FINITE EXTENSIONS PARTIALLY SPLITTING PGO-TORSORS

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Abstract. We determine the g.c.d. of degrees of all finite base field extensions, performing a prescribed partial splitting of a generic central simple algebra with a quadratic pair, or, equivalently, of a generic torsor under a split projective orthogonal group. We also compute this invariant for the generic central simple algebras with quadratic pairs *of trivial discriminant*, or, equivalently, for the generic torsors under the split projective *special* orthogonal groups.

Contents

1. Introduction

Given a field F and an integer $n \geq 1$, let A be a central simple F-algebra of degree $\deg(A) = 2n$ endowed with a *quadratic pair* σ (see [25, §5.B]). If the characteristic of F is different from 2, then σ is an orthogonal involution on A. More precisely, the first component of the quadratic pair σ is an orthogonal involution and the second component – a map with certain properties of the set of σ-symmetric elements to F – is determined by the first one. If $\operatorname{char} F = 2$, then the first component of σ is a symplectic involution on A whereas the second component constitutes an additional datum.

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The level of splitting of (A, σ) is measured by two parameters: the Schur index $i(A)$ of A, defined as the minimal reduced dimension of a nonzero right ideal in A, and the Witt index $i(\sigma)$ of σ, defined as the maximal reduced dimension of a σ-isotropic right ideal in A. (We refer to [25, Definition 6.5] for the definition of a σ-isotropic ideal.) The first parameter $i(A)$ is a 2-power dividing $2n$, the second parameter $i(\sigma)$ is a non-negative multiple of $i(A)$ non-exceeding n. Algebra A is *split*, if $i(A) = 1$; quadratic pair σ is *hyperbolic*, if $i(\sigma) = n$; the duo (A, σ) is *split*, if A is split and σ is hyperbolic.

The Witt index of a quadratic pair generalizes the notion of the Witt index of a quadratic form: if A is the algebra of endomorphisms of a $2n$-dimensional F-vector space V, then $i(A) = 1$, σ is adjoint to a non-degenerate quadratic form q on V, the involution of σ is adjoint to the bilinear form associated with q, and $i(\sigma)$ is the Witt index of q.

For every integer m satisfying $1 \leq m \leq n$, let X_m be the variety of pairs of σ-isotropic right ideals $I \subset J$ in A of reduced dimensions 1 and m. In particular, X_1 is the involution variety of (A, σ). The variety X_m has a rational point if and only if $i(A) = 1$ (i.e., A is split) and $i(\sigma) \geq m$ (i.e., σ is *m-isotropic*). Note that m-isotropic σ is also m'-isotropic for any non-negative $m' \leq m$ divisible by $i(A)$. Besides, let us note that the variety X_m is smooth and projective. It is also connected except when $m = n$ and the discriminant of σ is trivial; in the excepted case it consists of two connected components.

We are interested in information on the index $i(X_m)$, where the index of a variety is defined as the g.c.d. of degrees of its closed points. Note that the varieties X_m are partial splitting varieties of (A, σ) splitting the algebra A completely. More general varieties of pairs of embedded ideals, where the smaller ideal is allowed to have reduced dimension any 2-power dividing $2n$ (and the reduced dimension of the larger ideal), are also interesting but more difficult to study.

Clearly, every index in the sequence $i(A), i(X_1), \ldots, i(X_n)$ is a 2-power and divides the next one. If $X_m(K) \neq \emptyset$ for some $m = 1, \ldots, n-1$ and some extension field K of F, the quadratic pair σ_K is adjoint to a quadratic form of Witt index $\geq m$; there is a finite field extension L/K of degree dividing 2 over which the Witt index increases giving $X_{m+1}(L) \neq \emptyset$; it follows that the quotient $i(X_{m+1})/i(X_m)$ divides 2. By a similar reason, the quotient $i(X_1)/i(A)$ also divides 2. In terms of the base two logarithm $\mathfrak{l}(-)$ of $i(-)$, we have $\mathfrak{l}(A) \leq \mathfrak{l}(X_1) \leq \cdots \leq \mathfrak{l}(X_n)$ with every successive difference being 0 or 1.

In the case of $\mathfrak{l}(A) > 0$, it turns out (see Corollary 2.3) that the underlying division algebra of A contains a maximal subfield over which σ is 1-isotropic. In particular, $\mathfrak{l}(X_1) = \mathfrak{l}(A)$.

Now we allow the field F to vary among all extensions of a given field k and, for every m, look for the sharp upper bound $\mathfrak{l}(m)$ on $\mathfrak{l}(X_m)$. It can be obtained by the generic construction described below.

For the split projective (general) orthogonal group $G := \mathrm{PGO}(2n)$ over k, the isomorphism classes of G-torsors over any extension field F of k are in canonical bijection with the isomorphism classes of the duos (A, σ) as above (see [25, §29.F]). A *generic G-torsor*, defined as the generic fiber of the quotient map $\mathrm{GL}(N) \to \mathrm{GL}(N)/G$ for some embedding of G into the general linear group $\mathrm{GL}(N)$ with some N, yields the isomorphism class of a central simple F-algebra A with a quadratic pair σ, where F is the function field $k(\mathrm{GL}(N)/G)$ of the quotient variety. We call such duo (A, σ) *generic*. The Schur index

$i(A)$ is the 2-primary part of $2n$ and the Witt index $i(\sigma)$ is 0 (i.e., σ is anisotropic) – see Remark 5.4. The value $\mathfrak{l}(X_m)$ is the sharp upper bound $\mathfrak{l}(m)$ we are looking for (see Remark 3.4). Our goal becomes to determine the indexes $i(X_2),\dots,i(X_n)$ for generic (A,σ). (As explained above, the index $i(X_1)$ is already determined in a more general context.)

The last of them $i(X_n)$ is the *torsion index* of the group $\mathrm{PGO}(2n)$ (see §4). An evident lower bound for it is the torsion index of the orthogonal group $\mathrm{O}(2n)$ equal to 2^n. The computation of the torsion index for $\mathrm{PGO}^+(2n)$ – the connected component of the group $\mathrm{PGO}(2n)$, made in [33, Theorem 7.1], implies that the lower bound is the exact value in most cases. The precise statement is (see Proposition 4.2):

$$\mathfrak{l}(n)=\begin{cases} n+1, & \text{if } n \text{ is a 2-power} \geq 2;\\ n, & \text{otherwise.}\end{cases}$$

Since we know the values $\mathfrak{l}(1)$ and $\mathfrak{l}(n)$, we know the number of $i\in\{2,\dots,n\}$ satisfying $\mathfrak{l}(i-1)=\mathfrak{l}(i)$. We call such i a *relaxed position*. The number of relaxed positions is $\mathfrak{l}(A)-2$ for n a 2-power ≥ 2; otherwise the number of relaxed positions is $\mathfrak{l}(A)-1$. Determine the entire sequence $\mathfrak{l}(1),\dots,\mathfrak{l}(n)$ means to find all relaxed positions. This is done in our final main result – Theorem 6.6, stating that the relaxed positions are the positions of the form 2^s-1 for a 2-power divisor $2^s\geq 2$ of $2n$. It follows that

$$\boxed{\mathfrak{l}(m)=m+r-\min\{r,\lfloor\log_2(m+1)\rfloor\}}$$

for any $m\in\{1,\dots,n\}$, where 2^r is the 2-primary part of $2n$ and where $\lfloor\log_2(m+1)\rfloor$ is the integral part (floor) of the base 2 logarithm.

The preparatory main result of the present text is Theorem 3.3, which provides basic information needed to determine $\mathfrak{l}(m)$ for any given m. Namely, it describes a finite system of generators for the Chow ring $\mathrm{CH}(X_m)$ in the case of a generic (A,σ). Since the variety X_m is projective, its index is determined by its Chow group of 0-cycles $\mathrm{CH}_0(X_m)$: $i(X_m)$ is the positive generator of the image of the degree homomorphism $\mathrm{CH}_0(X_m)\to\mathbb{Z}$. To get the actual value of $\mathfrak{l}(m)$, calculations are needed. But at least one sees right away that the answer does not depend on the field k (and, in particular, on its characteristic) – see Corollary 3.10.

As a warm up, for $n\geq 2$, using the description of $\mathrm{CH}(X_1)$, given in Theorem 3.3, we show that $\mathfrak{l}(2)=\mathfrak{l}(1)+1=\mathfrak{l}(A)+1$ (see Corollary 5.5). In other terms, the position 2 is not relaxed (as confirmed later by Theorem 6.6).

In the final §7 we perform the similar study for the split projective special orthogonal group $\mathrm{PGO}^+(2n)$ replacing $\mathrm{PGO}(2n)$. By Theorem 7.1, the answer for $\mathrm{PGO}^+(2n)$ is almost the same as for $\mathrm{PGO}(2n)$.

Note that $\mathrm{PGO}^+(2n)$ is the adjoint split simple group of type D_n. The simply connected split simple group of type D_n – the group $\mathrm{Spin}(2n)$ – has been studied a lot (see [2] and [24]). After heavy computations of [10] and [12], a complete algorithm to determine the indexes of the corresponding partial splitting varieties has been obtained in [11, Theorem 2.3 with Remark 2.6]. However, for a general n, a formula for the corresponding indexes is still unavailable. Such an algorithm (still without the formula) for $\mathrm{Spin}(2n+1)$ – the simply connected split simple group of type B_n – has been obtained in [20].

Terminology. A variety is a separated scheme of finite type over field. An (affine) algebraic group is an affine group scheme of finite type over a field.

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2. Index of X_1

Proposition 2.1. *For a field F, let A be a central* division *F-algebra endowed with a quadratic pair σ. Assume that $i(A) = \deg(A) > 1$. Then A contains a maximal subfield over which σ is 1-isotropic.*

Proof. Let X_1 be the involution variety of (A, σ). The pull-back homomorphism of the Chow rings $\mathrm{CH}(X_1 \times X_1) \to \mathrm{CH}(X_1)$ with respect to the diagonal embedding $X_1 \hookrightarrow X_1 \times X_1$ maps the class of the diagonal to a 0-cycle class. For an arbitrary smooth variety in place of X_1, this class coincides with the top Chern class of the tangent bundle (see [6, Self-Intersection Formula following Corollary 6.3]). For a smooth projective variety, the degree of this class is known as the *index of self-intersection of the diagonal*. We claim that for X_1, this index is equal to $\deg(A)$.

Since the index does not change under base field extensions, it is enough to compute it over an algebraic closure, where the variety X_1 becomes a (split) projective quadric Q of even dimension $2d := \deg(A) - 2$. The claim follows from [4, Proof of Lemma 78.1].

Let us indicate one more way to prove the claim. Since the Chow motive of the smooth projective variety Q is *split* (i.e., isomorphic to a finite direct sum of shifted motives of the point), the Chow group $\mathrm{CH}(Q)$ is free of finite rank equal to the index of self-intersection of the diagonal (cf. [16, Lemma 2.21]). The motive of Q is split because the variety Q is cellular (see [4, §66]).

By [6, §12.2] (see also [14, Theorem 2.3 and Remark 2.4]), the diagonal's self-intersection class of X_1 can be represented by a non-negative 0-cycle. Therefore X_1 contains a closed point x such that $\deg(A)$ is the degree of the residue field K of x. Since $i(A_K) = 1$, K is isomorphic to a maximal subfield of A ([3, Corollary 3 of §9]. □

Remark 2.2 (The quaternion case). If A in Proposition 2.1 is quaternion, the discriminant of σ is non-trivial by [25, Example 7.4]. If σ is isotropic over a maximal subfield K of A, the discriminant of σ_K is trivial. Therefore K/F is the quadratic field extension given by the discriminant of σ.

Corollary 2.3. *For a field F, let A be a central simple F-algebra endowed with a quadratic pair σ. Assume that $i(A) > 1$. Then the division F-algebra Brauer-equivalent to A contains a maximal subfield over which σ is isotropic.*

Proof. Using [29, Theorem 6.3], one finds a symmetric idempotent $e \in A$ such that eAe is a division algebra Brauer-equivalent to A. The quadratic pair σ on A restricts to a quadratic pair on eAe. By Proposition 2.1, we can find a maximal subfield $K \subset eAe$ and a nonzero isotropic right ideal in $(eAe)_K$. It generates a nonzero isotropic right ideal in A_K. □

Remark 2.4. The following will be shown in Corollary 5.5: if $\deg A = 2n$ with $n \geq 3$ and the duo (A, σ) is generic, then for any maximal subfield K of the division algebra Brauer-equivalent to A the quadratic pair σ_K is not 2-isotropic.

Corollary 2.5. *For a field F, let A be a central simple F-algebra endowed with a quadratic pair σ and let X_1 be the corresponding involution variety. Assume that $i(A) > 1$. Then the index $i(X_1)$ of the variety X_1 is equal to $i(A)$.* □

Remark 2.6. Corollary 2.5 in particular applies to a generic (A, σ) with any $n \geq 1$ because $i(A)$ is divisible by 2. The resulting equality $i(A) = i(X_1)$ can be proved directly by looking at the diagonal's self-intersection index of X_1.

Remark 2.7. For any $m = 1, \ldots, n$, the diagonal's self-intersection index of X_m is equal to $2^m \cdot \binom{n}{m} \cdot m$ (c.f. [1, Proof of Theorem 3.2]). The 2-primary part of this product is an upper bound for $i(X_m)$, but (aside from the case of $m = 1$) we get better bounds by other means.

3. Index of X_m

For any (not necessarily generic) (A, σ) with $\deg(A) = 2n$ and any $m \in \{1, \ldots, n\}$, the variety X_m, being a variety of 2-flags, has two tautological vector bundles: $\mathcal{I}$ of rank $2n$ and $\mathcal{J}$ of rank $2nm$. The fiber of $\mathcal{I}$ over a point $I \subset J$ of X_m is I and the fiber of $\mathcal{J}$ is J. Both vector bundles are right A-modules. The tensor products $\mathcal{I} \otimes_A \mathcal{I}$ and $\mathcal{J} \otimes_A \mathcal{I}$, where the right factor is a left A-module via the involution of σ, are vector bundles on X_m of rank 1 and m. We write C for the subring in $\mathrm{CH}(X_m)$ generated by the Chern classes of $\mathcal{I}$, $\mathcal{I} \otimes_A \mathcal{I}$, and $\mathcal{J} \otimes_A \mathcal{I}$.

Remark 3.1. To make a thriftier definition of C, one can replace the third vector bundle by $(\mathcal{J}/\mathcal{I}) \otimes_A \mathcal{I}$. To make it closer to [33, Proof of Theorem 7.1], one can use the vector bundle $\mathcal{H}om_A(\mathcal{I}, \mathcal{J}/\mathcal{I})$ instead.

Remark 3.2. If A is the endomorphism algebra $\mathrm{End}(V)$ of a $2n$-dimensional vector space V and σ is adjoint to a quadratic form on V, the variety X_m is the variety of flags of totally isotropic subspaces $U \subset W$ in V of dimensions 1 and m. Such a variety comes equipped with two tautological vectors bundles $\mathcal{U}$ and $\mathcal{W}$ of ranks 1 and m. The vector bundle $\mathcal{I}$ is then the direct sum of $2n$ copies of $\mathcal{U}$. Besides,

$$\mathcal{I} \otimes_A \mathcal{I} = \mathcal{U} \otimes \mathcal{U}, \quad \mathcal{J} \otimes_A \mathcal{I} = \mathcal{W} \otimes \mathcal{U}, \quad (\mathcal{J}/\mathcal{I}) \otimes_A \mathcal{I} = (\mathcal{W}/\mathcal{U}) \otimes \mathcal{U}, \text{ and}$$
$$\mathcal{H}om_A(\mathcal{I}, \mathcal{J}/\mathcal{I}) = \mathcal{H}om(\mathcal{U}, \mathcal{W}/\mathcal{U}).$$

Theorem 3.3. *For generic (A, σ), the inclusion $C \subset \mathrm{CH}(X_m)$ is an equality.*

Remark 3.4. Theorem 3.3 shows that the Chow ring of X_m is smallest possible in the generic case. This is why the index of X_m is maximal, i.e., $i(X_m) = 2^{\mathfrak{l}(m)}$ for X_m given by a generic (A, σ). Accurately speaking, in the above argument, we compare Chow rings of two varieties defined over possibly different extension fields F and F' of k, by looking at their images under the change of field homomorphisms to a common splitting field containing F and F'.

Remark 3.5. Providing generators for $\mathrm{CH}(X_m)$, Theorem 3.3 does not describe relations between them. So, unlike [18, Theorem 6.1], it does not provide a complete description of the Chow ring.

Proof of Theorem 3.3. By definition (see, e.g., [25, §28.B]), the group $G := \mathrm{PGO}(2n)$ is the group of the automorphisms of the duo (A,σ), where A is the degree $2n$ standard split central simple F-algebra and σ is the quadratic pair adjoint to the dimension $2n$ standard split quadratic form. The group G acts transitively on the corresponding variety X_m, described in Remark 3.2. The stabilizer of a rational point is a parabolic subgroup $P \subset G$ with the quotient G/P variety isomorphic to X_m.

Now let E be a generic G-torsor giving a generic algebra with quadratic pair (A,σ). The variety X_m corresponding to this new (A,σ) is identified with the quotient E/P. By [17, Proof of Lemma 2.1], the natural ring homomorphism

$$\mathrm{CH}(\mathsf{B}P) \to \mathrm{CH}(E/P) = \mathrm{CH}(X_m)$$

is surjective. The Chow ring $\mathrm{CH}(\mathsf{B}H)$ of the classifying space of an affine algebraic group H (covariant in H), appearing here for $H = P$, is defined in [35, §2.2]. For any H-torsor over a smooth variety X, a homomorphism of graded rings $\mathrm{CH}(\mathsf{B}H) \to \mathrm{CH}(X)$, showing up here for $H = P$, $X = X_m$, and the P-torsor $E \to X_m$, is defined in [35, Theorem 2.8].

The algebraic group P is a semi-direct product of its unipotent radical and an algebraic group P' defined below. For $m < n$,

$$P' := (\mathbb{G}_\mathrm{m} \times \mathrm{GL}(m-1) \times \mathrm{O}(2n-2m))/\mu_2,$$

where μ_2 is embedded diagonally into the product $\mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m} \times \mu_2$ of the centers of the factors $\mathbb{G}_\mathrm{m}$, $\mathrm{GL}(m-1)$, and $\mathrm{O}(2n-2m)$. For $m = n$,

$$P' := (\mathbb{G}_\mathrm{m} \times \mathrm{GL}(n-1))/\mu_2,$$

where μ_2 is embedded diagonally into the product $\mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m}$ of the centers of the two factors. By [19, Proof of Proposition 6.1] (see also [22, Proposition 5.9]), for any m, the ring homomorphism $\mathrm{CH}(\mathsf{B}P') \to \mathrm{CH}(\mathsf{B}P)$, induced by the quotient group homomorphism $P \to P'$, is an isomorphism.

Let us finish the case of $m = n$ first. The automorphism $(a,b) \mapsto (a,ab)$ of the product $\mathbb{G}_\mathrm{m} \times \mathrm{GL}(n-1)$ induces an isomorphism

$$P' \simeq (\mathbb{G}_\mathrm{m}/\mu_2) \times \mathrm{GL}(n-1)$$

and the homomorphism $\mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m}$, $a \mapsto a^2$ yields an isomorphism $\mathbb{G}_\mathrm{m}/\mu_2 \simeq \mathbb{G}_\mathrm{m}$. In particular, the group P' is *special*, in the sense that every P'-torsor over an extension of the base field is trivial. Combining the above surjections and isomorphisms, we get a surjective homomorphism of graded rings

$$\mathrm{CH}(\mathsf{B}\,\mathbb{G}_\mathrm{m}) \otimes \mathrm{CH}(\mathsf{B}\,\mathrm{GL}(n-1)) \twoheadrightarrow \mathrm{CH}(X_n). \tag{3.6}$$

Recall that $\mathrm{CH}(\mathsf{B}\,\mathrm{GL}(n-1))$ is the polynomial ring over the integers in the Chern classes $c_1,\dots,c_{n-1}$. Surjection (3.6) maps to the first Chern class of $\mathcal{I} \otimes_A \mathcal{I}$ the generator of the first factor $\mathrm{CH}(\mathsf{B}\,\mathbb{G}_\mathrm{m}) = \mathrm{CH}(\mathsf{B}\,\mathrm{GL}(1)) = \mathbb{Z}[c_1]$. The generators of the second factor are mapped to the corresponding Chern classes of $(\mathcal{J}/\mathcal{I}) \otimes_A \mathcal{I}$. In the next paragraph it is explained why the images of the generators are as just claimed.

Since the generators are the Chern classes of certain elements in the representation ring

$$R(\mathbb{G}_{\mathrm{m}} \times \mathrm{GL}(n-1)) \simeq R(P') \tag{3.7}$$

in the sense of [23, §4] (namely, of the classes u and v of the tautological representations of $\mathbb{G}_{\mathrm{m}}$ and of $\mathrm{GL}(n-1)$), there images under (3.6) are Chern classes of the corresponding elements in the Grothendieck group $K_0(X_n)$. Since by [28] any extension of the base field yields an injection on the Grothendieck group, it suffices to determine the elements in question over an algebraic closure of the base field, i.e., in $K_0(\bar{X}_n)$. Recall that the group P' is initially defined as the quotient $(\mathbb{G}_{\mathrm{m}} \times \mathrm{GL}(n-1))/\mu_2$. The quotient homomorphism $\mathbb{G}_{\mathrm{m}} \times \mathrm{GL}(n-1) \to P'$ induces an embedding

$$R(P') \hookrightarrow R(\mathbb{G}_{\mathrm{m}} \times \mathrm{GL}(n-1)), \tag{3.8}$$

and the homomorphism $R(P') \to K_0(\bar{X}_n)$ is a restriction of the natural homomorphism

$$f\colon R(\mathbb{G}_{\mathrm{m}} \times \mathrm{GL}(n-1)) \to K_0(\bar{X}_n)$$

(cf. [18, §7]). The images of u and v under f are the classes of the vector bundles $\mathcal{U}$ and $\mathcal{W}/\mathcal{U}$ from Remark 3.2. Finally, the composition of the isomorphism (3.7) with the embedding (3.8) maps u and v to u^2 and vu.

We finished the proof of the case $m = n$. Moreover, we showed that the list of generators of C can be shorten by removing the Chern classes of $\mathcal{I}$ in this case.

From now on we assume that $m < n$. The group P' embeds into the special algebraic group

$$S := (\mathbb{G}_{\mathrm{m}} \times \mathrm{GL}(m-1) \times \mathrm{GL}(2n-2m))/\mu_2.$$

The quotient variety $S/P' = \mathrm{GL}(2n-2m)/\mathrm{O}(2n-2m)$ is the open subvariety of non-degenerate quadratic forms on an $(2n-2m)$-dimensional vector space V in the affine space of all quadratic forms on V (see [18, §3]). It follows by [35, Theorem 5.1] (see also [18, Proposition 5.1]) that the ring homomorphism $\mathrm{CH}(BS) \to \mathrm{CH}(BP')$, induced by the embedding $P' \hookrightarrow S$, is surjective.

The automorphism $(a, b, c) \mapsto (a, ab, ac)$ of the product

$$\mathbb{G}_{\mathrm{m}} \times \mathrm{GL}(m-1) \times \mathrm{GL}(2n-2m)$$

induces an isomorphism $S \simeq (\mathbb{G}_{\mathrm{m}}/\mu_2) \times \mathrm{GL}(m-1) \times \mathrm{GL}(2n-2m)$. The factor $\mathbb{G}_{\mathrm{m}}/\mu_2$ is isomorphic to $\mathbb{G}_{\mathrm{m}}$. As a result, we get a surjective ring homomorphism

$$\mathrm{CH}(B\,\mathbb{G}_{\mathrm{m}}) \otimes \mathrm{CH}(B\,\mathrm{GL}(m-1)) \otimes \mathrm{CH}(B\,\mathrm{GL}(2n-2m)) \twoheadrightarrow \mathrm{CH}(X_m), \tag{3.9}$$

which maps to the first Chern class of $\mathcal{I} \otimes_A \mathcal{I}$ the generator of the first factor. The generators of the second factor are mapped to the corresponding Chern classes of the vector bundle $(\mathcal{J}/\mathcal{I}) \otimes_A \mathcal{I}$ as one shows similarly to the case of $m = n$ considered above. Finally, the addition to the image of (3.9) made by the third factor coincides with the addition of the Chern classes of $\mathcal{I}$. □

Corollary 3.10. *The sequence $\mathfrak{l}(1), \ldots, \mathfrak{l}(n)$ does not depend on the initial field k.*

Proof. By Theorem 3.3, for any $m \in \{1, \ldots, n\}$, the index of the variety X_m, given by generic (A, σ), is the positive generator of the image of $C \subset \mathrm{CH}(X_m)$ under the degree homomorphism $\mathrm{CH}(X_m) \to \mathbb{Z}$. This image is not changed under base field extensions.

Therefore we may assume that the duo (A,σ) is split in which case the Chow ring $\mathrm{CH}(X_m)$ and its subring C do not depend on the base field. □

4. Index of X_n

The torsion index of an arbitrary affine algebraic group G over a field k is the least common multiple of the indexes $i(E)$, where E is a G-torsor over an extension field of k. The index of a G-torsor here is just the index of the underlying variety.

By Theorem A.2, the torsion index of G is the index of any generic G-torsor, defined as the generic fiber of the morphism

$$\mathrm{GL}(N) \to \mathrm{GL}(N)/G \tag{4.1}$$

given by an embedding of G into $\mathrm{GL}(N)$ for some N. The proof of Theorem A.2 is based on the so-called *versal* property of (4.1) obtained in Theorem A.1 and considered previously for smooth G in [30, §5]. For a split reductive G (including $G = \mathrm{PGO}^+(2n)$ considered below), Theorem A.2 is a slight extension of [7, Théorème 2], proven in [34, Theorem 1.1] (see also [21, Theorem 6.4]).

For G being the affine algebraic group $\mathrm{PGO}(2n)$ of automorphisms of the standard split degree $2n$ central simple algebra A_0 with the standard hyperbolic quadratic pair σ_0, a degree $2n$ algebra with quadratic pair (A,σ) corresponds to the G-torsor E of isomorphisms between (A,σ) and (A_0,σ_0). In particular, the torsor E is trivial if and only if the corresponding variety X_n has a rational point. It follows that $2^{\mathfrak{l}(n)}$ is the torsion index of G – the affine algebraic group of automorphisms of the standard split degree $2n$ central simple algebra with the standard hyperbolic quadratic pair. We determine this torsion index using its closed relation with the torsion index of the connected component G^+ of G, computed in [33]. Note that for $n=1$ the group $G^+ = \mathbb{G}_{\mathrm{m}}$, being special, has the torsion index 1.

Proposition 4.2. *One has $\mathfrak{l}(n) = n$ if n is 1 or not a 2-power; if n is a 2-power ≥ 2 then $\mathfrak{l}(n) = n+1$. In other terms, the torsion index of G is twice the torsion index of G^+.*

Proof. Let (A,σ) be the generic algebra with quadratic pair given by the generic G-torsor E over the function field $F = k(\mathrm{GL}(N)/G)$ obtained (F and E) from an embedding $G \hookrightarrow \mathrm{GL}(N)$. The embedding given by the composition

$$G^+ \hookrightarrow G \hookrightarrow \mathrm{GL}(N)$$

yields a generic G^+-torsor over the separable quadratic extension field K of F given by the discriminant of σ. The corresponding K-algebra with quadratic pair is (A_K,σ_K) so that $i((X_n)_K)$ is the torsion index of the group G^+. Since the residue field of any point on X_n contains K, the index of X_n is twice the index of $(X_n)_K$. □

5. Index of X_2

Given an integer $n \geq 2$ and an initial field k, let F be the function field $F = k(\mathrm{GL}(N)/G)$ for an embedding $G = \mathrm{PGO}(2n) \hookrightarrow \mathrm{GL}(N)$, and let (A,σ) be the corresponding generic degree $2n$ central simple F-algebra with a quadratic pair. In order to determine $i(X_2)$ we use the information on $\mathrm{CH}(X_1)$ provided in Theorem 3.3. We fix a splitting extension field $\bar{F}/F$ of (A,σ) (e.g., an algebraic closure of F), write $\bar{X}_1$ for $(X_1)_{\bar{F}}$

and write $\bar{\mathrm{CH}}(X_1)$ for the image of the change of field homomorphism $\mathrm{CH}(X_1) \to \mathrm{CH}(\bar{X}_1)$. Recall that $\bar{X}_1$ is a split projective quadric of dimension $2n-2$. Let $t \in \mathrm{CH}^1(\bar{X}_1)$ be the class of a hyperplane section.

Proposition 5.1. *The ring $\bar{\mathrm{CH}}(X_1)$ is generated by $2t$ and $\binom{2n}{i}t^i$, $i \geq 0$.*

Proof. In view of Theorem 3.3 and since $\mathcal{J} = \mathcal{I}$ for X_1, the ring $\mathrm{CH}(X_1)$ is generated by the Chern classes of $\mathcal{I}$ and $\mathcal{I} \otimes_A \mathcal{I}$ alone. As one can see from Remark 3.2, under the change of field homomorphism $\mathrm{CH}(X_1) \to \mathrm{CH}(\bar{X}_1)$, the element $2t$ is the image of the first Chern class of $\mathcal{I} \otimes_A \mathcal{I}$ whereas $\binom{2n}{i}t^i$ (for any $i \geq 0$) is the image of the ith Chern class of $\mathcal{I}$. □

Lemma 5.2. *Given an integer $d \geq 1$, let B be the subring in the polynomial ring $\mathbb{Z}[h]$ generated by $\binom{d}{i}h^i$, $i \geq 0$. Then for any $i \geq 0$, the ith graded component of B is generated by $\frac{d}{(i,d)}h^i$, where (i,d) is the g.c.d. of i and d.*

Proof. Since $h^d \in B$, is suffices to prove the statement with $B \subset \mathbb{Z}[h]$ replaced by $B/(h^d) \subset \mathbb{Z}[h]/(h^d)$. Let Y be the Severi-Brauer variety of a generic central simple algebra of degree d – the algebra, given by a generic $\mathrm{PGL}(d)$-torsor, where

$$\mathrm{PGL}(d) = \mathrm{GL}(d)/\,\mathbb{G}_\mathrm{m} = \mathrm{SL}(d)/\mu_d$$

is the projective general linear group. We identify $\mathrm{CH}(\bar{Y})$ with $\mathbb{Z}[h]/(h^d)$, by letting h correspond to the hyperplane class in $\mathrm{CH}^1(\bar{Y})$ of the projective space $\bar{Y}$. By [21, §8.1], $B \subset \mathbb{Z}[h]/(h^d)$ is then identified with tjhe image $\bar{\mathrm{CH}}(Y)$ of $\mathrm{CH}(Y)$ in $\mathrm{CH}(\bar{Y})$. (By [17, Theorem 3.1], one knows that $\bar{\mathrm{CH}}(Y) = \mathrm{CH}(Y)$ but this is not needed here.) Finally, by [13, Theorem 1], for any $i \geq 0$, the group $\bar{\mathrm{CH}}^i(Y)$ is generated by $\frac{d}{(i,d)}h^i$. □

Corollary 5.3. *For any $i \geq 0$, the group $\bar{\mathrm{CH}}^i(X_1)$ is generated by $a_i t^i$, where the integer a_i is the g.c.d. of 2^i and $\frac{2n}{(i,2n)}$.* □

Remark 5.4. Corollary 5.3 in particular affirms that $\bar{\mathrm{CH}}(X_1)$ is contained in the subring of $\mathrm{CH}(\bar{X}_1)$ generated by h. This implies that $i(\sigma) = 0$, i.e, the quadratic pair σ of a generic duo (A, σ) is anisotropic. Besides, since the integer $\frac{2n}{(i,2n)}$ is even for every $i \geq 1$ which is not divisible by the 2-primary part of $2n$, $\bar{\mathrm{CH}}^i(X_1)$ is contained in the subgroup of $\mathrm{CH}^i(\bar{X}_1)$ generated by $2h^i$ for such i. This implies that $i(A)$ is the 2-primary part of $2n$.

Corollary 5.5. *For any $n \geq 2$, one has*

$$\mathfrak{l}(2) = \mathfrak{l}(1) + 1.$$

In other terms, position 2 is not relaxed.

Proof. The case of $n = 2$ is already covered (by Proposition 4.2 and §2). For $n \geq 3$, the element $t^{2n-3} \in \mathrm{CH}(\bar{X}_1)$ equals twice the class of a line.

The only alternate to the claimed equality $i(X_2) = 2i(A)$ is $i(X_2) = i(A)$ in which case

$$(i(A)/2) \cdot t^{2n-3} \in \bar{\mathrm{CH}}(X_1). \qquad (5.6)$$

Recall that $i(A)$ is the 2-primary part of $2n$. However $2n-3$ is odd and $2^{2n-3} \geq 2n$ for $n \geq 3$. Therefore the integer a_{2n-3} is divisible by $i(A)$ so that (5.6) fails according to Corollary 5.3. □

6. Relaxed positions

In this section, for any given $n \geq 2$, we locate all relaxed positions – see Theorem 6.6. Therefore, as explained in §1, we determine the entire sequence $\mathfrak{l}(1), \dots, \mathfrak{l}(n)$.

We start by collecting information on $\mathrm{CH}(X)$ with $X := \bar{X}_m$ for general $n \geq 1$ and $m \in \{1, \dots, n\}$. As before, the bar over X_m means that we consider it over an extension of the base field splitting the duo (A, σ) defining X_m.

Recall (see Remark 3.2) that X is the variety of 2-flags of totally isotropic subspaces $U \subset W$ of dimensions 1 and m of a (split) quadratic form on a $2n$-dimensional vector space V. We write $t \in \mathrm{CH}^1(X)$ for the first Chern class of the tautological line bundle $\mathcal{U}$ and we write $f_i \in \mathrm{CH}^i(X)$, $i \geq 1$ for the Chern classes of the rank $2n-m$ vector bundle $\mathcal{V}/\mathcal{W}$, where $\mathcal{V}$ is the rank $2n$ trivial vector bundle on X, given by V, whereas $\mathcal{W}$ is the tautological rank m vector bundle on X. The notation t, f_i, and introduced below notation u_i correspond (vaguely) to [33, Proof of Theorem 7.1], where a component of $\bar{X}_n$ was treated. Some of the facts established below generalize results from [33, Proof of Theorem 7.1]. Some of them are proved similarly, some others – differently.

The projection to the second component of 2-flags makes X a rank $m-1$ projective bundle over the variety of totally isotropic m-planes in V. From this viewpoint, one sees that the 0-dimensional component of the subring $R \subset \mathrm{CH}(X)$, generated by $t, f_1, \dots, f_{2n-m}$, is spanned by the product $t^{m-1} f_{2n-m-1} \dots f_{2n-2m} \in \mathrm{CH}_0(X)$. The degree of this 0-cycle class is 2^m. Note that $f_{2n-m} = 0$ (see [18, Theorem 2.1]).

Let $u_i \in \mathrm{CH}^i(X)$, $i \geq 1$ be the Chern classes of the rank $2n-m$ vector bundle $(\mathcal{V}/\mathcal{W}) \otimes \mathcal{U}$. Clearly, the subring generated by t and all u_i coincides with R. The benefit of the generators u_i is as follows: by Theorem 3.3, the subring in $\mathrm{CH}(X)$, we need to compute the index $\mathfrak{l}(m)$, is the subring $R' \subset R$ generated by all u_i along with $2t$ and all $\binom{2n}{i} t^i$. More precisely,

Lemma 6.1. *The product $2^{\mathfrak{l}(m)-m} \cdot t^{m-1} f_{2n-m-1} \dots f_{2n-2m}$ generates the 0-dimensional component of R'.* □

We write R_t for the subring in R generated by t. We write R'_t for the subring in R' generated by $2t$ and all $\binom{2n}{i} t^i$, $i \geq 0$. By Lemma 5.2, for any $i \geq 0$, the codimension i graded component of R'_t is spanned by $a_i t^i$, where a_i is the g.c.d. of 2^i and $2n/(i, 2n)$.

Lemma 6.2. *In R one has $t^{2n-1} = 0$ and*

$$(-t)u_{2n-m-1} + (-t)^2 u_{2n-m-2} + \dots + (-t)^{2n-m-1} u_1 + (-t)^{2n-m} = 0.$$

In $R'/2R'$, for $i > n-m$, the square u_i^2 is a (homogeneous) linear combinations of u_j^2, $j \leq n-m$, with coefficients in R'_t; in particular, u_i^2 is divisible by $t^{2i-2n+2m}$ in $R/2R$. The ring R as an R_t-module as well as the ring R' as an R'_t-module is generated by products of at most $m-1$ of u_i. Finally, $u_{2n-m} = 0$ in $R' \subset R$.

Proof. Projection to the first component of 2-flags maps X to the quadric Q of dimension $2n-2$. The element t is in the image of the pull-back from the quadric. This shows that $t^{2n-1}=0$.

Since $(\mathcal{V}/\mathcal{W})\otimes\mathcal{U}$, tensored by the dual $\mathcal{U}^*$ of $\mathcal{U}$, turns back to $\mathcal{V}/\mathcal{W}$, we get the second relation of Lemma 6.2 rewriting the relation $f_{2n-m}=0$ in terms of $-t=c_1(\mathcal{U}^*)$ and $u_i=c_i((\mathcal{V}/\mathcal{W})\otimes\mathcal{U})$.

The isomorphism $\mathcal{V}/\mathcal{W}^\perp=\mathcal{W}^*$, where the vector bundle $\mathcal{W}^\perp$ is obtained from $\mathcal{W}$ by taking the orthogonal complement, implies the equality

$$[\mathcal{V}/\mathcal{W}]+[(\mathcal{V}/\mathcal{W})^*]=[\mathcal{W}^\perp/\mathcal{W}]+[\mathcal{V}]$$

of classes $[-]$ in the Grothendieck group. Tensoring with $\mathcal{U}$, we get

$$[(\mathcal{V}/\mathcal{W})\otimes\mathcal{U}]+[(\mathcal{V}/\mathcal{W})^*\otimes\mathcal{U}]=[(\mathcal{W}^\perp/\mathcal{W})\otimes\mathcal{U}]+[\mathcal{V}\otimes\mathcal{U}]. \tag{6.3}$$

Note that the quotient $\mathcal{W}^\perp/\mathcal{W}$ and the tensor product $(\mathcal{W}^\perp/\mathcal{W})\otimes\mathcal{U}$ are vector bundles of rank $2n-2m$ and therefore the Chern classes of the latter with numbers higher than $2n-2m$ vanish. Since the modulo 2 Chern classes of $\mathcal{U}$ and of $\mathcal{U}^*$ coincide, the Chern classes modulo 2 of the left side in (6.3) coincide with the Chern classes of

$$[(\mathcal{V}/\mathcal{W})\otimes\mathcal{U}]+[((\mathcal{V}/\mathcal{W})\otimes\mathcal{U})^*].$$

The even Chern class with number $2i$ here is equal to u_i^2 and the odd Chern classes vanish. The ith Chern class of $\mathcal{V}\otimes\mathcal{U}$ equals $\binom{2n}{i}t^i$; for odd i it vanishes modulo 2. This proves the statement on u_i^2 with $i>n-m$ of Lemma 6.2.

The next statement of Lemma 6.2 follows from the following general result on the Chow ring of the grassmannian Γ of d-planes in a vector bundle $\mathcal{E}$ over a smooth variety Y (see [6, §14.6]): the Chow ring $\mathrm{CH}(\Gamma)$ as a module over the ring $\mathrm{CH}(Y)$ is generated by the products of at most d Chern classes of $\mathcal{E}/\mathcal{T}$, where $\mathcal{T}$ is the tautological rank d vector bundle on Γ. In our case, Y is the quadric Q, $\mathcal{E}$ is the vector bundle $(\mathcal{U}^\perp/\mathcal{U})\otimes\mathcal{U}$ over Q, and $d=m-1$. As a Q-scheme, X is a closed subscheme in Γ. The restriction to X of the tautological vector bundle $\mathcal{T}$ is then $(\mathcal{W}/\mathcal{U})\otimes\mathcal{U}$ so that the restriction of the quotient $\mathcal{E}/\mathcal{T}$ becomes $(\mathcal{U}^\perp/\mathcal{W})\otimes\mathcal{U}$. Recall that u_i were defined as the Chern classes of $(\mathcal{V}/\mathcal{W})\otimes\mathcal{U}$. Since the line bundle

$$\frac{(\mathcal{V}/\mathcal{W})\otimes\mathcal{U}}{(\mathcal{U}^\perp/\mathcal{W})\otimes\mathcal{U}}=(\mathcal{V}/\mathcal{U}^\perp)\otimes\mathcal{U}=\mathcal{U}^*\otimes\mathcal{U}$$

is trivial, they coincide with the pull-backs of the Chern classes of the vector bundle $\mathcal{E}/\mathcal{T}$. Since the rank of this vector bundle is $2n-m-1$, we get on the way the relation $u_{2n-m}=0$ – the final statement of Lemma 6.2.

At this point we proved that the ring R as an R_t-module is generated by products of at most $m-1$ of u_i. To prove the similar statement on the ring R' as an R'_t-module, one takes into account that R'_t is the image of the change of field homomorphism

$$\mathrm{CH}(X_1)\to\mathrm{CH}(\bar{X}_1)=\mathrm{CH}(Q)$$

and R' is the image of the change of field homomorphism

$$\mathrm{CH}(X_m)\to\mathrm{CH}(\bar{X}_m)=\mathrm{CH}(X)$$

provided that the varieties X_1 and X_m are given by a generic duo (A,σ). The variety X_m, considered as an X_1-scheme via the projection $X_m \to X_1$, is a closed X_1-subscheme in the grassmannian of $(m-1)$-planes of the rank $2n-2$ vector bundle $(\mathcal{I}^\perp/\mathcal{I})\otimes_A \mathcal{I}$ over X_1, where, as in §3, $\mathcal{I}$ is the tautological (rank $2n$) vector bundle on X_1. The orthogonal complement $\mathcal{I}^\perp$ is defined as in [25, Definition 6.1]. It follows that the ring $\mathrm{CH}(X_m)$, viewed as a $\mathrm{CH}(X_1)$-module, is generated by the products of at most $m-1$ Chern classes of the vector bundle $(\mathcal{I}^\perp/\mathcal{J})\otimes_A \mathcal{I}$. Therefore the ring $\bar{\mathrm{CH}}(X_m) = R'$ is generated as a module over the ring $\bar{\mathrm{CH}}(X_1) = R'_t$ by the products of at most $m-1$ of u_i. □

Corollary 6.4. *The product*

$$t^{2n-2}u_{2n-m-2}\cdots u_{2n-2m} \tag{6.5}$$

generates the 0*-dimensional component of* $R/2R$.

Proof. As follows from Lemma 6.2, the quotient $R/2R$ is additively generated by two groups of generators, taken together. The first group consists of products of a power of t non-exceeding $2n-2$ by a product of at most $m-1$ elements u_i with various $i \le 2n-m-2$, in which no u_i with $i > n-m$ is repeated. The second group consists of product of at most $m-1$ elements u_i with various $i \le 2n-m-1$, in which no u_i with $i > n-m$ is repeated. (The element t does not show up in the second group.) The product (6.5) is the unique generator from this list of highest possible codimension. Since it happens to have dimension 0, it is ought to be the unique nonzero element of the 0-dimensional component of $R/2R$. □

Theorem 6.6. *For a given* $n \ge 2$, *let* 2^r *be the* 2*-primary part of* $2n$. *An integer* $m \in \{2,\dots,n\}$ *constitutes a relaxed position if and only if* $m = 2^s - 1$ *for some* $s \le r$.

Proof. As already discussed in §1, the total number of relaxed position is $r-2$ or $r-1$ depending on whether n is a 2-power or not. So, the number of relaxed positions, announced in Theorem 6.6, is the right one: s runs over the set $\{2,\dots,r-1\}$ if $n = 2^{r-1}$, otherwise s runs over the set $\{2,\dots,r\}$. Consequently, we only need to check that the positions 2^s-1 are relaxed.

We pick up some $m = 2^s - 1 < n$ with $2 \le s \le r$. In order to show that m is relaxed, it suffices to check that $\mathfrak{l}(m-1) \ge \mathfrak{l}(m)$. We check separately the two inequalities

$$\mathfrak{l}(m) \le m + r - s \quad \text{and} \quad \mathfrak{l}(m-1) \ge m + r - s, \tag{6.7}$$

starting with the first one.

By Lemma 5.2, the ring R' contains the product $2^{r-s}t^{2n-m-1}$ because the 2-primary part of

$$\frac{2n}{(2n-m-1,2n)} = \frac{2n}{2^s}$$

is 2^{r-s} and $2n-m-1 \ge r-s$. We rewrite the product

$$R' \ni 2^{r-s}t^{2n-m-1}u_{2n-m-1}\cdots u_{2n-2m+1}$$

in $R/2^{r-s+1}R$, eliminating u_{2n-m-1} with the help of the second relation of Lemma 6.2:

$$2^{r-s}t^{2n-m-1}u_{2n-m-1}\dots u_{2n-2m+1} = \\ 2^{r-s}t^{2n-m-2}u_{2n-m-2}\dots u_{2n-2m+1}\cdot(tu_{2n-m-1}) = \\ 2^{r-s}t^{2n-m-2}u_{2n-m-2}\dots u_{2n-2m+1}\cdot(t^2u_{2n-m-2}+\dots+t^mu_{2n-2m}+\dots) = \\ 2^{r-s}\cdot(6.5)\in R/2^{r-s+1}R,$$

where (6.5) is the generator from Corollary 6.4. To explain the last equality in the chain, let us mention that all summands except one, we have on the left of the equality, reduce to zero as being divisible by $t^{2n-1}=0$. (Remember that u_i^2 is divisible in $R/2R$ by $t^{2i-2n+2m}$ for $i>n-m$.)

It follows by Lemma 6.1 that $\mathfrak{l}(m)-m\le r-s$ which is the first inequality of (6.7). To prove the second one, we set $m:=2^s-2$. The second inequality in (6.7) then reads:

$$\mathfrak{l}(m)-m\ge r-s+1.$$

We prove it by checking that every element in the 0-dimensional component of the ring R' vanishes in $R/2^{r-s+1}R$.

The ring $R'/2R'$ as an R'_t-module is generated by product of at most $m-1$ elements u_i with various $i\le 2n-m-1$, in which no u_i with $i>n-m$ is repeated. The product $u_{2n-m-1}\dots u_{2n-2m+1}$ has the highest possible codimension and its dimension is equal to $2n-m-1$. Therefore any element in the 0-dimensional component of $R'/2R'$ is divisible by a_it^i for some $i\ge 2n-m-1$. Replacing i by $2n-i$, we get the divisor $a_{2n-i}t^{2n-i}$ for some $i\le m+1=2^s-1$. Since a_{2n-i} is the g.c.d. of 2^{2n-i} and $2n/(2n-i,2n)=2n/(i,2n)$, we see that the integer a_{2n-i} is divisible by 2^{r-s+1}. It follows that every element in the 0-dimensional component of the ring R' vanishes in $R/2^{r-s+1}R$. □

7. PGO$^+$

In this section we shift our attention from the group $G=\mathrm{PGO}(2n)$ to its connected component – the projective *special* orthogonal group $G^+=\mathrm{PGO}^+(2n)$, which showed up already in §4. For $n\ge 3$, G^+ is the adjoint split simple group of Dynkin type D_n (see [25, Theorem 25.12]).

Via the embedding $G^+\hookrightarrow G$, a G^+-torsor yields a G-torsor and a central simple algebra A with a quadratic pair σ of trivial discriminant $\mathrm{disc}(\sigma)$. Every duo (A,σ) of degree $2n$ with trivial $\mathrm{disc}(\sigma)$ comes from a G^+-torsor this way.

Fixing an initial field k, for $m\in\{1,\dots,n\}$, we define the integer $\mathfrak{l}^+(m)$ by letting $2^{\mathfrak{l}^+(m)}$ be the maximum of $i(X_m)$ when F runs over all extension fields of k and (A,σ) runs over all F-duos of degree $2n$ with trivial $\mathrm{disc}(\sigma)$. By [34, Theorem 1.1] (see also [21, Theorem 6.4]), extending [7, Théorème 2], $2^{\mathfrak{l}^+(m)}$ is the index of the variety X_m given by a generic G^+-torsor. Our goal is to determine the sequence $\mathfrak{l}^+(1),\dots,\mathfrak{l}^+(n)$.

Clearly, this sequence is (non-strictly) increasing: $\mathfrak{l}^+(1)\le\dots\le\mathfrak{l}^+(n)$. Moreover, every successive difference is 0 or 1.

By [26, §4.4], the Schur index $i(A)$ of A is the 2-primary part 2^r of $2n$. By §2, $2^{\mathfrak{l}^+(1)}=i(A)$. It follows that $\mathfrak{l}^+(1)=r=\mathfrak{l}(1)$.

Since the discriminant of any quadratic pair vanishes in an at most quadratic extension of the base field, for every m the integer $\mathfrak{l}^+(m)$ equals $\mathfrak{l}(m)$ or $\mathfrak{l}(m)-1$.

Finally, $\mathfrak{l}^+(n)$ is the torsion index of the group G^+. Due to [33], where it is computed, we know that $\mathfrak{l}^+(n)=\mathfrak{l}(n)-1$. It is also clear that $\mathfrak{l}^+(n-1)=\mathfrak{l}^+(n)$ and so $\mathfrak{l}^+(n-1)=\mathfrak{l}(n)-1=\mathfrak{l}(n-1)$.

The main result here is

Theorem 7.1. *One has $\mathfrak{l}^+(m)=\mathfrak{l}(m)$ for every $m\in\{1,\dots,n-1\}$.*

To prove Theorem 7.1, we compare the Chow ring $\mathrm{CH}(X_m)$ of the variety X_m, given by a generic G-torsor, with the Chow ring $\mathrm{CH}(X_m^+)$ of the similar variety, given by a generic G^+-torsor obtained via the composition $G^+\hookrightarrow G\hookrightarrow \mathrm{GL}(N)$. Then X_m^+ is simply X_m, considered over the quadratic discriminant extension field of the base of X_m. In particular, $\mathrm{CH}(X_m^+)$ is a $\mathrm{CH}(X_m)$-algebra via the change of field ring homomorphism. A finite system of generators for the ring $\mathrm{CH}(X_m)$ is provided in Theorem 3.3.

We write X for the variety X_m^+ over an algebraic closure F of its base field. As per Remark 3.2, the variety X can be viewed as the variety of totally isotropic subspaces $U\subset W$ of dimensions 1 and m of a $2n$-dimensional split quadratic form. In particular, X is endowed with tautological vector bundles $\mathcal{U}$ and $\mathcal{W}$ of ranks 1 and m.

Let τ be the involution of the Chow ring $\mathrm{CH}(X)$, induced by any element of $G(F)\setminus G^+(F)$. Note that by [15, Corollary 4.2], the action of $G^+(F)$ on $\mathrm{CH}(X)$ is trivial.

Proposition 7.2. *For $m<n$, the Chow ring $\mathrm{CH}(X_m^+)$, viewed as a $\mathrm{CH}(X_m)$-module, is generated by 1 and an element e whose image in $\mathrm{CH}(X)$ equals $2^{n-m-1}t^{n-m}+t'$, where t is the first Chern class of the line bundle $\mathcal{U}$ and $t'\in\mathrm{CH}(X)$ satisfies $\tau(t')=-t'$.*

Proof. We prove Proposition 7.2 going along the lines of the proof of Theorem 3.3 and performing suitable modifications.

For a second, let us consider the standard split duo (A,σ), where A is the degree $2n$ standard split central simple F-algebra and σ is the quadratic pair adjoint to the dimension $2n$ standard split quadratic form. For $m<n$, the connected component G^+ of the group G of automorphisms of (A,σ) acts transitively on the corresponding variety X_m. The stabilizer of a rational point is a parabolic subgroup $P\subset G^+$ with the quotient G^+/P isomorphic to X_m. For $m\le n-2$, P is a parabolic subgroup whose Dynkin type is given by the 1st and mth vertices of the Dynkin diagram. For $m=n-1$, Dynkin type of P is given by the vertices with the numbers 1, $n-1$, and n; the two connected components of the variety X_n yield two parabolic subgroups: one of the type $\{1,\ n-1\}$, the other of the type $\{1,\ n\}$.

Now, switching to the variety X_m^+, given by a generic G^+-torsor E, we identify X_m^+ with the quotient E/P. By [17, Proof of Lemma 2.1], the natural graded ring homomorphism $\mathrm{CH}(BP)\to\mathrm{CH}(E/P)=\mathrm{CH}(X_m^+)$ is surjective.

The algebraic group P is a semi-direct product of its unipotent radical and the reductive group

$$P'=(\mathbb{G}_\mathrm{m}\times\mathrm{GL}(m-1)\times\mathrm{SO}(2n-2m))/\mu_2,$$

where μ_2 is embedded diagonally into the product of the centers $\mathbb{G}_\mathrm{m}\times\mathbb{G}_\mathrm{m}\times\mu_2$ of the factors $\mathbb{G}_\mathrm{m}$, $\mathrm{GL}(m-1)$, and $\mathrm{SO}(2n-2m)$. By [19, Proof of Proposition 6.1] (see also [22, Proposition 5.9]), the ring homomorphism $\mathrm{CH}(\mathsf{B}P') \to \mathrm{CH}(\mathsf{B}P)$, induced by the quotient group homomorphism $P \to P'$, is an isomorphism.

The difference with the proof of Theorem 3.3, that just showed up, is the special orthogonal group $\mathrm{SO}(2n-2m)$ in the definition of P' here, replacing the orthogonal group $\mathrm{O}(2n-2m)$ we had in the definition of P' there. This difference results in the appearance of the additional – compared to $\mathrm{CH}(X_m)$ – generator e of $\mathrm{CH}(X_m^+)$.

Via the standard representation $\mathrm{SO}(2n-2m) \hookrightarrow \mathrm{GL}(2n-2m)$, the group P' embeds into the special algebraic group

$$S := (\mathbb{G}_\mathrm{m} \times \mathrm{GL}(m-1) \times \mathrm{GL}(2n-2m))/\mu_2.$$

By [5] (see [8, Corollary 5.4] for generalization to an arbitrary base field), the Chow ring of the quotient variety $S/P' = \mathrm{GL}(2n-2m)/\mathrm{SO}(2n-2m)$ is generated by a single element of codimension $n-m$ and of square 0. By [32, §14] (see also [35, Theorem 5.1]), the natural graded ring homomorphism $\mathrm{CH}(\mathsf{B}P') \to \mathrm{CH}(S/P')$ is surjective and, as a $\mathrm{CH}(\mathsf{B}S)$-module, the ring $\mathrm{CH}(\mathsf{B}P')$ is generated by 1 and any lift of the generator of the ring $\mathrm{CH}(S/P')$. The image e of such a lift in $\mathrm{CH}(X_m^+)$ generates together with 1 the $\mathrm{CH}(X_m)$-module $\mathrm{CH}(X_m^+)$ as follows from the commutative square

$$\begin{array}{ccc} \mathrm{CH}(\mathsf{B}P') & \xrightarrow{\text{onto}} & \mathrm{CH}(X_m^+) \\ \uparrow & & \uparrow \\ \mathrm{CH}(\mathsf{B}S) & \xrightarrow{\text{onto}} & \mathrm{CH}(X_m). \end{array}$$

To finish the proof of Proposition 7.2, it remains to check the statement on the image of e in $\mathrm{CH}(X)$. To make it true, a specific choice of the lift in the construction of e needs to be made.

The automorphism $(a,b,c) \mapsto (a,ab,c)$ of the product $\mathbb{G}_\mathrm{m} \times \mathrm{GL}(m-1) \times \mathrm{SO}(2n-2m)$ yields an isomorphism $P' = \mathrm{GL}(m-1) \times Q$, where $Q := (\mathbb{G}_\mathrm{m} \times \mathrm{SO}(2n-2m))/\mu_2$. It follows by [35, §6] that

$$\mathrm{CH}(\mathsf{B}P') = \mathrm{CH}(\mathsf{B}\,\mathrm{GL}(m-1)) \otimes \mathrm{CH}(\mathsf{B}Q).$$

As follows from the commutative square

$$\begin{array}{ccc} \mathrm{CH}(\mathsf{B}P') & \longrightarrow & \mathrm{CH}(S/P') \\ \uparrow{\scriptstyle \text{injection}} & & \| \\ \mathrm{CH}(\mathsf{B}Q) & \longrightarrow & \mathrm{CH}(\mathrm{GL}(2n-2m)/\mathrm{SO}(2n-2m)), \end{array}$$

we can (and do) choose a lift to $\mathrm{CH}(\mathsf{B}P')$ of the generator of $\mathrm{CH}(S/P')$ inside the subring $\mathrm{CH}(\mathsf{B}Q) \subset \mathrm{CH}(\mathsf{B}P')$.

The variety X is identified with the quotient G^+/P whose Chow ring does not depend on the base field. The composition $\mathrm{CH}(\mathsf{B}P') = \mathrm{CH}(\mathsf{B}P) \to \mathrm{CH}(X_m^+) \to \mathrm{CH}(X)$, we need to consider, is given by the natural homomorphism $\mathrm{CH}(\mathsf{B}P) \to \mathrm{CH}(G^+/P)$, which factors (see [2, Lemma 2.2]) as

$$\mathrm{CH}(\mathsf{B}P) \to \mathrm{CH}(\mathsf{B}T)^W \to \mathrm{CH}(G^+/P),$$

where T is the standard split maximal torus of G^+ (contained in $P' \subset P$) and W is the corresponding Weyl group. The torus T, sitting inside the product $P' = \mathrm{GL}(m-1) \times Q$, decomposes in the direct product $T_{\mathrm{GL}} \times T_Q$ of the standard maximal split tori of the factors $\mathrm{GL}(m-1)$ and Q. The product $W_{\mathrm{GL}} \times W_Q$ of the corresponding Weyl groups is the Weyl group W. The composition $\mathrm{CH}(BQ) \hookrightarrow \mathrm{CH}(BP') \to \mathrm{CH}(BT)^W$ factors through

$$\mathrm{CH}(BQ) \to \mathrm{CH}(BT_Q)^{W_Q}. \tag{7.3}$$

The ring on the right in (7.3) has been studied in [9, Lemma 4.3 and its proof]. It has been shown that as a module over $\mathrm{CH}(B(\mathbb{G}_\mathrm{m} \times \mathrm{GL}(2n-2m))/\mu_2)$ it is generated by 1 and the element

$$E := 2^{n-m-1} y^{n-m} + 2^{n-m-1} y_1 \dots y_{n-m}. \tag{7.4}$$

Here we view the torus T_Q as the quotient by μ_2 of the standard split maximal torus of the product $\mathbb{G}_\mathrm{m} \times \mathrm{SO}(2n-2m)$. The Chow ring of the classifying space of the latter torus is identified with the polynomial ring $\mathbb{Z}[y, y_1, \dots, y_{n-m}]$ so that $\mathrm{CH}(BT_Q)$ becomes the subring in the polynomial ring generated by the polynomials $2y$, $y_1 + y$, $\dots$, $y_{n-m} + y$. Note that

$$E = \frac{1}{2}\Big((2y)^{n-m} + \prod_{i=1}^{n-m} \big(2(y_i + y) - 2y\big)\Big)$$

is indeed in $\mathrm{CH}(BT_Q)$.

We may choose a lift to $\mathrm{CH}(BQ)$ of the generator of $\mathrm{CH}(S/P')$ the way that it maps to E under (7.3). The image in $\mathrm{CH}(X)$ of the first summand in the definition (7.4) of E equals $2^{n-m-1} t^{n-m}$. The image t' of the second summand satisfies $\tau(t') = -t'$. $\square$

Proof of Theorem 7.1. In order to prove Theorem 7.1, it suffices to show that $\mathfrak{l}^+(m) \geq \mathfrak{l}(m)$ for every $m \in \{1, \dots, n-1\}$ of the form $m = 2^s - 2$ with $s \in \{2, \dots, r\}$. Below in the proof we assume that m is of such form; in particular, $n \geq 3$.

As in §6, let $R' \subset \mathrm{CH}(X)$ be the image of the change of field homomorphism

$$\mathrm{CH}(X_m) \to \mathrm{CH}(X_m^+) \to \mathrm{CH}(X).$$

Abusing notation, we denote the image of $e \in \mathrm{CH}(X_m^+)$ in $\mathrm{CH}(X)$ by the same symbol. Let $\deg\colon \mathrm{CH}(X) \to \mathbb{Z}$ be the degree homomorphism (vanishing on the Chow ring's homogeneous components of positive dimensions). By Proposition 7.2, the image of $\mathrm{CH}(X_m^+)$ in $\mathrm{CH}(X)$ equals $R' + R' \cdot e$. Therefore

$$\deg(R' + R' \cdot e) = 2^{\mathfrak{l}^+(m)}\mathbb{Z}.$$

Since $\deg(R') = 2^{\mathfrak{l}(m)}\mathbb{Z}$, to prove Theorem 7.1 it suffices to check that

$$\deg(R' \cdot e) \subset 2^{\mathfrak{l}(m)}\mathbb{Z}. \tag{7.5}$$

Recall that $e = 2^{n-m-1} t^{n-m} + t'$, where $\tau(t') = -t'$. For any $x \in R'$, one has $\tau(x) = x$ and therefore $\tau(xt') = -xt'$. Since $\deg(y) = \deg(\tau(y))$ for any $y \in \mathrm{CH}(X)$, it follows that $\deg(xt') = 0$. Therefore

$$\deg(R' \cdot e) = 2^{n-m-1} \deg(R' \cdot t^{n-m}). \tag{7.6}$$

Recall that X is the variety of totally isotropic subspaces $U \subset W$ of dimensions 1 and m of a $2n$-dimensional split quadratic form. In particular, X is endowed with tautological vector bundles $\mathcal{U}$ and $\mathcal{W}$ of ranks 1 and m. As in §6, we write $R \subset \mathrm{CH}(X)$ for the subring generated by their Chern classes.

The inclusion $R' \cdot t^{n-m} \subset R$ implies that $\deg(R' \cdot t^{n-m-1}) \subset \deg(R)$. Besides, $\deg(R) = 2^m\mathbb{Z}$ as explained in §6. It follows from (7.6) that

$$\deg(R' \cdot e) \subset 2^{n-1}\mathbb{Z}.$$

If n is not a 2-power, then $n - 1 = \mathfrak{l}(n-1) \geq \mathfrak{l}(m)$, giving (7.5).

If n is a 2-power, i.e., $n = 2^{r-1}$, and $m \neq n-2$, then $n - 1 = \mathfrak{l}(n-3) \geq \mathfrak{l}(m)$, giving (7.5) once again.

We finish by considering the remaining case: $n = 2^{r-1}$ and $m = n - 2$. Here we have $\mathfrak{l}(m) = n$ and the desired inclusion (7.5) reads as $2\deg(R' \cdot t^2) \subset 2^n\mathbb{Z}$, or, after cancelling by 2, as

$$\deg(R' \cdot t^2) \subset 2^{n-1}\mathbb{Z}.$$

Acting like in the end of the proof of Theorem 6.6, one checks that every element in the 2-dimensional component of R' vanishes in $R/2R$. Therefore every element in the 0-dimensional component of $R' \cdot t^2$ vanishes in $R/2R$ as well, giving the inclusion

$$\deg(R' \cdot t^2) \subset 2\deg(R) = 2 \cdot 2^m\mathbb{Z} = 2^{n-1}\mathbb{Z}. \qquad \square$$

Appendix: Index of a generic torsor

by Alexander S. Merkurjev

Let F be a field and G be an affine group scheme over F of finite type. Choose an embedding $G \hookrightarrow \mathrm{GL}(N)$ for some N and consider the quotient variety $X := \mathrm{GL}(N)/G$ (existing by [27, Theorem 7.18]). The quotient morphism $f\colon \mathrm{GL}(N) \to X$ is a G-torsor (see [27, Corollary 5.27]).

Theorem A.1. *The G-torsor $f\colon \mathrm{GL}(N) \to X$ is* versal*, i.e., for every G-torsor $h\colon E \to \operatorname{Spec} K$ over a field extension K/F with K an infinite field and every nonempty open subset $U \subset X$, there is a point $x \in U(K)$ such that h is isomorphic to the pullback $x^*(f)$ of f with respect to the morphism $x\colon \operatorname{Spec} K \to X$.*

Proof. Note that the morphism $\mathrm{GL}(N) \times E \to \operatorname{Spec} K$ is a $(\mathrm{GL}(N) \times G)$-torsor over K, where the action of $\mathrm{GL}(N) \times G$ on $\mathrm{GL}(N) \times E$ is defined by the formula

$$(a, g)(b, e) = (gba^{-1}, ge).$$

It follows that there exists a G-torsor $\mathrm{GL}(N) \times E \to (\mathrm{GL}(N) \times E)/G$ (with respect to the diagonal G-action) and $(\mathrm{GL}(N) \times E)/G \to \operatorname{Spec} K$ is a $\mathrm{GL}(N)$-torsor.

Consider the following diagram with the two pullback squares and the upper horizontal maps given by projections:

$$\begin{array}{ccccc}
\mathrm{GL}(N) & \longleftarrow & \mathrm{GL}(N) \times E & \longrightarrow & E \\
{\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow \\
X & \xleftarrow{\;u\;} & (\mathrm{GL}(N) \times E)/G & \xrightarrow{\;s\;} & \operatorname{Spec} K
\end{array}$$

Since every $\mathrm{GL}(N)$-torsor over a field is trivial, the variety $(\mathrm{GL}(N) \times E)/G$ is isomorphic to $\mathrm{GL}(N)$ over K. Since K is an infinite field, the K-points in $\mathrm{GL}(N)$ are dense, so that there is a point $t\colon \operatorname{Spec} K \to (\mathrm{GL}(N) \times E)/G$ with the image in $u^{-1}(U)$ such that the composition $s \circ t$ is the identity of $\operatorname{Spec} K$. It follows that

$$h = (s \circ t)^*(h) = t^*(s^*(h)) = t^*(g) = t^*u^*(f) = (u \circ t)^*(f)$$

and we can take $x = u \circ t$. □

Let $I := \mathrm{GL}(N)$ and let $f\colon I \to X$ be the versal G-torsor of Theorem A.1. Set $L = F(X)$. The generic fiber $E_{\mathrm{gen}} \to \operatorname{Spec}(L)$ of f is a generic G-torsor.

Theorem A.2. *Let $h\colon E \to \operatorname{Spec} K$ be a G-torsor over a field extension K/F. Then the index $i(E)$ of E divides $i(E_{\mathrm{gen}})$.*

Proof. Suppose first that F is an infinite field. Let $z \in E_{\mathrm{gen}}$ be a closed point of degree d. It sufficed to show that $i(E)$ divides d. The residue field $\widetilde{L} := L(z)$ is a field extension of L of degree d.

Write $r\colon \widetilde{X} \to X$ for the normalization of X in $\widetilde{L}$, so $F(\widetilde{X}) = \widetilde{L}$. The composition

$$z = \operatorname{Spec} \widetilde{L} \hookrightarrow E_{\mathrm{gen}} \to I$$

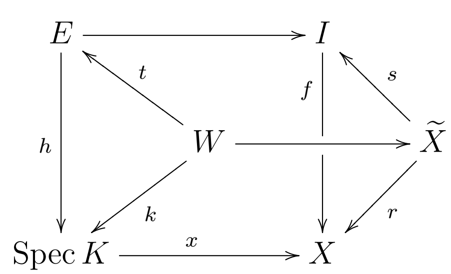
yields a rational morphism $s\colon \widetilde{X} \dashrightarrow I$ such that the composition $\widetilde{X} \dashrightarrow I \xrightarrow{f} X$ coincides with r.

Lemma A.3. *Let $\widetilde{U} \subset \widetilde{X}$ be a nonempty open subset. Then there is a nonempty open subset $U \subset X$ such that $r^{-1}(U) \subset \widetilde{U}$.*

Proof. We may assume that $X = \operatorname{Spec} A$ and $\widetilde{X} = \operatorname{Spec} \widetilde{A}$ are affine schemes and $\widetilde{U} = D_{\widetilde{X}}(\tilde{a})$ for a nonzero $\tilde{a} \in \widetilde{A}$ is a principal open set. Since $\widetilde{A}$ is integral over A, $\tilde{a}$ divides a nonzero element $a \in A$. It follows that $r^{-1}(D_X(a)) = D_{\widetilde{X}}(a) \subset D_{\widetilde{X}}(\tilde{a}) = \widetilde{U}$. □

Let $\widetilde{U} \subset \widetilde{X}$ be the open set of definition of the rational morphism s. By Lemma A.3, shrinking X (together with $\widetilde{X}$ and I) we may assume that s is regular (everywhere defined). Moreover, shrinking X further, we may also assume that the finite morphism $r\colon \widetilde{X} \to X$ is flat (see [31, Proposition 29.27.1]).

Recall that $h\colon E \to \operatorname{Spec} K$ is a G-torsor over a field extension K/F. By Theorem A.1, there is a morphism $x\colon \operatorname{Spec} K \to X$ over F such that the pullback of $f\colon I \to X$ with respect to x is isomorphic to h. Write $W \to \operatorname{Spec} K$ for the pullback of $r\colon \widetilde{X} \to X$ with respect to x. We have the following diagram:



Since r is flat and finite of degree d, W is finite of degree d over K. In other words, the class $[W]$ of W in the group of 0-cycles $Z_0(W)$ on W has degree d (over K). It follows that the image of $[W]$ under the push-forward homomorphism $t_*\colon Z_0(W) \to Z_0(E)$ is a 0-cycle of degree d, therefore, $i(E)$ divides $d = \deg(z)$.

Now suppose that F is a finite field. Given a prime number p, let F'/F be an infinite algebraic field extension such that the degree of every subextension in F'/F finite over F is prime to p. Set $X' := X_{F'} := X \times_F \operatorname{Spec} F'$.

Lemma A.4. *The ratio $i(X)/i(X')$ is an integer prime to p.*

Proof. Let $w : X' \to X$ be the first projection. The composition

$$Z_0(X) \xrightarrow{w^*} Z_0(X') \xrightarrow{\deg_{X'}} \mathbb{Z},$$

where $\deg_{X'}$ is the degree map of X', coincides with the degree map $\deg_X$ of X. Therefore, $i(X)/i(X')$ is an integer.

Let $c \in Z_0(X')$ be a 0-cycle on X'. There is a finite subextension F''/F of F'/F such that c comes from $Z_0(X_{F''})$. The composition

$$Z_0(X_{F''}) \xrightarrow{w_*} Z_0(X) \xrightarrow{\deg_X} \mathbb{Z}$$

coincides with $[F'' : F] \cdot \deg_X$, hence $i(X)$ divides $[F'' : F] \cdot \deg(c)$ and hence $i(X)$ divides $[F'' : F] \cdot i(X')$. Since $[F'' : F]$ is prime to p, the integer $i(X)/i(X')$ is prime to p. □

By Lemma A.4,

$$v_p(i(E)) = v_p(i(E')) \quad \text{and} \quad v_p(i(E_{\text{gen}})) = v_p(i(E'_{\text{gen}})),$$

where v_p is the p-adic valuation, $E' := E \times_F \operatorname{Spec} F'$, and $E'_{\text{gen}} := E_{\text{gen}} \times_F \operatorname{Spec} F'$. Note that E'_{gen} is a generic torsor for the group scheme $G' := G \times_F \operatorname{Spec} F'$ over the field F'. Since F' is an infinite field, by the first part of the proof, $i(E')$ divides $i(E'_{\text{gen}})$. It follows that $v_p(i(E)) \le v_p(i(E_{\text{gen}}))$ for every p, hence $i(E)$ divides $i(E_{\text{gen}})$. □

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