ORTHOGONAL AND SYMPLECTIC GRASSMANNIANS OF DIVISION ALGEBRAS

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Abstract. We consider a central division algebra (over a field) endowed with a quadratic pair or with a symplectic involution and prove 2-incompressibility of certain varieties of isotropic right ideals of the algebra. This covers a recent conjecture raised by M. Zhykhovich. The remaining related projective homogeneous varieties are 2-compressible in general.

Let F be a field, $n \geq 1$, D a central division F-algebra of degree 2^n endowed with a *quadratic pair* σ (orthogonal case) or with a *symplectic involution* σ (symplectic case). For definitions as well as for basic facts about involutions on central simple algebras, we refer to [11]. We recall that in the characteristic $\neq 2$ case the notion of quadratic pair is equivalent to the notion of orthogonal involution.

For any integer i, we write X_i for the variety of *isotropic* (with respect to σ) right ideals in D of reduced dimension i. For any i, the variety X_i is smooth and projective. It is nonempty if and only if $0 \leq i \leq 2^{n-1}$ (X_0 is simply $\operatorname{Spec} F$) and is equidimensional in this case. Moreover, it is geometrically integral except the orthogonal case with $i = 2^{n-1}$. The variety $X_{2^{n-1}}$ in the orthogonal case is connected if and only if the *discriminant* of σ is nontrivial; otherwise it has two connected components.

For any i, the variety X_i is a closed subvariety of the generalized Severi-Brauer variety $\mathrm{SB}_i(D)$ – the variety of all right ideals in D of reduced dimension i. We recall that according to [10], for any $r = 0, 1, \ldots, n-1$, the variety $\mathrm{SB}_{2^r}(D)$ is 2-*incompressible*. This means, roughly speaking, that any correspondence $\mathrm{SB}_{2^r}(D) \rightsquigarrow \mathrm{SB}_{2^r}(D)$ of odd multiplicity is dominant. In particular, any rational map $\mathrm{SB}_{2^r}(D) \dashrightarrow \mathrm{SB}_{2^r}(D)$ is dominant.

The following theorem is the main result of this note. It extends to the symplectic case as well as to the characteristic 2 case a recent conjecture due to M. Zhykhovich, [16].

Theorem 1. *For any $r = 0, 1, \ldots, n-1$, excluding $r = n-1$ in the orthogonal case, the variety X_{2^r} is 2-incompressible.*

The proof will be given right after some preparation work. It extensively uses the notion of *upper motives* introduced in [10] and [9].

Example 12 shows that Theorem 1 precisely detects the types of those projective homogeneous varieties under the connected component $\mathrm{Aut}^0(D, \sigma)$ of the algebraic group $\mathrm{Aut}(D, \sigma)$, which are 2-incompressible *in general*, i.e., for any F, D and σ. Note that $\mathrm{Aut}^0(D, \sigma)$ is an absolutely simple adjoint affine algebraic group of type $\mathcal{D}_{2^{n-1}}$ in the orthogonal case and $\mathcal{C}_{2^{n-1}}$ in the symplectic case.

Date: 21 May 2012. Revised: 3 December 2012.

Key words and phrases. Algebraic groups, quadratic forms, projective homogeneous varieties, Chow groups and motives. *Mathematical Subject Classification (2010):* 14L17; 14C25.

The $r = 0$ case of Theorem 1 is already known:

Theorem 2 ([16])**.** *The Chow motive with coefficients in $\mathbb{F}_2$ of the variety X_1 is indecomposable. In particular, the variety X_1 is 2-incompressible so that Theorem 1 holds for $r = 0$.*

Sketch of proof. The degree of any closed point on X_1 is divisible by 2^n. Therefore, by [10, Lemma 2.21], the *rank* of any summand of the Chow motive (with coefficients in $\mathbb{F}_2$) of X_1 is divisible by 2^n. On the other hand, the rank of the total motive of X_1 is 2^n (note that $X_1 = \mathrm{SB}_1(D)$ in the symplectic case). The nilpotence principle [4, §92] concludes the proof. $\square$

We start the preparation for the proof of Theorem 1.

Lemma 3. *For any $r = 0, 1, \ldots, n-1$, excluding $r = n-1$ in the orthogonal case, the Schur index of D over the function field of X_{2^r} is equal to 2^r.*

Proof. The generic point of X_{2^r} is given by certain right ideal in $D_{F(X_{2^r})}$ of reduced dimension 2^r. Therefore, the index of $D_{F(X_{2^r})}$ divides 2^r. In particular, we may assume that $r \geq 1$ (so that $n \geq 3$ in the orthogonal case).

In the symplectic case, by the index reduction formula [12, III on Page 593], the index of $D_{F(X_{2^r})}$ is $\min\{2^n, 2^r\} = 2^r$.[1]

In the orthogonal case, if the discriminant of σ is trivial, we apply the index reduction formula [12, IV on Page 593].[1] Since $n \geq 2$, the index of $D_{F(X_{2^r})}$ is at least the minimum of $\{2^r,\ 2^{2^{n-1}-2^r-1}\}$. Since $r \leq n-2$, we have that $2^{n-1} - 2^r - 1 \geq 2^{n-2} - 1$. Since $2^{n-2} - 1 \geq n-2 \geq r$, the minimum is 2^r. We are done with the case of trivial discriminant.

Finally, if the discriminant of σ is nontrivial, we apply the index reduction formula [13, (9.49)]. We are in the case of this index reduction formula because $n \geq 3$ so that $2^r \leq 2^{n-2} < 2^{n-1} - 1$.[2] It follows that the index of $D_{F(X_{2^r})}$ is at least $\min\{2^r,\ 2^{2^{n-1}-2^r}\} = 2^r$. $\square$

In the case excluded in Lemma 3, we have the following partial information:

Lemma 4. *In the orthogonal case, assume that the discriminant of σ is trivial as well as a component of its Clifford algebra. Then the Schur index of D over the function field of an appropriate component of the variety $X_{2^{n-1}}$ is equal to 2^{n-1}. The Schur index of D over the function field of the other component of the variety $X_{2^{n-1}}$ is equal to 1.*

Proof. Apply the index reduction formula of [12, page 594].[2] $\square$

Here is an incompressibility result close to but outside of Theorem 1:

Lemma 5. *For $n \geq 2$ in the orthogonal case, assume that the discriminant of σ is trivial as well as a component of its Clifford algebra. The component of $X_{2^{n-1}}$ whose function field does not completely split D is a 2-incompressible variety.*

Proof. This is a particular case of Proposition A4. The condition of Proposition A4 concerning the absence of a multiplicity 1 correspondence to $\mathrm{SB}_{2^{n-2}}(D)$ holds by Lemma 4. $\square$

[1]For the characteristic 2 case see [12, Remark 5.10].

[2]For the characteristic 2 case see [13, Remark 9.2].

Remark 6. As shown in Example 12, the other component of $X_{2^{n-1}}$ may be 2-compressible.

The proof of Theorem 1 for $r = n-1$ (in the symplectic case) is very close to the proof of Lemma 5:

Lemma 7. *Theorem 1 holds for $r = n-1$.*

Proof. This is a particular case of Proposition A4. The condition of Proposition A4 concerning the absence of a multiplicity 1 correspondence to $\mathrm{SB}_{2^{n-2}}(D)$ holds by Lemma 3. □

We are working with Chow groups modulo 2. In particular, *multiplicities of correspondences*, [4, §75], take values in $\mathbb{F}_2 := \mathbb{Z}/2\mathbb{Z}$. The following result concerns both orthogonal and symplectic cases with a common proof:

Lemma 8. *Assume that for some $r = 0, 1, \ldots, n-2$ there is no multiplicity 1 correspondence $X_{2^r} \rightsquigarrow X_{2^{r+1}}$. Then the variety X_{2^r} is 2-incompressible.*

Proof. This is a particular case of Proposition A4. The condition of Proposition A4 concerning the absence of a multiplicity 1 correspondence to $\mathrm{SB}_{2^{r-1}}(D)$ holds by Lemma 3. □

Lemma 9. *In the orthogonal case, assume that $n \geq 3$ and that for some $r \in \{0, 1, \ldots, n-2\}$ there exists a multiplicity 1 correspondence $X_{2^r} \rightsquigarrow X_{2^{n-1}}$. Then the discriminant of σ is trivial as well as a component of its Clifford algebra.*

Proof. We set $X := X_{2^r}$. Since there exists a multiplicity 1 correspondence $X \rightsquigarrow X_{2^{n-1}}$, the variety $X_{2^{n-1}\,F(X)}$ has an odd degree closed point. Therefore the quadratic pair $\sigma_{F(X)}$ becomes *hyperbolic* over a finite odd degree field extension (which actually means that already $\sigma_{F(X)}$ is hyperbolic) and, in particular, its discriminant is trivial. Since the variety X is geometrically integral, the field F is algebraically closed in $F(X)$ and therefore the discriminant of σ is trivial.

Let Y be the Severi-Brauer variety of D. The quadratic pair $\sigma_{F(Y)}$ is adjoint to some quadratic form q of dimension 2^n and of trivial discriminant. Since the form q becomes hyperbolic over $F(Y)(X)$ (and $n \geq 3$), its Clifford algebra is trivial by the index reduction formula [12, IV on Page 593]. It follows (cf. [8, page 385]) that a component of the Clifford algebra of σ is trivial (the other component is Brauer-equivalent to D by [11, (9.14)]). □

Before proving the general case of Theorem 1, as a warm up, we prove its extreme case (among yet unproved ones) opposite to the case of Theorem 2:

Proposition 10. *Theorem 1 holds for $r = n-2$.*

Proof. The case of $n = 2$ being done by Theorem 2, we assume that $n \geq 3$ below. By Lemma 8, we may assume that there exists a multiplicity 1 correspondence $X_{2^{n-2}} \rightsquigarrow X_{2^{n-1}}$. If we are in the orthogonal case, it follows by Lemma 9 (with $r = n-2$) that the discriminant of σ is trivial as well as a component of its Clifford algebra. It follows by Lemma 4 in the orthogonal case and by Lemma 3 in the symplectic case that there exists a field extension E/F (the function field of an appropriate component of $X_{2^{n-1}}$) such that the involution σ_E is hyperbolic and the Schur index of D_E is 2^{n-1}.

For $X := X_{2^{n-2}}$, we have $\operatorname{ind} D_{F(X)} = 2^{n-2}$ by Lemma 3. By Lemma A3 (applied twice), the complete motivic decomposition of $X_{F(X)}$ contains four Tate summands: $\mathbb{F}_2$, $\mathbb{F}_2(4^{n-2})$, $\mathbb{F}_2(\dim X - 4^{n-2})$, $\mathbb{F}_2(\dim X)$. Note that $\dim X = 2^{2n-3} + 2^{2n-5} - 2^{n-3}$ in the orthogonal case and $\dim X = 2^{2n-3} + 2^{2n-5} + 2^{n-3}$ in the symplectic case, so that in any case $\dim X > 2^{2n-3}$ and therefore $4^{n-2} < \dim X - 4^{n-2}$, showing that the four Tate summands have pairwise different shifts.

Each of the remaining summands of the complete motivic decomposition of $X_{F(X)}$ is of even rank because by Lemma A3 it is a summand of the motive of an anisotropic variety. (The definition of anisotropic variety is given right before Lemma Lemma A3; the rank of any summand of the motive of an anisotropic variety is even by [10, Lemma 2.21]). For the upper motive $U(X)$, we are going to show that $U(X)_{F(X)}$ contains all the 4 Tate summands; this will imply that X is 2-incompressible, cf. [7, Theorem 5.1].

By definition of $U(X)$, $U(X)_{F(X)}$ contains the Tate summand $\mathbb{F}_2$.

By Corollary A6, $U(X)_{F(X)}$ contains the Tate summand $\mathbb{F}_2(\dim X - 4^{n-2})$.

Let C be a central division E-algebra (of degree 2^{n-1}) Brauer-equivalent to D_E. Since there exist multiplicity 1 correspondences $X_E \leftrightsquigarrow \mathrm{SB}_{2^{n-2}}(C)$, the upper motive of the variety X_E is isomorphic to the upper motive of $\mathrm{SB}_{2^{n-2}}(C)$, [10, Corollary 2.15]. Since the variety $\mathrm{SB}_{2^{n-2}}(C)$ is 2-incompressible and has dimension 4^{n-2}, $U(X_E)_{E(X)}$ contains the Tate summand $\mathbb{F}_2(4^{n-2})$. In particular, $U(X)_{E(X)}$ contains this Tate summand. Since the field extension $E(X)/F(X)$ is purely transcendental, $U(X)_{F(X)}$ contains the Tate summand $\mathbb{F}_2(4^{n-2})$.

Finally, since $U(X)$ has even rank, $U(X)_{F(X)}$ contains the remaining (fourth) Tate summand $\mathbb{F}_2(\dim X)$. □

For the general case of Theorem 1 we need one more observation:

Lemma 11. *For some $r = 0, 1, \ldots, n-2$, let us consider the biggest i such that there exists a multiplicity 1 correspondence $X_{2^r} \rightsquigarrow X_i$. Then $i = 2^s$ for some $s \in \{r, r+1, \ldots, n-1\}$.*

Proof. Assuming that $i > 2^s$ for some $s = r, r+1, \ldots, n-2$, we show that $i \geq 2^{s+1}$. Since $i > 2^s$, we have $s \leq n-2$. Therefore $\operatorname{ind} D_L = 2^s$ for $L := F(X_{2^s})$ by Lemma 3.

Let I be an isotropic right ideal of reduced dimension 2^s in D_L. Let $C := \operatorname{End}_A I$ so that C is a central division L-algebra of degree 2^s Brauer-equivalent to D. Let A be a central simple L-algebra obtained out of I by Construction A2. Let X be the variety of isotropic right ideals in A of reduced dimension 2^r. The upper motives of X and of $\mathrm{SB}_{2^r}(C)$ are isomorphic. Since $\mathrm{SB}_{2^r}(C)$ is 2-incompressible and has dimension $d := 2^r(2^s - 2^r)$, the motive of $X_{L(X)}$ contains the Tate motive $\mathbb{F}_2(d)$ as a summand. It follows by Lemma A3 that the maximum of the Witt index of the quadratic pair (resp., symplectic involution) on A_E for E running over finite odd-degree field extensions of $L(X)$ is at least 2^s. Therefore the maximum of the Witt index of σ_E is at least $2^s + 2^s = 2^{s+1}$ and it follows that $i \geq 2^{s+1}$. □

Proof of Theorem 1. By Lemma 7 we may assume that $r \leq n-2$.

We set $X := X_{2^r}$. Let i be the maximal integer such that there exists a multiplicity 1 correspondence $X \rightsquigarrow X_i$. By Lemma 11, $i = 2^s$ for some $s \in \{r, r+1, \ldots, n-1\}$.

By Lemma 3, $\operatorname{ind} D_{F(X)} = 2^r$. By Lemma A3 (applied 2^{s-r} times), the complete motivic decomposition of the variety $X_{F(X)}$ contains the Tate summands with the shifts

$j4^r$ and $\dim X - j4^r$ for $j = 0, 1, \dots, 2^{s-r} - 1$ (precisely one Tate summand for each shifting number). Note that $(2^{s-r} - 1)4^r < \dim X - (2^{s-r} - 1)4^r$ so that the shifting numbers are pairwise different. Each of the remaining summands in the complete motivic decomposition of $X_{F(X)}$ is of even rank. For the upper motive $U(X)$ it suffices to show that $U(X)_{F(X)}$ contains the Tate summand $\mathbb{F}_2(\dim X)$.

By Corollary A6, $U(X)_{F(X)}$ contains the Tate summand $\mathbb{F}_2(\dim X - (2^{s-r} - 1)4^r)$.

Let us check now that $\operatorname{ind} D_{F(Y)} = 2^s$ for an appropriate component Y of the variety X_{2^s}. Indeed, if $s < n - 1$, this is so (for $Y = X_{2^s}$) by Lemma 3. Is $s = n - 1$, it suffices to apply Lemmas 9 and 4.

Let C be a central division $F(Y)$-algebra of degree 2^s Brauer-equivalent to $D_{F(Y)}$. The upper motives of the varieties $X_{F(Y)}$ and $S := \mathrm{SB}_{2^r}(C)$ are isomorphic. Passing to the dual motives and shifting, we get that

$$U(X_{F(Y)})^*(\dim X) \simeq U(S)^*(\dim X).$$

Since the variety S is 2-incompressible, the motive $U(S)_{F(Y)(X)}$ contains the Tate summands $\mathbb{F}_2$ and $\mathbb{F}_2(\dim S)$. Consequently, $U(S)^*_{F(Y)(X)}(\dim X)$ contains the Tate summands $\mathbb{F}_2(\dim X)$ and $\mathbb{F}_2(\dim X - \dim C)$. In particular, $U(X)^*_{F(Y)(X)}(\dim X)$ contains both Tate summands. Since the field extension $F(Y)(X)/F(X)$ is purely transcendental, $U(X)^*_{F(X)}(\dim X)$ contains both Tate summands. Note that $\dim S = (2^{s-r} - 1)4^r$ and $U(X)^*(\dim X)$ is an indecomposable summand of $M(X)$. Since $U(X)_{F(X)}$ also contains the Tate summand $\mathbb{F}_2(\dim X - \dim S)$, the Krull-Schmidt principle of [3] (see also [9]) tells us that $U(X) \simeq U(X)^*(\dim X)$ and therefore $U(X)_{F(X)}$ contains $\mathbb{F}_2(\dim X)$ as desired. □

The following Example shows that for $G = \mathrm{Aut}^0(D, \sigma)$, the varieties listed in Theorem 1 are the only projective G-homogeneous varieties which are 2-incompressible in general. We recall that in the symplectic case, an arbitrary projective G-homogeneous variety is isomorphic to the variety $X_{l_1 \dots l_k}$ of flags of isotropic right ideals in D of some fixed reduced dimensions $1 \leq l_1 < \dots < l_k \leq 2^{n-1}$ with some $k \geq 1$. In the orthogonal case with nontrivial $\operatorname{disc} \sigma$, an arbitrary projective G-homogeneous variety is isomorphic to the flag variety $X_{l_1 \dots l_k}$ with the additional restriction $l_k < 2^{n-1}$. If $\operatorname{disc} \sigma$ is trivial, one has to add components of $X_{l_1 \dots l_k}$ with $l_k = 2^{n-1}$.

Example 12. For any given $n \geq 2$, let us consider over an appropriate field F of characteristic $\neq 2$, a central division F-algebra D of degree 2^n endowed with an orthogonal or a symplectic involution σ such that (D, σ) is the tensor product of n quaternion algebras with involutions. By [7, Theorem 3.8] based on the celebrated result of K. Becher [1], for any field extension L/F, the involution σ_L is anisotropic or hyperbolic. An arbitrary projective G-homogeneous variety is isomorphic to the variety $X_{l_1 \dots l_k}$ with some $k \geq 1$ and some $1 \leq l_1 < \dots < l_k \leq 2^{n-1}$. Its upper motive is isomorphic to $U(X_{2^r})$, where 2^r is the largest 2-power dividing $l_1, \dots, l_k$. Hence $\{l_1, \dots, l_k\} = \{2^r\}$ if the variety $X_{l_1 \dots l_k}$ is 2-incompressible (because $\dim X_{l_1 \dots l_k} > \dim X_{2^r}$ otherwise). This accomplishes Example 12 in the symplectic case.

In the orthogonal case, e.g. by Lemma 9, the discriminant of σ is trivial as well as a component of its Clifford algebra. Let X be the component of the variety $X_{2^{n-1}}$ whose function field splits D (cf. Lemma 4). We have $U(X) \simeq U(X_1)$. Since $\dim X_1 = 2^n - 2$

and $\dim X = 2^{n-2}(2^{n-1}-1)$, we have $\dim X_1 < \dim X$ provided that $n \neq 3$. Therefore the variety X is 2-compressible for $n \neq 3$.

Appendix. Quadric-like behavior

In this Appendix we establish some results on grassmannians of isotropic ideals which are very close (in the statement as well as in the proof) to results on projective quadrics in the spirit of [15].

Let F be a field, A a central simple F-algebra endowed with a quadratic pair $\sigma = (\tau, f)$ (where τ is an involution on A and f a linear map of the vector space of τ-symmetric elements of A to F, satisfying the conditions of [11][Definition 5.4]) or with a symplectic involution σ. In order to get unified formulas, we will also write τ for σ in the symplectic case.

For a right ideal $J \subset A$, its orthogonal complement $J^\perp$ is defined as the (right) annihilator of the left ideal $\tau(J)$. This is a right ideal of reduced dimension $\operatorname{rdim} J^\perp = \deg A - \operatorname{rdim} J$, [11, Proposition 6.2].

A right ideal J is *nondegenerate* if $J \cap J^\perp = 0$. The following two construction in the orthogonal characteristic 2 case are close to [5, Page 379] (and are well known and easily obtained otherwise).

Construction A1. Given a nondegenerate right ideal $J \subset A$, the right A-module A is a direct sum of the submodules J and $J^\perp$. The image $e \in J$ of $1 \in A$ with respect to the projection $A \to J$ is a symmetric (with respect to the involution τ) idempotent generating J: $\tau(e) = e$, $e^2 = e$, and $J = eA$. The F-algebra $\operatorname{End}_A J$ is identified with the subalgebra eAe of A (see [11, Corollary 1.13]) stable under the involution τ. (Note that the unit of the algebra eAe is the element e which may differ from the unit 1 of A so that the unital algebra eAe is, in general, not a unital subalgebra of A.) In the symplectic case, the F-algebra eAe turns out to be endowed this way with a symplectic involution – the restriction of $\sigma = \tau$. In the orthogonal case, the restriction of the involution τ together with the restriction of the linear map f is a quadratic pair on eAe. Note that the degree of the algebra eAe is equal to the reduced dimension of the ideal J.

In contrast to [11], we define the (Witt) index $\operatorname{ind}\sigma$ of σ as the maximum of reduced dimension of an isotropic right ideal in A. The information given by the Witt index of σ in the sense of [11], or equivalently by the Tits index of the algebraic group $\operatorname{Aut}^0(A,\sigma)$, is equivalent to the information given by $\operatorname{ind}\sigma$ and $\operatorname{ind} A$.

Construction A2. Given an isotropic right ideal I in A, we have $I \subset I^\perp$. Let us choose an ideal $J \subset I^\perp$ such that $I^\perp = I \oplus J$. The ideal J is nondegenerate so that, using Construction A1, we get the algebra eAe with restriction of σ. Note that $\deg(eAe) = \operatorname{rdim} J = \deg A - 2\operatorname{rdim} I$. The (Witt) index of this restriction is equal to $\operatorname{ind}\sigma - \operatorname{rdim} I$. Construction A1 applied to the ideal $J^\perp$ produces an algebra with hyperbolic quadratic pair / symplectic involution.

A variety is called *anisotropic* here if every its closed point has even degree. The following statement in the case of $\operatorname{ind} A = 1$ is the motivic decomposition [4, Proposition 70.1] of smooth projective quadrics, observed originally by M. Rost:

Lemma A3. *Let I be an isotropic ideal of reduced dimension* $\operatorname{ind} A$ *in A. Let X be the variety of isotropic right ideals of reduced dimension* $\operatorname{ind} A$ *in A. Let B be an algebra eAe given by Construction A2. Let Y be the variety of isotropic right ideals of reduced dimension* $\operatorname{ind} A = \operatorname{ind} B$ *in B (Y is nonempty iff* $\deg A \geq 4 \operatorname{ind} A$*). Then there exists a motivic decomposition of X with summands $\mathbb{F}_2$, $\mathbb{F}_2(\dim X)$, and – in the case of nonempty Y – $M(Y)(2\operatorname{ind} A) = M(Y)((\dim X - \dim Y)/2)$ such that each of the remaining summands of the decomposition is the motive of an anisotropic variety.*

Proof. For the case of characteristic $\neq 2$ see [6, Corollary 15.14]. The general case is in [2]. □

For any integer i, we write X_i for the variety of isotropic right ideals in A of reduced dimension i. Such a variety has at most one component except for $i = (\deg A)/2$ in the orthogonal case with trivial discriminant.

Proposition A4 and Corollary A6 below are analogues of computation of canonical 2-dimension of smooth projective quadrics [4, Theorem 90.2]. We refer to [7] for definition and basic properties of canonical dimension.

Proposition A4. *For some $r \geq 0$, assume that a component X of the variety X_{2^r} is anisotropic and has no multiplicity 1 correspondence to $X_{2^{r+1}}$. For $r \geq 1$ we additionally assume that there is no multiplicity 1 correspondence $X \rightsquigarrow \mathrm{SB}_{2^{r-1}}(A)$. Then the variety X is 2-incompressible.*

Proof. The index of $A_{F(X)}$ is 2^r (it divides 2^r in general and, if $r \geq 1$, does not divide 2^{r-1} because of absence of a multiplicity 1 correspondence $X \rightsquigarrow \mathrm{SB}_{2^{r-1}}(A)$) and the $F(X)$-variety Y as in Lemma A3 is anisotropic (because of absence of a multiplicity 1 correspondence $X \rightsquigarrow X_{2^{r+1}}$). It follows that all summands of the complete motivic decomposition of the variety $X_{F(X)}$ but $\mathbb{F}_2$ and $\mathbb{F}_2(\dim X)$ have even ranks. On the other hand, since X is anisotropic, the motive $U(X)$ is also of even rank. It follows that $U(X)_{F(X)}$ contains $\mathbb{F}_2(\dim X)$. Therefore X is 2-incompressible. □

Lemma A5. *Excluding*

- *the symplectic case with split A and*
- *the orthogonal case with split A in characteristic 2,*

for any multiple m of $\operatorname{ind} A$ *satisfying $0 \leq m \leq \deg A$, there exists a nondegenerate right ideal in A of reduced dimension m.*

Proof. In the orthogonal case, the statement has nothing to do with the component f of the quadratic pair σ. So, we work with the involution τ (in the orthogonal case as well as in the symplectic case). We write $A = \operatorname{End}_D V$ for some central division algebra D with a fixed involution of the same type as τ and a right D-module V with a hermitian form h such that τ is adjoint to h. By [14, Theorem 6.3 of Chapter 7], since the case of symplectic τ with split A is excluded (note that in characteristic 2, the involution τ is symplectic in the orthogonal case as well), h can be diagonalized. □

Corollary A6. *Let us exclude the symplectic case with split A as well as the orthogonal case in characteristic 2 with split A. Let r be such that* $\operatorname{ind} A = 2^r$*. Assume that a component X of the variety X_{2^r} is anisotropic and – if $r > 0$ – has no multiplicity*

1 *correspondence to* $\mathrm{SB}_{2^{r-1}}(A)$. *Let* i *be the maximal integer such that there exists a multiplicity* 1 *correspondence* $X \rightsquigarrow X_{(i+1)2^r}$. *Then the canonical* 2*-dimension* $\operatorname{cdim}_2 X$ *of* X *is equal to* $\dim X - i4^r$. *In particular,* $U(X)_{F(X)}$ *contains the Tate motive* $\mathbb{F}_2(\dim X - i4^r)$ *as a summand.*

Proof. For $i = 0$ simply apply Proposition A4. Below we assume that $i > 0$. In particular, $X = X_{2^r}$.

Let $J \subset A$ be a nondegenerate right ideal of reduced dimension $\deg A - i2^r$ (existing by Lemma A5). Let B be the corresponding nonunital subalgebra of A (obtained by Construction A1) and let Y_{2^r} be the variety of isotropic right ideals of reduced dimension 2^r in B. Since there is a multiplicity 1 correspondence $X \rightsquigarrow X_{(i+1)2^r}$, there is a multiplicity 1 correspondence from X to an appropriate component Y of Y_{2^r}. Note that there also is a multiplicity 1 correspondence $Y \rightsquigarrow X$ so that $U(X) \simeq U(Y)$ and it follows by [7, Theorem 5.1] that $\operatorname{cdim}_2 X = \operatorname{cdim}_2 Y$.

The variety Y satisfies conditions of Proposition A4: it has no multiplicity 1 correspondence neither to $Y_{2^{r+1}}$ nor to $\mathrm{SB}_{2^{r-1}}(B)$. Therefore Y is 2-incompressible. It follows that $U(Y)_{F(Y)}$ as well as $U(X)_{F(X)}$ contain $\mathbb{F}_2(\dim Y) = \mathbb{F}_2(\dim X - i4^r)$ as a summand. $\square$

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