisible by $p^{\beta_2}$, say $i = lp^{\beta_2} + 1$. Making this substitution for $i$ and dividing by $p^{\beta_2}$ we obtain:

$$v_p\left(p^{\beta_1} - p^{\beta_1 - \beta_2} - (l+1)\right) \geq lp^{\beta_2} + 1\ . \tag{iii}$$

We split further consideration into two cases: $\beta_1 > \beta_2$ and $\beta_1 = \beta_2$.

Suppose that $\beta_1 > \beta_2$. Then the number $(l+1)$ is divisible by $p$. If so then $l \geq p - 1 \geq 1$ and consequently $(l+1)$ is divisible by $p^2$ (if $\beta_1 - \beta_2 \geq 2$).

If we continue in such the way we will get that $(l+1)$ is divisible by $p^{\beta_1 - \beta_2}$. Hence

$$v_p\left(p^{\beta_1} - p^{\beta_1 - \beta_2} - (l+1)\right) \geq$$
$$\geq (p^{\beta_1 - \beta_2} - 1)p^{\beta_2} + 1 = p^{\beta_1} - p^{\beta_2} + 1 > p^{\beta_1 - 1} \geq \beta_1\ .$$

It is a contrudiction since the first expression in this chain of inequalities is less than $\beta_1$ by a trivial reason.

Suppose that $\beta_1 = \beta_2$. Note that $\alpha$ is now even, so the case $p = 2$ is excluded. The inequality (iii) looks as follows:

$$v_p\left(p^\beta - (l+2)\right) \geq lp^\beta + 1$$

where $\beta = \beta_1$ $(= \beta_2)$. It follows that $(l+2)$ is divisible by $p$, so $l \geq p - 2 \geq 1$. Thus

$$v_p\left(p^\beta - (l+2)\right) \geq p^\beta + 1 > \beta\ .$$

However the first expression in this chain of inequalities is less than $\beta$ by a trivial reason. This contrudiction ends the proof.

*Remark 3.2.* The above proof shows that the generic algebra $D(p^\alpha, p)$ remains indecomposable over any extension of its center having degree prime to $p$ (compare with (2.2)).

*Remark 3.3.* In the case $p = 2$, it was in fact also shown that if an algebra $D(2^{2\beta}, 2)$ decomposes in a product of two smaller algebras then degree of each factor is $2^\beta$. One can use this fact to show indecomposability of $D(16, 2)$: if it would decompose then each of the factors decomposes once again by the Albert theorem; it contradicts to the previous sentence.

## 4. Torsion in $\mathbf{CH^2}$

In this section we consider the algebra $D = D(8,2)$ and the variety $X = \mathrm{SB}(D)$.

**Lemma 4.1.** *The group* $\deg \mathrm{CH}^3 X$ *is generated by 8.*

*Proof.* Apply (1.3).

Consider the Grothendieck group $K'_0(X) = K_0(X)$ which we will denote simply by $K(X)$. K-theory of Severi-Brauer varieties was computed by Quillen [13]. In particular, any restriction map $K(X) \to K(X_E)$ is known to be an inclusion. Take an extension $E/F$ which splits $X$. The ring $K(X_E)$ is generated by the class $h$ of a hyperplane with the only relation: $h^8 = 0$. The subgroup $K(X) \subset K(X_E)$ is generated by $(\operatorname{ind} D^{\otimes i})(1-h)^i$ with all $i \geq 0$. Since $\operatorname{ind} D^{\otimes i}$ equals 8 for an odd $i$ and 1 for an even $i$ the generators are $8(1-h)^i$ and $(1-h)^{2i}$.

**Lemma 4.2.** *The group $K(X)$ contains an element $x$ of the kind*

$$x = 4h^3 + \cdots$$

*where the dots stay for a sum (with integer coefficients) of the elements $h^i$ with $i > 3$.*

*Proof.* For instance,

$$x = 3 - 8(1-h) + 6(1-h)^2 - (1-h)^4 \; .$$

Consider the topological filtration

$$K(X) = K(X)^{(0)} \supset K(X)^{(1)} \supset \dots$$

on $K(X)$ [13]. One can give another proof of the last lemma which contains no computations:

*Proof.* Take a decomposable division algebra $D'$ of degree 8 and exponent 2 (defined over an arbitrary field, not nesessary over $F$ !). The decomposition of $D'$ gives a 3-codimensional cycle on $X' = \mathrm{SB}(D')$ of degree

$$\deg(\mathbb{P}^3 \times \mathbb{P}^1) = \begin{pmatrix} 4 \\ 1 \end{pmatrix} = 4 \; .$$

Class $x' \in K(X')$ of this cycle has the kind $4h^3 + \cdots$. Since the Grothendieck group of $X'$ "coincide" with that of $X$ we are done.

**Lemma 4.3.** *For the element $x$ from (4.2), it holds: $x \notin K(X)^{(3)}$.*

*Proof.* Suppose that $x \in K(X)^{(3)}$ and take any preimage of

$$\bar{x} \in K(X)^{(3/4)} = K(X)^{(3)}/K(X)^{(4)}$$

with respect to the canonical epimorphism $\mathrm{CH}^3 X \twoheadrightarrow K(X)^{(3/4)}$. The degree of the cycle obtained equals 4 what contradicts to (4.1).

**Lemma 4.4.** *In the same notations, one has: $x \in K(X)^{(2)}$ and $2x \in K(X)^{(3)}$.*

*Proof.* First of all, since the Severi-Brauer variety $X$ has a splitting field $E/F$ of degree 8 and $x_E \in K(X_E)^{(3)}$ the transfer arguments shows that

$$8x \in K(X)^{(3)} .$$

Since by the previous lemma $x \notin K(X)^{(3)}$ the element $\bar{x} \in K(X)^{(0/3)}$ is non-trivial and of finite order. Since the groups $K(X)^{(0/1)}$ and $K(X)^{(1/2)}$ are torsion-free $x \in K(X)^{(2)}$.

To see that $2x \in K(X)^{(3)}$ take a quadratic extension $L/F$ which partially splits $X$. The group $K(X_L)^{(2/3)}$ is torsion-free [7] whence $x_L \in K(X_L)^{(3)}$. Now take the transfer.

**Theorem 4.5.** *Let $X$ be the Severi-Brauer variety of the generic algebra $D(8,2)$. The 2-codimensional Chow group $\mathrm{CH}^2 X$ contains a non-trivial torsion.*

*Proof.* According to (4.3) and (4.4) the element

$$\bar{x} \in K(X)^{(2/3)}$$

has order 2. To get a non-trivial torsion in $\mathrm{CH}^2 X$ we simply take any preimage of $\bar{x}$ with respect to the surjection $\mathrm{CH}^2 X \twoheadrightarrow K(X)^{(2/3)}$ (which is even an isomorphism).

*Remark 4.6.* To complete the picture we mention here that if an algebra of degree 8 and exponent 2 decomposes (and hence is a triquaternion algebra) then $\mathrm{CH}^2$ of the corresponding Severi-Brauer variety is torsion-free [8].

*Remark 4.7.* For any generic algebra $D(p^\alpha, p)$ with exception $p = 2 = \alpha$ certain quotients of the topological filtration on the Grothendieck group of the corresponding Severi-Brauer variety have torsion (and hence the Chow group has torsion too). It can be shown by the same method as it was for $D(8,2)$ (the later algebra is of particular interest just because the torsion for this algebra appears in codimension 2).

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