MOTIVIC DECOMPOSITION OF COMPACTIFICATIONS OF CERTAIN GROUP VARIETIES

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Abstract. Let D be a central simple algebra of prime degree over a field and let E be an $\mathbf{SL}_1(D)$-torsor. We determine the complete motivic decomposition of certain compactifications of E. We also compute the Chow ring of E.

Contents

1. Introduction

Let p be a prime number. For any integer $n \geq 2$, a *Rost motive of degree* n is a direct summand $\mathcal{R}$ of the Chow motive with coefficients in $\mathbb{Z}_{(p)}$ (the localization of the integers at the prime ideal (p)) of a smooth complete geometrically irreducible variety X over a field F such that for any extension field K/F with a closed point on X_K of degree prime to p, the motive $\mathcal{R}_K$ is isomorphic to the direct sum of Tate motives

$$\mathbb{Z}_{(p)} \oplus \mathbb{Z}_{(p)}(b) \oplus \mathbb{Z}_{(p)}(2b) \oplus \cdots \oplus \mathbb{Z}_{(p)}((p-1)b),$$

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where $b = (p^{n-1} - 1)/(p - 1)$. The isomorphism class of $\mathcal{R}$ is determined by X, [19, Proposition 3.4]; $\mathcal{R}$ is indecomposable as long as X has no closed points of degree prime to p.

A smooth complete geometrically irreducible variety X over F is a *p-generic splitting variety* for an element $s \in H^n_{ét}(F, \mathbb{Z}/p\mathbb{Z}(n-1))$, if s vanishes over a field extension K/F if and only if X has a closed point of degree prime to p over K. A *norm variety* of s is a p-generic splitting variety of dimension $p^{n-1} - 1$.

A Rost motive living on a p-generic splitting variety of an element s is determined by s up to isomorphism and called the Rost motive of s. In characteristic 0, any *symbol* s admits a norm variety possessing a Rost motive. This played an important role in the proof of the Bloch-Kato conjecture (see [31]). It is interesting to understand the complement to the Rost motive in the motive of a norm variety X for a given s; this complement, however, depends on X and is not determined by s anymore.

For $p = 2$, there are nice norm varieties known as norm quadrics. Their complete motivic decomposition is a classical result due to M. Rost. A norm quadric X can be viewed as a compactification of the affine quadric U given by $\pi = c$, where π is a quadratic $(n-1)$-fold Pfister form and $c \in F^\times$. The summands of the complete motivic decomposition of X are given by the degree n Rost motive of X and shifts of the degree $n-1$ Rost motive of the projective Pfister quadric $\pi = 0$. It is proved in [16, Theorem A.4] that $\mathrm{CH}(U) = \mathbb{Z}$ if the equation $\pi = c$ has no solutions over F. In the present paper we extend these results to arbitrary prime p (and $n = 3$).

For arbitrary p, there are nice norm varieties in small degrees. For $n = 2$, these are the Severi-Brauer varieties of degree p central simple F-algebras. Any of them admits a degree 2 Rost motive which is simply the total motive of the variety.

The first interesting situation occurs in degree $n = 3$. Let D be a degree p central division F-algebra, $G = \mathbf{SL}_1(D)$ the special linear group of D, and E a principle homogeneous space under G. The affine variety E is given by the equation $\mathrm{Nrd} = c$, where Nrd is the reduced norm of D and $c \in F^\times$. Any smooth compactification of E is a norm variety of the element $s := [D] \cup (c) \in H^3_{ét}(F, \mathbb{Z}/p\mathbb{Z}(2))$. It has been shown by N. Semenov in [26] for $p = 3$ (and $\operatorname{char} F = 0$) that the motive of a certain smooth equivariant compactification of E decomposes in a direct sum, where one of the summands is the Rost motive of s, another summand is a motive ε vanishing over any field extension of F splitting D, and each of the remaining summands is a shift of the motive of the Severi-Brauer variety of D. All these summands (but ε) are indecomposable and ε was expected to be 0.

Another proof of this result (covering arbitrary characteristic) has been provided in [30] along with the claim that $\varepsilon = 0$, but the proof of the claim was incomplete.

In the present paper we prove the following main result (see Theorem 10.3):

Theorem 1.1. *Let F be a field, D a central division F-algebra of prime degree p, X a smooth compactification of an $\mathbf{SL}_1(D)$-torsor, and $M(X)$ its Chow motive with $\mathbb{Z}_{(p)}$-coefficients. Assume that $M(X)$ over the function field of the Severi-Brauer variety S of D is isomorphic to a direct sum of Tate motives. Then $M(X)$ (over F) is isomorphic to the direct sum of the Rost motive of X and several shifts of $M(S)$. This is the unique decomposition of $M(X)$ into a direct sum of indecomposable motives.*

We note that the compactification in [26] (for $p = 3$) has the property required in Theorem 1.1 (see Example 10.6).

In Section 6 we show that the condition that $M(X)$ is *split* (i.e., isomorphic to a finite direct sum of Tate motives) over $F(S)$ is satisfied for all smooth $G \times G$-equivariant compactifications of $G = \mathbf{SL}_1(D)$. Moreover, we prove that the motive $M(X)$ is split for all smooth equivariant compactifications X of split semisimple groups (see Theorem 6.8).

We also compute the Chow ring of G in arbitrary characteristic as well as the Chow ring of E in characteristic 0 (see Theorem 9.7 and Corollary 10.8):

Theorem 1.2. *Let D be a central division algebra of prime degree p and $G = \mathbf{SL}_1(D)$.*
1) *There is an element $h \in \mathrm{CH}^{p+1}(G)$ such that*

$$\mathrm{CH}(G) = \mathbb{Z} \cdot 1 \oplus (\mathbb{Z}/p\mathbb{Z})h \oplus (\mathbb{Z}/p\mathbb{Z})h^2 \oplus \cdots \oplus (\mathbb{Z}/p\mathbb{Z})h^{p-1}.$$

2) *Let E be a nonsplit G-torsor. If $\operatorname{char} F = 0$, then $\mathrm{CH}(E) = \mathbb{Z}$.*

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2. K-cohomology

Let X be a smooth variety over F. We write $A^i(X, K_n)$ for the K-cohomology groups as defined in [25]. In particular, $A^i(X, K_i)$ is the Chow group $\mathrm{CH}^i(X)$ of classes of codimension i algebraic cycles on X.

Let G be a simply connected semisimple algebraic group. The group $A^1(G, K_2)$ is additive in G, i.e., if G and G' are two simply connected group, then the projections of $G \times G'$ onto G and G' yield an isomorphism (see [13, Part II, Proposition 7.6 and Theorem 9.3])

$$A^1(G, K_2) \oplus A^1(G', K_2) \xrightarrow{\sim} A^1(G \times G', K_2).$$

The following lemma readily follows.

Lemma 2.1. *1) The map*

$$A^1(G, K_2) \to A^1(G \times G, K_2) = A^1(G, K_2) \oplus A^1(G, K_2)$$

induced by the product homomorphism $G \times G \to G$ is equal to $(1, 1)$.

2) The map $A^1(G, K_2) \to A^1(G, K_2)$ induced by the morphism $G \to G$, $x \mapsto x^{-1}$ is equal to -1.

Proof. 1) It suffices to note that the isomorphism

$$A^1(G \times G', K_2) \xrightarrow{\sim} A^1(G, K_2) \oplus A^1(G', K_2)$$

inverse to the one mentioned above, is given by the pull-backs with respect to the group embeddings $G, G' \hookrightarrow G \times G'$.

2) The composition of the embedding of varieties $G \hookrightarrow G \times G$, $g \mapsto (g, g^{-1})$ with the product map $G \times G \to G$ is trivial. □

If G is an absolutely simple simply connected group, then $A^1(G, K_2)$ is an infinite cyclic group with a canonical generator q_G (see [13, Part II, §7]).

3. BGQ spectral sequence

Let X be a smooth variety over F. We consider the Brown-Gersten-Quillen *coniveau spectral sequence*

$$E_2^{s,t} = A^s(X, K_{-t}) \Rightarrow K_{-s-t}(X) \tag{3.1}$$

converging to the K-groups of X with the topological filtration [23, §7, Th. 5.4].

Example 3.2. Let $G = \mathbf{SL}_n$. By [29, §2], we have $\mathrm{CH}(G) = \mathbb{Z}$. It follows that all the differentials of the BGQ spectral sequence for G coming to the zero diagonal are trivial.

Lemma 3.3 ([20, Theorem 3.4])**.** *If δ is a differential of finite order in the spectral sequence (3.1) on the q-th page $E_q^{*,*}$, then for every prime divisor p of the order of δ, the integer $p-1$ divides $q-1$.*

Let p be a prime integer, D a central division algebra over F of degree p and $G = \mathbf{SL}_1(D)$. As D is split by a field extension of degree p, it follows from Example 3.2 that all Chow groups $\mathrm{CH}^i(G)$ are p-periodic for $i > 0$ and the order of every differential in the BGQ spectral sequence for G coming to the zero diagonal divides p. The edge homomorphism $K_1(G) \to E_2^{0,-1} = A^0(G, K_1) = F^\times$ is a surjection split by the pull-back with respect to the structure morphism $G \to \operatorname{Spec} F$. Therefore, all the differentials starting at $E_*^{0,-1}$ are trivial.

It follows then from Lemma 3.3 that the only possibly nontrivial differential coming to the terms $E_q^{i,-i}$ for $q \geq 2$ and $i \leq p+1$ is

$$\partial_G : A^1(G, K_2) = E_p^{1,-2} \to E_p^{p+1,-p-1} = \mathrm{CH}^{p+1}(G).$$

By [29, Theorem 6.1] (see also [22, Theorem 5.1]), $K_0(G) = \mathbb{Z}$, hence the factors

$$K_0(G)^{(i)}/K_0(G)^{(i+1)} = E_\infty^{i,-i}$$

of the topological filtration on $K_0(G)$ are trivial for $i > 0$. It follows that the map ∂_G is surjective. As the group $A^1(G, K_2)$ is cyclic with the generator q_G, the group $\mathrm{CH}^{p+1}(G)$ is cyclic of order dividing p. It is shown in [33, Theorem 4.2] (see also Theorem 8.2) that the differential ∂_G is nontrivial. We have proved the following lemma.

Lemma 3.4. *If D is a central division algebra of degree p, then $\mathrm{CH}^{p+1}(G)$ is a cyclic group of order p generated by $\partial_G(q_G)$.* □

4. Specialization

Let A be a discrete valuation ring with residue field F and quotient field L. Let $\mathcal{X}$ be a smooth scheme over A and set $X = \mathcal{X} \otimes_A F$, $X' = \mathcal{X} \otimes_A L$. By [11, Example 20.3.1], there is a *specialization* ring homomorphism

$$\sigma : \mathrm{CH}^*(X') \to \mathrm{CH}^*(X).$$

Example 4.1. Let X be a variety over F, $L = F(t)$ the rational function field. Consider the valuation ring $A \subset L$ of the parameter t and $\mathcal{X} = X \otimes_F A$. Then $X' = X_L$ and we have a specialization ring homomorphism $\sigma : \mathrm{CH}^*(X_L) \to \mathrm{CH}^*(X)$.

A section of the structure morphism $\mathcal{X} \to \operatorname{Spec} A$ gives two rational points $x \in X$ and $x' \in X'$. By definition of the specialization, $\sigma([x']) = [x]$.

Let F be a field of finite characteristic. By [2, Ch. IX, §2, Propositions 5 and 1], there is a complete discretely valued field L of characteristic zero with residue field F. Let A be the valuation ring and D a central simple algebra over F. By [14, Theorem 6.1], there is an Azumaya algebra $\mathcal{D}$ over A such that $D \simeq \mathcal{D} \otimes_A F$. The algebra $D' = \mathcal{D} \otimes_A L$ is a central simple algebra over L. Then we have a specialization homomorphism

$$\sigma : \mathrm{CH}^*(\mathbf{SL}_1(D')) \to \mathrm{CH}^*(\mathbf{SL}_1(D))$$

satisfying $\sigma([e']) = [e]$, where e and e' are the identities of the groups.

5. A source of split motives

We work in the category of Chow motives over a field F, [9, §64]. We write $M(X)$ for the motive (with integral coefficients) of a smooth complete variety X over F.

A motive is *split* if it is isomorphic to a finite direct sum of Tate motives $\mathbb{Z}(a)$ (with arbitrary shifts a). Let X be a smooth proper variety such that the motive $M(X)$ is split, i.e., $M(X) = \coprod_i \mathbb{Z}(a_i)$ for some a_i. The *generating* (*Poincaré*) polynomial $P_X(t)$ of X is defined by

$$P_X(t) = \sum_i t^{a_i}.$$

Note that the integer a_i is equal to the rank of the (free abelian) Chow group $\mathrm{CH}^i(X)$.

Example 5.1. Let G be a split semisimple group and $B \subset G$ a Borel subgroup. Then

$$P_{G/B}(t) = \sum_{w \in W} t^{l(w)},$$

where W is the Weyl group of G and $l(w)$ is the length of w (see [8, §3]).

Proposition 5.2 (P. Brosnan, [4, Theorem 3.3])**.** *Let X be a smooth projective variety over F equipped with an action of the multiplicative group $\mathbb{G}_m$. Then*

$$M(X) = \coprod_i M(Z_i)(a_i),$$

where the Z_i are the (smooth) connected components of the subscheme of $X^{\mathbb{G}_m}$ of fixed points and $a_i \in \mathbb{Z}$. Moreover, the integer a_i is the dimension of the positive eigenspace of the action of $\mathbb{G}_m$ on the tangent space $\mathcal{T}_z$ of X at an arbitrary closed point $z \in Z_i$. The dimension of Z_i is the dimension of $(\mathcal{T}_z)^{\mathbb{G}_m}$.

Let T be a split torus of dimension n. The choice of a $\mathbb{Z}$-basis in the character group T^* allows us to identify T^* with $\mathbb{Z}^n$. We order $\mathbb{Z}^n$ (and hence T^*) lexicographically.

Suppose T acts on a smooth variety X and let $x \in X$ be an T-fixed rational point. Let $\chi_1, \chi_2, \ldots, \chi_m$ be all characters of the representation of T in the tangent space $\mathcal{T}_x$ of X at x. Write a_x for the number of positive (with respect to the ordering) characters among the χ_i's.

Corollary 5.3. *Let X be a smooth projective variety over F equipped with an action of a split torus T. If the subscheme X^T of T-fixed points in X is a disjoint union of finitely many rational points, the motive of X is split. Moreover,*

$$P_X(t) = \sum_{x \in X^T} t^{a_x}.$$

Proof. Induction on the dimension of T. □

Example 5.4. Let T be a split torus of dimension n and X a smooth projective toric variety (see [12]). Let σ be a cone of dimension n in the fan of X and $\{\chi_1, \chi_2, \dots, \chi_n\}$ a (unique) $\mathbb{Z}$-basis of T^* generating the dual cone $\sigma^\vee$. The standard T-invariant affine open set corresponding to σ is $V_\sigma := \operatorname{Spec} F[\sigma^\vee]$. The map $V_\sigma \to \mathbb{A}^n$, taking x to $(\chi_1(x), \chi_2(x), \dots, \chi_n(x))$ is a T-equivariant isomorphism, where $t \in T$ acts on the affine space $\mathbb{A}^n$ by componentwise multiplication by $\chi_i(t)$. The only one T-equivariant point $x \in V_\sigma$ corresponds to the origin under the isomorphism, so we can identify the tangent space $\mathcal{T}_x$ with $\mathbb{A}^n$, and the χ_i's are the characters of the representation of T in the tangent space $\mathcal{T}_x$. Let a_σ be the number of positive χ_i's with respect to a fixed lexicographic order on T^*. Every T-fixed point in X belongs to V_σ for a unique σ. It follows that the motive $M(X)$ is split and

$$P_X(t) = \sum_{\sigma} t^{a_\sigma},$$

where the sum is taken over all dimension n cones in the fan of X.

6. Compactifications of algebraic groups

A *compactification* of an affine algebraic group G is a projective variety containing G as a dense open subvariety. A $G \times G$*-equivariant* compactification of G is a projective variety X equipped with an action of $G \times G$ and containing the homogeneous variety $G = (G \times G)/\operatorname{diag}(G)$ as an open orbit. Here the group $G \times G$ acts on G by the left-right translations.

Let G be a split semisimple group over F. Write G_{ad} for the corresponding adjoint group. The group G_{ad} admits the so-called *wonderful* $G_{ad} \times G_{ad}$-equivariant compactification $\mathbf{X}$ (see [3, §6.1]). Let $T \subset G$ be a split maximal torus and T_{ad} the corresponding maximal torus in G_{ad}. The closure $\mathbf{X}'$ of T_{ad} in $\mathbf{X}$ is a toric T_{ad}-variety with fan consisting of all Weyl chambers in $(T_{ad})_* \otimes \mathbb{R} = T_* \otimes \mathbb{R}$ and their faces.

Let B be a Borel subgroup of G containing T and B^- the opposite Borel subgroup. There is an open $B^- \times B$-invariant subscheme $\mathbf{X}_0 \subset \mathbf{X}$ such that the intersection $\mathbf{X}'_0 := \mathbf{X}_0 \cap \mathbf{X}'$ is the standard open T_{ad}-invariant subscheme of the toric variety $\mathbf{X}'$ corresponding to the negative Weyl chamber Ω that is a cone in the fan of $\mathbf{X}'$. Note that the Weyl group W of G acts simply transitively on the set of all Weyl chambers.

A $G \times G$-equivariant compactification X of G is called *toroidal* if X is normal and the quotient map $G \to G_{ad}$ extends to a morphism $\pi : X \to \mathbf{X}$ (see [3, §6.2]). The closed subscheme $X' := \pi^{-1}(\mathbf{X}')$ of X is a projective toric T-variety. Note that the diagonal subtorus $\operatorname{diag}(T) \subset T \times T$ acts trivially on X'. The fan of X' is a subdivision of the fan consisting of the Weyl chambers and their faces. The scheme X is smooth if and only if so is X'.

Conversely, if F is a perfect field, given a smooth projective toric T-variety with a W-invariant fan that is a subdivision of the fan consisting of the Weyl chambers and their faces, there is a unique smooth $G\times G$-equivariant toroidal compactification X of G with the toric variety X' isomorphic to the given one (see [3, §6.2] and [15, §2.3]). By [5] and [7], such a smooth toric variety exists for every split semisimple group G. In other words, the following holds.

Proposition 6.1. *Every split semisimple group G over a perfect field admits a smooth $G\times G$-equivariant toroidal compactification.* □

Let X be a smooth $G\times G$-equivariant toroidal compactification of G over F. Recall that the toric T-variety X' is smooth projective. Set $X_0 := \pi^{-1}(\mathbf{X}_0)$ and $X_0' := \pi^{-1}(\mathbf{X}_0') = X'\cap X_0$. Then the T-invariant subset $X_0'\subset X'$ is the union of standard open subschemes V_σ of X' (see Example 5.4) corresponding to all cones σ in the negative Weyl chamber Ω. The subscheme $(V_\sigma)^T$ reduces to a single rational point if σ is of largest dimension. In particular, the subscheme $(X_0')^T$ of T-fixed points in X_0' is a disjoint union of k rational points, where k is the number of cones of maximal dimension in Ω. It follows that $|(X')^T| = k|W|$, the number of all cones of maximal dimension in the fan of X'.

Let U and U^- be the unipotent radicals of B and B^- respectively.

Lemma 6.2 ([3, Proposition 6.2.3])**.** 1) *Every $G\times G$-orbit in X meets X_0' along a unique T-orbit.*
2) *The map*
$$U^-\times X_0'\times U\to X_0,\quad (u,x,v)\mapsto uxv^{-1},$$
is a $T\times T$-equivariant isomorphism.
3) *Every closed $G\times G$-orbit in X is isomorphic to $G/B\times G/B$.*

Proposition 6.3. *The scheme $X^{T\times T}$ is the disjoint union of Wx_0W over all $x_0\in(X_0')^T$ and Wx_0W is a disjoint union of $|W|^2$ rational points.*

Proof. Take $x\in X^{T\times T}$. Let $\mathbf{x}$ be the image of x under the map $\pi: X\to\mathbf{X}$. Computing dimensions of maximal tori of the stabilizers of points in the wonderful compactification $\mathbf{X}$, we see that $\mathbf{x}$ lies in the only closed $G\times G$-orbit $\mathbf{O}$ in $\mathbf{X}$ (e.g., [10, Lemma 4.2]). By Lemma 6.2(3), applied to the compactification $\mathbf{X}$ of G_{ad}, $\mathbf{O}\simeq G/B\times G/B$. In view of Lemma 6.2(1), $\mathbf{O}\cap\mathbf{X}_0'$ is a closed T-orbit in $\mathbf{X}_0'$ and therefore, reduces to a single rational T-invariant point in $\mathbf{X}_0'$. The group $W\times W$ acts simply transitively on the set of $T\times T$-fixed point in $G/B\times G/B$. It follows that $|W\mathbf{x}W| = |W|^2$ and $W\mathbf{x}W$ intersects $\mathbf{X}_0'$. Therefore, WxW intersects $X^{T\times T}\cap X_0' = (X_0')^T$, that is the disjoint union of k rational points. Hence x is a rational point, $x\in W(X_0')^TW$ and $|WxW| = |W|^2$.

Note that for a point $x_0\in(X_0')^T$, the $G\times G$-orbit of x_0 intersects X_0' by the T-orbit $\{x_0\}$ in view of Lemma 6.2(1). It follows that different Wx_0W do not intersect and therefore, $X^{T\times T}$ is the disjoint union of Wx_0W over all $x_0\in(X_0')^T$. □

Let X be a smooth $G\times G$-equivariant toroidal compactification of a split semisimple group G of rank n. By Proposition 6.3, every $T\times T$-fixed point x in X is of the form $x = w_1x_0w_2^{-1}$, where $w_1, w_2\in W$ and $x_0\in(X_0')^T$. Recall that X_0' is the union of the standard affine open subsets V_σ of the toric T-variety X' over all cones σ of dimension n in the Weyl chamber Ω. Let σ be a (unique) cone in Ω such that $x_0\in V_\sigma$.

By Lemma 6.2(2), the map

$$f : U^- \times V_\sigma \times U \to X, \quad (u_1, y, u_2) \mapsto w_1 u_1 x_0 u_2^{-1} w_2^{-1}$$

is an open embedding. We have $f(1, x_0, 1) = x$. Thus, f identifies the tangent space $\mathcal{T}_x$ of x in X with the space $\mathfrak{u}^- \oplus \mathfrak{a} \oplus \mathfrak{u}$, where $\mathfrak{u}$ and $\mathfrak{u}^-$ are the Lie algebras of U and U^- respectively and $\mathfrak{a}$ is the tangent space of V_σ at x'. The torus $T \times T$ acts linearly on the tangent space $\mathcal{T}_x$ leaving invariant $\mathfrak{u}^-$, $\mathfrak{a}$ and $\mathfrak{u}$. For convenience, we write $T \times T$ as $S := T_1 \times T_2$ in order to distinguish the components. Let Φ_1^- and Φ_2^- be two copies of the set of negative roots in T_1^* and T_2^* respectively. The set of characters of the S-representation $\mathfrak{u}^-$ (respectively, $\mathfrak{u}$) is $w_1(\Phi_1^-)$ (respectively, $w_2(\Phi_2^-)$).

Let $\{\chi_1, \chi_2, \dots, \chi_n\}$ be a (unique) $\mathbb{Z}$-basis of T^* generating the dual cone $\sigma^\vee$. By Example 5.4, the set of characters of the S-representation $\mathfrak{a}$ is

$$\{(w_1(\chi_i), -w_2(\chi_i))\}_{i=1}^n \subset S^* = T_1^* \oplus T_2^*.$$

Let Π_1 and Π_2 be (ordered) systems of simple roots in Φ_1 and Φ_2 respectively. Consider the lexicographic ordering on $S^* = T_1^* \oplus T_2^*$ corresponding to the basis $\Pi_1 \cup \Pi_2$ of S^*. As $\chi_i \neq 0$, we have $(w_1(\chi_i), -w_2(\chi_i)) > 0$ if and only if $w_1(\chi_i) > 0$. For every $w \in W$, write $b(\sigma, w)$ for the number of all i such that $w(\chi_i) > 0$. Note that the number of positive roots in $w(\Phi^-)$ is equal to the length $l(w)$ of w. By Corollary 5.3, we have

$$P_X(t) = \sum_{w_1, w_2 \in W,\ \sigma \subset \Omega} t^{l(w_1) + b(\sigma, w_1) + l(w_2)} = \Big(\sum_{w \in W,\ \sigma \subset \Omega} t^{l(w) + b(\sigma, w)} \Big) \cdot P_{G/B}(t), \tag{6.4}$$

as by Example 5.1,

$$P_{G/B}(t) = \sum_{w \in W} t^{l(w)}.$$

We have proved the following theorem.

Theorem 6.5. *Let X be a smooth $G \times G$-equivariant toroidal compactification of a split semisimple group G. Then the motive $M(X)$ is split into a direct sum of $s|W|$ Tate motives, where s is the number of cones of maximal dimension in the fan of the associated toric variety X'. Moreover,*

$$P_X(t) = \Big(\sum_{w \in W,\ \sigma \subset \Omega} t^{l(w) + b(\sigma, w)} \Big) \cdot P_{G/B}(t).$$

In particular, the motive $M(X)$ is divisible by $M(G/B)$.

Example 6.6. Let G be a semisimple adjoint group and X the wonderful compactification of G. Then the negative Weyl chamber Ω is the cone $\sigma = \Omega$ in the fan of X'. The dual cone $\sigma^\vee$ is generated by $-\Pi$. Hence $b(w, \sigma)$ is equal to the number of *simple* roots α such that $w(\alpha) \in \Phi^-$.

Example 6.7. Let $G = \mathbf{SL}_3$, $\Pi = \{\alpha_1, \alpha_2\}$. Bisecting each of the six Weyl chambers we get a smooth projective fan with 12 two-dimensional cones. The two cones dual to the ones in the negative Weyl chamber are generated by $\{-\alpha_1, (\alpha_1 - \alpha_2)/3\}$ and $\{-\alpha_2, (\alpha_2 - \alpha_1)/3\}$ respectively. Let X be the corresponding $G \times G$-equivariant toroidal compactification of G. By (6.4),

$$P_X(t) = (t^5 + t^4 + 4t^3 + 4t^2 + t + 1)(t^3 + 2t^2 + 2t + 1).$$

Now consider arbitrary (not necessarily toroidal) $G \times G$-equivariant compactifications.

Theorem 6.8. *Let X be a smooth $G\times G$-equivariant compactification of a split semisimple group G over F. Then the subscheme $X^{T\times T}$ is a disjoint union of finitely many rational points. In particular, the motive $M(X)$ is split.*

Proof. By [3, Proposition 6.2.5], there is a $G \times G$-equivariant toroidal compactification $\widetilde{X}$ of G together with a $G\times G$-equivariant morphism $\varphi : \widetilde{X} \to X$. Let $x \in X^{T\times T}$. By Borel's fixed point theorem, the fiber $\varphi^{-1}(x)$ has a $T \times T$-fixed point, so the map $\widetilde{X}^{T\times T} \to X^{T\times T}$ is surjective. By Proposition 6.3, $\widetilde{X}^{T\times T}$ is a disjoint union of finitely many rational points, hence so is $X^{T\times T}$. □

Example 6.9. Let Y be a smooth $H \times H$-equivariant compactification of the group $H = \mathbf{SL}_n$ over F. In particular the projective linear group $\mathbf{PGL}_n$ acts on Y by conjugation. Let D be a central simple F-algebra of degree n and J the corresponding $\mathbf{PGL}_n$-torsor. The twist of H by J is the group $G = \mathbf{SL}_1(D)$, hence the twist X of Y is a smooth $G\times G$-equivariant compactification of G. If E is a G-torsor, one can twist X by E to get a smooth compactification of E. By Theorem 6.8, the motives of these compactifications are split over every splitting field of D.

7. Some computations in $\mathrm{CH}(\mathbf{SL}_1(D))$

Let D be a central simple algebra of prime degree p over F and $G = \mathbf{SL}_1(D)$.

Lemma 7.1. *Let X be a smooth compactification of G. Then D is split by the residue field of every point in $X \setminus G$.*

Proof. Let Y be the projective (singular) hypersurface given in the projective space $\mathbb{P}(D\oplus F)$ by the equation $\mathrm{Nrd} = t^p$, where Nrd is the reduced norm form. The group G is an open subset in Y, so we can identify the function fields $F(X) = F(G) = F(Y)$. Let $x \in X \setminus G$. As x is smooth in X, there is a regular system of local parameters around x and therefore a valuation v of $F(G)$ over F with residue field $F(x)$. Since Y is projective, v dominates a point $y \in Y \setminus G$. Over the residue field $F(y)$ the norm form Nrd is isotropic, hence D is split over $F(y)$. Since v dominates y, the field $F(y)$ is contained in $F(v) = F(x)$. Therefore, D is split over $F(x)$. □

Lemma 7.2. *If D is a division algebra, then the group $\mathrm{CH}_0(G) = \mathrm{CH}^{p^2-1}(G)$ is cyclic of order p generated by the class of the identity e of G.*

Proof. The group of R-equivalence classes of points in $G(F)$ is equal to $\mathrm{SK}_1(D)$ (see [32, Ch. 6]) and hence is trivial by a theorem of Wang. It follows that we have $[x] = [e]$ in $\mathrm{CH}_0(G)$ for every rational point $x \in G(F)$. If $x \in G$ is a closed point, then $[x'] = [e]$ in $\mathrm{CH}_0(G_K)$, where $K = F(x)$ and x' is a rational point of G_K over x. Taking the norm homomorphism $\mathrm{CH}_0(G_K) \to \mathrm{CH}_0(G)$ for the finite field extension K/F, we have $[x] = [K : F] \cdot [e]$ in $\mathrm{CH}_0(G)$. It follows that $\mathrm{CH}_0(G)$ is a cyclic group generated by $[e]$.

As $p \cdot \mathrm{CH}_0(G) = 0$ it suffices to show that $[e] \neq 0$ in $\mathrm{CH}_0(G)$. Let Y be the compactification of G as in the proof of Lemma 7.1 and let $Z = Y \setminus G$. As D is a central division algebra, the degree of every closed point of Z is divisible by p by Lemma 7.1.

It follows that the class $[e]$ in $\mathrm{CH}_0(Y)$ does not belong to the image of the push-forward homomorphism i in the exact sequence

$$\mathrm{CH}_0(Z) \xrightarrow{i} \mathrm{CH}_0(Y) \to \mathrm{CH}_0(G) \to 0.$$

Therefore, $[e] \neq 0$ in $\mathrm{CH}_0(G)$. □

Consider the morphism $s : G \times G \to G$, $s(x, y) = xy^{-1}$. Note that s is flat as the composition of the automorphism $(x, y) \mapsto (xy^{-1}, y)$ of the variety $G \times G$ with the projection $G \times G \to G$.

Let $h = \partial_G(q_G) \in \mathrm{CH}^{p+1}(G)$.

Lemma 7.3. *We have* $s^*(h) = h \times 1 - 1 \times h$ *in* $\mathrm{CH}^{p+1}(G \times G)$.

Proof. By Lemma 2.1, we have $s^*(q_G) = q_G \times 1 - 1 \times q_G$ in $A^1(G \times G, K_2)$. The differentials ∂_G commute with flat pull-back maps, hence we have

$$\begin{aligned} s^*(h) = s^*(\partial_G(q_G)) = \partial_{G\times G}(s^*(q_G)) = \partial_{G\times G}(q_G \times 1 - 1 \times q_G) = \\ \partial_G(q_G) \times 1 - 1 \times \partial_G(q_G) = h \times 1 - 1 \times h. \end{aligned}$$

□

Proposition 7.4. *Let c be an integer with $h^{p-1} = c[e]$ in $\mathrm{CH}^{p^2-1}(G)$. Then*

$$c\Delta_G = \sum_{i=0}^{p-1} h^i \times h^{p-1-i},$$

where Δ_G is the class of the diagonal $\mathrm{diag}(G)$ *in* $\mathrm{CH}^{p^2-1}(G \times G)$.

Proof. The diagonal in $G \times G$ is the pre-image of e under s. Hence by Lemma 7.3,

$$c\Delta_G = cs^*([e]) = s^*(h^{p-1}) = (h \times 1 - 1 \times h)^{p-1} = \sum_{i=0}^{p-1} h^i \times h^{p-1-i}$$

as $\binom{p-1}{i} \equiv (-1)^i$ modulo p and $ph = 0$. □

8. Rost's theorem

We have proved in Lemma 3.4 that if D is a central division algebra, then $\partial_G(q_G) \neq 0$ in $\mathrm{CH}^{p+1}(G)$. This result is strengthened in Theorem 8.2 below.

Lemma 8.1. *If there is an element $h \in \mathrm{CH}^{p+1}(G)$ such that $h^{p-1} \neq 0$, then $\partial_G(q_G)^{p-1} \neq 0$.*

Proof. By Lemma 3.4, h is a multiple of $\partial_G(q_G)$. □

Theorem 8.2 (M. Rost)**.** *Let D be a central division algebra of degree p, $G = \mathbf{SL}_1(D)$. Then $\partial_G(q_G)^{p-1} \neq 0$ in $\mathrm{CH}^{p^2-1}(G) = \mathrm{CH}_0(G)$.*

Proof. Case 1: Assume first that $\mathrm{char}(F) = 0$, F contains a primitive p-th root of unity and D is a cyclic algebra, i.e., $D = (a, b)_F$ for some $a, b \in F^\times$.

Let $c \in F^\times$ be an element such that the symbol

$$u := (a, b, c) \in H^3_{ét}(F, \mathbb{Z}/p\mathbb{Z}(3)) \simeq H^3_{ét}(F, \mathbb{Z}/p\mathbb{Z}(2))$$

is nontrivial modulo p. Consider a norm variety X of u.

Then u defines a *basic correspondence* in the cokernel of the homomorphism

$$\mathrm{CH}^{p+1}(X) \to \mathrm{CH}^{p+1}(X \times X)$$

given by the difference of the pull-backs with respect to the projections. A representative in $\mathrm{CH}^{p+1}(X \times X)$ of the basic correspondence is a *special correspondence.* Let $z \in \mathrm{CH}^{p+1}(X_{F(X)})$ be its pull-back. The modulo p degree

$$c(X) := \deg(z^{p-1}) \in \mathbb{Z}/p\mathbb{Z}$$

is independent of the choice of the special correspondence. The construction of $c(X)$ is natural with respect to morphisms of norm varieties (see [24]).

It is shown in [24] that there is an X such that $c(X) \neq 0$. We claim that $c(X') \neq 0$ for every norm variety X' of u. As $F(X')$ splits u and X is p-generic, X has a closed point over $F(X')$ of degree prime to p, or equivalently, there is a prime correspondence $X' \rightsquigarrow X$ of multiplicity prime to p. Resolving singularities, we get a smooth complete variety X'' together with the two morphisms $f : X'' \to X$ of degree prime to p and $g : X'' \to X'$. It follows by [28, Corollary 1.19] that X'' is a norm variety of u. Moreover, $c(X'') = \deg(f)c(X) \neq 0$ in $\mathbb{Z}/p\mathbb{Z}$. As $c(X'') = \deg(g)c(X')$, $c(X')$ is also nonzero. The claim is proved.

Let X be a smooth compactification of the G-torsor E given by the equation $\mathrm{Nrd} = t$ over the rational function field $L = F(t)$ given by a variable t. By the above, since $\{a, b, t\} \neq 0$, we have an element $z \in \mathrm{CH}^{p+1}(X_{L(X)})$ such that $\deg(z^{p-1}) \neq 0$ in $\mathbb{Z}/p\mathbb{Z}$. The torsor E is trivial over $L(X)$, i.e. $E_{L(X)} \simeq G_{L(X)}$. Then the restriction of z to the torsor gives an element $y \in \mathrm{CH}^{p+1}(G_{L(X)})$ with $y^{p-1} \neq 0$. The field extension $L(X)/F$ is purely transcendental. By Section 4 and Lemma 7.2, every specialization homomorphism $\sigma : \mathrm{CH}^{p^2-1}(G_{L(X)}) \to \mathrm{CH}^{p^2-1}(G)$ is an isomorphism taking the class of the identity to the class of the identity. Specializing, we get an element $h \in \mathrm{CH}^{p+1}(G)$ with $h^{p-1} \neq 0$. It follows from Lemma 8.1 that $\partial_G(q_G)^{p-1} \neq 0$.

Case 2: Suppose that $\mathrm{char}(F) = 0$ but F may not contain p-th roots of unity and D is an arbitrary division algebra of degree p (not necessarily cyclic). There is a finite field extension K/F of degree prime to p containing a primitive p-th root of unity and such that the algebra $D \otimes_F K$ is cyclic (and still nonsplit). By Case 1, $\partial_G(q_G)_K^{p-1} \neq 0$ over K. Therefore $\partial_G(q_G)^{p-1} \neq 0$.

Case 3: F is an arbitrary field. Choose a field L of characteristic zero and a central simple algebra D' of degree p over L as in Section 4 and let $G' = \mathbf{SL}_1(D')$. By Case 2, there is an element $h' \in \mathrm{CH}^{p+1}(G')$ such that $(h')^{p-1} \neq 0$. Applying a specialization σ (see Section 4), we have $h^{p-1} \neq 0$ for $h = \sigma(h')$. By Lemma 8.1 again, $\partial_G(q_G)^{p-1} \neq 0$. □

Let D be a central division algebra of degree p over F and X a smooth compactification of G. Let $\bar{h} \in \mathrm{CH}^{p+1}(X)$ be an element such that $\bar{h}|_G = \partial_G(q_G) \in \mathrm{CH}^{p+1}(G)$. Let $i = 0, 1, \dots, p-1$. The element $\bar{h}^i$ defines the following two morphisms of Chow motives:

$$f_i : M(X) \to \mathbb{Z}((p+1)i), \qquad g_i : \mathbb{Z}((p+1)(p-1-i)) \to M(X).$$

Let

$$R = \mathbb{Z} \oplus \mathbb{Z}(p+1) \oplus \mathbb{Z}(2p+2) \oplus \cdots \oplus \mathbb{Z}(p^2-1).$$

We thus have the following two morphisms:

$$f : M(X) \to R, \qquad g : R \to M(X).$$

The composition $f \circ g$ is c times the identity, where $c = \deg \bar{h}^{p-1}$. As c is prime to p by Theorem 8.2, switching to the *Chow motives with coefficients in* $\mathbb{Z}_{(p)}$, we have a decomposition

$$M(X) = R \oplus N \tag{8.3}$$

for some motive N.

9. The category of D-motives

Let D be a central simple algebra of prime degree p over F. For a field extension L/F, let $N_i^D(L)$ be the subgroup of the Milnor K-group $K_i^M(L)$ generated by the norms from finite field extensions of L that split the algebra D.

Consider the Rost cycle module (see [25]):

$$L \mapsto K_*^D(L) := K_*^M(L)/N_*^D(L),$$

and the corresponding cohomology theory with the "Chow groups"

$$\mathrm{CH}_D^i(X) := A^i(X, K_i^D).$$

Note that $\mathrm{CH}_D^i(X) = 0$ if D is split over $F(x)$ for all points $x \in X$.

Let $S = \mathrm{SB}(D)$ be the Severi-Brauer variety of right ideals of D of dimension p. We have $\dim S = p - 1$.

Lemma 9.1. *For a variety X over F, the group $\mathrm{CH}_D(X)$ is naturally isomorphic to the cokernel of the push-forward homomorphism $pr_* : \mathrm{CH}(X \times S) \to \mathrm{CH}(X)$ given by the projection $pr : X \times S \to X$.*

Proof. The composition

$$\mathrm{CH}(X \times S) \xrightarrow{pr_*} \mathrm{CH}(X) \to \mathrm{CH}_D(X)$$

factors through the trivial group $\mathrm{CH}_D(X \times S)$ and therefore, is zero. This defines a surjective homomorphism

$$\alpha : \mathrm{Coker}(pr_*) \to \mathrm{CH}_D(X).$$

The inverse map is obtained by showing that the quotient map $\mathrm{CH}(X) \to \mathrm{Coker}(pr_*)$ factors through $\mathrm{CH}_D(X)$.

The kernel of the homomorphism $\mathrm{CH}(X) \to \mathrm{CH}_D(X)$ is generated by $[x]$ with $x \in X$ such that the algebra $D_{F(x)}$ is split and by $p[x]$ with arbitrary $x \in X$. The fiber of pr over x has a rational point y in the first case and a degree p closed point y in the second. The generators are equal to $pr_*([y])$ in both cases. It follows that they vanish in $\mathrm{Coker}\, pr_*$. □

Let $G = \mathbf{SL}_1(D)$.

Corollary 9.2. *The natural map $\mathrm{CH}^i(G) \to \mathrm{CH}_D^i(G)$ is an isomorphism for all $i > 0$.*

Proof. The algebra D is split over S. More precisely, $D_X = \operatorname{End}_X(I^\vee)$ for the rank p canonical vector bundle I over S (see [27, Lemma 2.1.4]). By [29, Theorem 4.2], the pull-back homomorphism $\mathrm{CH}^*(S) \to \mathrm{CH}^*(G\times S)$ is an isomorphism. Therefore, $\mathrm{CH}^j(G\times S) = 0$ if $j > p-1 = \dim(S)$. □

Let X be a smooth compactification of G. Write $X^k = X \times X \times \cdots \times X$ (k times).

Lemma 9.3. *The restriction homomorphism* $\mathrm{CH}^*_D(X^k) \to \mathrm{CH}^*_D(G^k)$ *is an isomorphism.*

Proof. Let $Z = X^k \setminus G^k$. By Lemma 7.1, the residue field of every point in Z splits D, hence $\mathrm{CH}_D(Z) = 0$. The statement follows from the exactness of the localization sequence

$$\mathrm{CH}_D(Z) \to \mathrm{CH}_D(X^k) \to \mathrm{CH}_D(G^k) \to 0. \qquad \square$$

It follows from Lemma 9.3 and Corollary 9.2 that $\mathrm{CH}^i_D(X) \simeq \mathrm{CH}^i(G)$ for $i > 0$.

Consider the category of motives of smooth complete varieties over F associated to the cohomology theory $\mathrm{CH}^*_D(X)$ (see [21]). Write $M^D(X)$ for the motive of a smooth complete variety X. We call $M^D(X)$ the *D-motive of* X. Recall that the group of morphisms between $M^D(X)$ and $M^D(Y)$ for Y of pure dimension d is equal to $\mathrm{CH}^d_D(X\times Y)$. Let $\mathbb{Z}^D$ the motive of the point $\operatorname{Spec} F$.

Recall that we write $M(X)$ for the usual Chow motive of X. We have a functor $N \mapsto N^D$ from the category of Chow motives to the category of D-motives.

Proposition 9.4. *Let N be a Chow motive. Then $N^D = 0$ if and only if N is isomorphic to a direct summand of $N \otimes M(S)$.*

Proof. As $M^D(S) = 0$, we have $N^D = 0$ if N is isomorphic to a direct summand of $N \otimes M(S)$.

Conversely, suppose $N^D = 0$. Let $N = (X, \rho)$, where X is a smooth complete variety of pure dimension d and $\rho \in \mathrm{CH}^d(X \times X)$ is a projector. By Lemma 9.1, we have $\rho = f_*(\theta)$ for some $\theta \in \mathrm{CH}^{d+p-1}(X \times (X \times S))$, where $f : X \times X \times S \to X \times X$ is the projection. Then

$$f_*\big((\rho \otimes \mathrm{id}_S) \circ \theta \circ \rho\big) = \rho$$

and $(\rho \otimes \mathrm{id}_S) \circ \theta \circ \rho$ can be viewed as a morphism $N \to N \otimes M(S)$ splitting on the right the natural morphism $N \otimes M(S) \to N$. □

The morphisms f and g in Section 8 give rise to the morphisms $f^D : M^D(X) \to R^D$ and $g^D : R^D \to M^D(X)$ of D-motives.

Proposition 9.5. *The morphism $f^D : M^D(X) \to R^D$ is an isomorphism in the category of D-motives.*

Proof. As $\mathrm{CH}^{p^2-1}_D(X \times X) \simeq \mathrm{CH}^{p^2-1}_D(G \times G)$ by Lemma 9.3, the composition $g^D \circ f^D$ is multiplication by $c \in \mathbb{Z}$ from Proposition 7.4. By Theorem 8.2, c is not divisible by p. Finally, $p\,\mathrm{CH}_D(G \times G) = 0$. □

If D is a central division algebra, it follows from Proposition 9.5 and Corollary 9.2 that for every $i > 0$,

$$\mathrm{CH}^i(G) = \mathrm{CH}^i_D(X) = \mathrm{CH}^i_D(R) = \begin{cases} (\mathbb{Z}/p\mathbb{Z})h^j, & \text{if } i = (p+1)j \le p^2-1; \\ 0, & \text{otherwise,} \end{cases} \tag{9.6}$$

where $h = \partial_G(q_G)$.

We can compute the Chow ring of G.

Theorem 9.7. *Let D be a central division algebra of prime degree p, $G = \mathbf{SL}_1(D)$ and $h = \partial_G(q_G) \in \mathrm{CH}^{p+1}(G)$. Then*

$$\mathrm{CH}(G) = \mathbb{Z}\cdot 1 \oplus (\mathbb{Z}/p\mathbb{Z})h \oplus (\mathbb{Z}/p\mathbb{Z})h^2 \oplus \cdots \oplus (\mathbb{Z}/p\mathbb{Z})h^{p-1}.$$

Proof. If F is a perfect field, G admits a smooth compactification X by Proposition 6.1. The statement follows from (9.6). In general, we proceed as follows.

A variety X over F is called *D-complete* is there is a compactification $\overline{X}$ of X such that D is split by the residue field of every point in $\overline{X} \setminus X$. Note that the restriction map $\mathrm{CH}_D(\overline{X} \times U) \to \mathrm{CH}_D(X \times U)$ is an isomorphism for every variety U. By the proof of Lemma 7.1, G is a D-complete variety.

We extend the category of D-motives by adding the motives $M^D(X)$ of smooth D-complete varieties X. If X and Y are two smooth D-complete varieties with Y equidimensional of dimension d, we define $\mathrm{Hom}(M^D(X), M^D(Y)) := \mathrm{CH}^d_D(X\times Y)$. The composition homomorphism

$$\mathrm{CH}^d_D(X \times Y) \otimes \mathrm{CH}^r_D(Y \times Z) \to \mathrm{CH}^r_D(X \times Z)$$

is given by

$$\alpha \otimes \beta \mapsto p_{13*}\big(p_{12}^*(\alpha) \cdot p_{23}^*(\beta)\big),$$

where p_{ij} are the three projections of $X \times Y \times Z$ on X, Y and Z, and the push-forward map p_{13*} is defined as the composition

$$p_{13*} : \mathrm{CH}^{d+r}_D(X \times Y \times Z) \simeq \mathrm{CH}^{d+r}_D(X \times \overline{Y} \times Z) \to \mathrm{CH}^r_D(X \times Z).$$

Here $\overline{Y}$ is a compactification of Y satisfying the condition in the definition of a D-complete variety and the second map is the push-forward homomorphism for the proper projection $X \times \overline{Y} \times Z \to X \times Z$.

By Proposition 7.4 and Theorem 8.2, the powers of $h = \partial_G(q_G)$ yield the following decomposition of D-motives (with coefficients in $\mathbb{Z}_{(p)}$):

$$M^D(G) \simeq \mathbb{Z}^D \oplus \mathbb{Z}^D(p+1) \oplus \cdots \oplus \mathbb{Z}^D(p^2-1).$$

The result follows as $\mathrm{CH}^i(G) = \mathrm{CH}^i_D(G)$ for $i > 0$ by Corollary 9.2. □

10. Motivic decomposition of compactifications of $\mathbf{SL}_1(D)$

Let D be a central division F-algebra of degree a power of a prime p and $S = \mathrm{SB}(D)$. We work with motives with $\mathbb{Z}_{(p)}$-coefficients in this section.

Proposition 10.1. *Let X be a connected smooth complete variety over F such that the motive of X is split over every splitting field of D and D is split over $F(X)$. Then the motive of X is a direct sum of shifts of the motive of S.*

Proof. Note that the variety X is *generically split*, that is, its motive is split over $F(X)$. In particular, X satisfies the nilpotence principle, [30, Proposition 3.1]. Therefore, it suffices to prove the result for motives with coefficients in $\mathbb{F}_p$: any lifting of an isomorphism of the motives with coefficients in $\mathbb{F}_p$ to the coefficients $\mathbb{Z}_{(p)}$ will be an isomorphism since it will become an isomorphism over any splitting field of D.

For $\mathbb{F}_p$-coefficients, here is the argument. The (isomorphism class of the) upper motive $U(X)$ is well-defined and, by the arguments as in the proof of [18, Theorem 3.5], the motive of X is a sum of shifts of $U(X)$. Besides, $U(X) \simeq U(S)$, cf. [18, Corollary 2.15]. Finally, $U(S) = M(S)$ because the motive of S is indecomposable, [18, Corollary 2.22]. □

From now on, the degree of the division algebra D is p. Recall that we work with motives with coefficients in $\mathbb{Z}_{(p)}$. So, we set

$$R = \mathbb{Z}_{(p)} \oplus \mathbb{Z}_{(p)}(p+1) \oplus \mathbb{Z}_{(p)}(2p+2) \oplus \cdots \oplus \mathbb{Z}_{(p)}(p^2-1)$$

now.

Theorem 10.2. *Let F be a field, D a central division F-algebra of prime degree p, $G = \mathbf{SL}_1(D)$, X a smooth compactification of G, and $M(X)$ its Chow motive with $\mathbb{Z}_{(p)}$-coefficients. Assume that $M(X)$ is split over every splitting field of D (see Example 6.9). Then the motive $M(X)$ (over F) is isomorphic to the direct sum of R and a direct sum of shifts of $M(S)$.*

Proof. By (8.3), $M(X) = R \oplus N$ for a motive N and by Proposition 9.5, $N^D = 0$. It follows from Proposition 9.4 that N is isomorphic to a direct summand of $N \otimes M(S)$. In its turn, $N \otimes M(S)$ is a direct summand of $M(X \times S)$. In view of Proposition 10.1, $M(X \times S)$ is a direct sum of shifts of $M(S)$. By the uniqueness of the decomposition [6, Corollary 35] and indecomposability of $M(S)$ [18, Corollary 2.22], the motive N is a direct sum of shifts of $M(S)$. □

Theorem 10.3. *Let E be an $\mathbf{SL}_1(D)$-torsor and X a smooth compactification of E such that the motive $M(X)$ is split over every splitting field of D (see Example 6.9). Then X satisfies the nilpotence principle. Besides, the motive $M(X)$ is isomorphic to the direct sum of the Rost motive $\mathcal{R}$ of X and a direct sum of shifts of $M(S)$. The above decomposition is the unique decomposition of $M(X)$ into a direct sum of indecomposable motives.*

Proof. By saying that X satisfies the nilpotence principle, we mean that it does it for any coefficient ring, or, equivalently, for $\mathbb{Z}$-coefficients. However, since the integral motive of X is split over a field extension of degree p, it suffices to check that X satisfies the nilpotence principle for $\mathbb{Z}_{(p)}$-coefficients, where we can simply refer to [9, Theorem 92.4] and Theorem 10.2 (applied to X over $F(X)$).

It follows that it suffices to get the motivic decomposition of Theorem 10.3 for $\mathbb{Z}_{(p)}$-coefficients replaced by $\mathbb{F}_p$-coefficients. For $\mathbb{F}_p$-coefficients we use the following modification of [17, Proposition 4.6]:

Lemma 10.4. *Let S be a geometrically irreducible variety with the motive satisfying the nilpotence principle and becoming split over an extension of the base field. Let M be a summand of the motive of some smooth complete variety X. Assume that there exists a field extension L/F and an integer $i \in \mathbb{Z}$ such that the change of field homomorphism $\mathrm{Ch}(X_{F(S)}) \to \mathrm{Ch}(X_{L(S)})$ is surjective and the motive $M(S)(i)_L$ is an indecomposable summand of M_L. Then $M(S)(i)$ is an indecomposable summand of M.*

Proof. It was assumed in [17, Proposition 4.6] that the field extension $L(S)/F(S)$ is purely transcendental. But this assumption was only used to ensure that the change of field homomorphism $\mathrm{Ch}(X_{F(S)}) \to \mathrm{Ch}(X_{L(S)})$ is surjective. Therefore the old proof works. □

We apply Lemma 10.4 to our S and X (with $L = F(X)$). First we take $M = M(X)$ and using Theorem 10.2, we extract from $M(X)$ our first copy of shifted $M(S)$. Then we apply Lemma 10.4 again, taking for M the complementary summand of$M(X)$. Continuing this way, we eventually extract from $M(X)$ the same number of (shifted) copies of $M(S)$ as we have by Theorem 10.2 over $F(X)$. Let $\mathcal{R}$ be the remaining summand of $M(X)$. By uniqueness of decomposition, we have $\mathcal{R}_{F(X)} \simeq R$ so that $\mathcal{R}$ is the Rost motive. It is indecomposable (over F), because the degree of every closed point on X is divisible by p.

The uniqueness of the constructed decomposition follows by [1, Theorem 3.6 of Chapter I], because the endomorphism rings of $M(S)$ and of $\mathcal{R}$ are local (see [19, Lemma 3.3]). □

Remark 10.5. If X is an equivariant toroidal compactification of $\mathbf{SL}_1(D)$, the number of motives $M(S)$ in the decomposition of Theorem 10.3 is equal to $s(p-1)!-1$, where s is the number of cones of maximal dimension in the fan of the associated toric variety (see Theorem 6.5).

Example 10.6. Let X be the (non-toroidal) equivariant compactification of $\mathbf{SL}_1(D)$ with $p = 3$ considered in [26]. Since $P_X(t) = t^8 + t^7 + 2t^6 + 3t^5 + 4t^4 + 3t^3 + 2t^2 + t + 1$, we have

$$M(X) \simeq \mathcal{R} \oplus M(S)(1) \oplus M(S)(2) \oplus M(S)(3) \oplus M(S)(4) \oplus M(S)(5).$$

Example 10.7. Let X be the toroidal equivariant compactification of $\mathbf{SL}_1(D)$ with $p = 3$ considered in Example 6.7 in the split case. We have

$$M(X) \simeq \mathcal{R} \oplus M(S)(1)^{\oplus 3} \oplus M(S)(2)^{\oplus 5} \oplus M(S)(3)^{\oplus 7} \oplus M(S)(4)^{\oplus 5} \oplus M(S)(5)^{\oplus 3}.$$

Corollary 10.8. *Let E be a nonsplit $\mathbf{SL}_1(D)$-torsor. Assume that* $\operatorname{char} F = 0$. *Then* $\mathrm{CH}(E) = \mathbb{Z}$.

Proof. Since $p\,\mathrm{CH}^{>0}(E) = 0$, it suffices to prove that $\mathrm{CH}^{>0}(E) = 0$ for $\mathbb{Z}$-coefficients replaced by $\mathbb{Z}_{(p)}$-coefficients. Below CH stands for Chow group with $\mathbb{Z}_{(p)}$-coefficients.

We prove that $\mathrm{CH}(E) = \mathrm{CH}_D(E)$ by the argument of Corollary 9.2. It remains to show that $\mathrm{CH}_D^{>0}(E) = 0$.

Let X be a compactification of E as in Theorem 10.3. Since $\mathrm{CH}_D(X)$ surjects onto $\mathrm{CH}_D(E)$ and $\mathrm{CH}_D(S) = 0$, it suffices to check that $\mathrm{CH}_D^{>0}(\mathcal{R}) = 0$. Actually, we have $\mathrm{CH}_D(\mathcal{R}) \simeq \mathrm{CH}_D(E)$ (see Section 9). Moreover, the D-motive of $\mathcal{R}$ is isomorphic to $M^D(E)$.

The Chow group $\mathrm{CH}^{>0}(\mathcal{R})$ has been computed in [19, Appendix RM] (the characteristic assumption is needed here). The generators of the torsion part, provided in [19, Proposition SC.21], vanish in $\mathrm{CH}_D(\mathcal{R})$ by construction. The remaining generators are norms from a degree p splitting field of D so that they vanish in $\mathrm{CH}_D(\mathcal{R})$, too. Hence $\mathrm{CH}_D^{>0}(\mathcal{R}) = 0$ as required. □

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