

# INCOMPRESSIBILITY OF PRODUCTS OF WEIL TRANSFERS OF GENERALIZED SEVERI-BRAUER VARIETIES

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ABSTRACT. We generalize the result of [11] on incompressibility of Galois Weil transfer of generalized Severi-Brauer varieties to direct products of varieties of such type; as shown in [11], this is needed to compute essential dimension of representations of finite groups. We also provide a generalization to non-Galois (separable) Weil transfer.

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## 1. INTRODUCTION

Let  $F$  be a field and let  $L/F$  be a finite Galois field extension. A central division  $L$ -algebra  $D$  is *balanced*, if for every  $g \in \text{Gal}(L/F)$ , the conjugate  $L$ -algebra  $g(D)$ , obtained from  $D$  by the base change  $g : L \rightarrow L$ , is Brauer-equivalent to a tensor power of  $D$ .

For a balanced  $D$  as above, we refer as  *$\mathcal{RSB}$ -variety* to the Weil transfer  $\mathcal{R}_{L/F}$  of a generalized Severi-Brauer variety  $\text{SB}(j, D)$  of  $D$  ( $j = 1, \dots, \deg D$ ). Let  $p$  be a prime integer. An  *$\mathcal{RSB}$ -variety* is a  *$p\mathcal{RSB}$ -variety*, if the integers  $[L : F]$ ,  $\deg D$ , and  $j$  are  $p$ -powers.

The following problem has been raised in [11] for needs of computing essential  $p$ -dimension of a representation of a finite group: compute canonical  $p$ -dimension of an arbitrary finite direct product of  *$\mathcal{RSB}$ -varieties*. This problem has been solved in [11] in

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the case of a single  $\mathcal{RSB}$ -variety. The core result producing the solution was the following statement on  $p$ -incompressibility of  $p\mathcal{RSB}$ -varieties (we refer to [11] for an introduction and further references on  $p$ -incompressibility, canonical  $p$ -dimension, Weil transfer, etc.):

**Theorem 1.1** ([11, Theorem 11.2]). *Any  $p\mathcal{RSB}$ -variety is  $p$ -incompressible.*

In the present paper, we generalize Theorem 1.1 in two directions. First, we replace Galois field extensions in the definition of  $p\mathcal{RSB}$ -varieties by separable ones. Although this generalization is not motivated anymore by the study of representations, it is a natural step to do. The statement remains the same (see Theorem 2.3), as does the main outline of the proof. The main change is in the definition of a balanced algebra over a separable extension. All this is done in §2.

As a second (and principal) generalization, we establish a criterion of  $p$ -incompressibility of a product of  $p\mathcal{RSB}$ -varieties, see Corollary 3.6. The basic result here is a  $p$ -incompressibility criterion for a product of a  $p\mathcal{RSB}$ -variety by an arbitrary projective homogeneous (under an action of a semisimple affine algebraic group) variety given in Theorem 3.1.

As a particular case of Corollary 3.6, we recover in §4a (with a new proof and a simplified statement) an old result [10, Theorem 2.1] on  $p$ -incompressibility of products of Severi-Brauer varieties. We can also now determine the canonical  $p$ -dimension of any  $p$ -primary Weil transfer of a product of this type, see §4b. In particular, we show that a  $p$ -primary Weil transfer of a  $p$ -incompressible product of this type is also  $p$ -incompressible. Moreover, the latter statement also holds for generalized Severi-Brauer varieties in place of the usual ones, see §4c.

In the last section (§5), we drop the balance assumption to do a complete analysis of a quadratic field extension. Note however that the balanced case, motivated by the representations, also looks interesting (not only for a quadratic  $L/F$ ) from the following viewpoint: for a  $p\mathcal{RSB}$ -variety  $Y$ , the integer  $\text{cd}_p(Y) = \dim Y$  turns out to be much bigger than  $\text{cd}_p(Y_L) = (\dim Y)/[L : F]$  so that the role of the field extension and the Weil transfer shows up to be crucial. In the imbalanced case however, the variety  $Y_L$  may, for instance, happen to be  $p$ -incompressible (c.f. Example 5.3), trivially implying  $p$ -incompressibility of  $Y$  itself.

Most of our terminology and notation being introduced “on the move”, we only mention here that smooth projective varieties  $X$  and  $Y$  over  $F$  are called *equivalent* if there exists a rational map  $X \dashrightarrow Y$  and a rational map  $Y \dashrightarrow X$ . Equivalent varieties have the same canonical ( $p$ -)dimension, [11, Lemma 3.3(a)].

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## 2. NON-GALOIS EXTENSIONS

In this section we generalize Theorem 1.1 to the case of an arbitrary finite separable (not necessarily Galois) field extension  $L/F$  of degree a power of  $p$ .

Here is the “Galois to separable” generalization of [11, Lemma 11.1] (with practically the same proof). Here and everywhere else in the paper, the motives we are talking about

are Chow motives with coefficients in  $\mathbb{F}_p$  (the finite field of  $p$  elements), see e.g. [2, §64]. Weil transfers of motives have been introduced in [5]. *Corestriction* of motives is from [7, §3].

**Lemma 2.1.** *Let  $L/F$  be an arbitrary finite separable field extension and let  $E/F$  be a finite Galois field extension containing  $L$ . For and any  $m \geq 1$ , let  $M_1, \dots, M_m$  be  $m$  motives over  $L$ . Then the motive  $\mathcal{R}_{L/F}(M_1 \oplus \dots \oplus M_m)$  decomposes into a direct sum*

$$\mathcal{R}_{L/F}(M_1 \oplus \dots \oplus M_m) \simeq \mathcal{R}_{L/F}M_1 \oplus \dots \oplus \mathcal{R}_{L/F}M_m \oplus N,$$

where  $N$  is a direct sum of corestrictions to  $F$  of motives over fields  $K$  with  $F \subsetneq K \subset E$ .  $\square$

Let  $L/F$  be a finite separable field extension. We generalize the notion of a *balanced* central simple  $L$ -algebra  $D$  (which we already have in the Galois case, see §1). A central simple  $L$ -algebra  $D$  is *balanced*, if for a Galois field extension  $E/F$  containing  $L$  and for any  $F$ -embedding of  $L$  into  $E$  with an image  $L_0$ , the algebra over the composite  $LL_0$  of the subfields  $L, L_0 \subset E$  obtained from  $D$  by the base change  $L \rightarrow L_0 \subset LL_0$  is Brauer-equivalent to a tensor power of  $D_{LL_0}$ . This definition does not depend on the choice of  $E$ ;  $E$  can be taken to be a Galois closure of  $L/F$ .

From now on, we extend the notion of a  $p\mathcal{RSB}$ -variety, introduced in §1, by allowing separable (not necessarily Galois) Weil transfer:

**Definition 2.2.** Let  $p$  be a prime number. An  $F$ -variety  $Y$  is a  $p\mathcal{RSB}$ -variety, if there exists a finite  $p$ -primary separable field extension  $L/F$  with a balanced division  $L$ -algebra  $D$  of degree  $p^n$  such that  $X \simeq \mathcal{R}_{L/F} \text{SB}(p^i, D)$  for some  $i = 0, \dots, n$ .

**Theorem 2.3.** *For any prime number  $p$ , any  $p\mathcal{RSB}$ -variety is  $p$ -incompressible.*

*Proof.* Let  $p$  be a prime number,  $L/F$  a finite separable field extension of degree a power of  $p$ ,  $D$  a balanced central division  $L$ -algebra of degree  $p^n$  for some  $n \geq 0$ , and  $X$  the generalized Severi-Brauer variety  $\text{SB}(p^i, D)$  of  $D$  with some  $i = 0, 1, \dots, n$ . Let us prove that the variety  $\mathcal{R}_{L/F}X$ , given by the Weil transfer of  $X$ , is  $p$ -incompressible.

Let  $E$  be a finite Galois field extension of  $F$  containing  $L$  (for instance,  $E$  can be a Galois closure of  $L/F$ ). The question on canonical  $p$ -dimension of  $\mathcal{R}_{L/F}X$  easily reduces to the case where  $[E : F]$  is also a power of  $p$ . Indeed, let  $G$  be the Galois group of  $E/F$  and let  $H$  be its subgroup corresponding to  $L$ . Let  $G'$  be a Sylow  $p$ -subgroup of  $G$  containing a Sylow  $p$ -subgroup  $H'$  of  $H$ . We get the following diagram of subgroups and the corresponding diagram of subfields, where the sign  $p^?$  marks  $p$ -primary while the sign  $p/\not$  marks indexes/degrees coprime to  $p$ :

$$\begin{array}{ccc} & \{1\} & \\ & |_{p^?} & \\ & H' & \\ p/\not & \swarrow & \searrow p^? \\ H & & G' \\ & \nwarrow p^? & \nearrow p/\not \\ & G & \end{array} \qquad \begin{array}{ccc} & E & \\ & |_{p^?} & \\ & L' & \\ p/\not & \swarrow & \searrow p^? \\ L & & F' \\ & \nwarrow p^? & \nearrow p/\not \\ & F & \end{array}$$

Since the degree  $[F' : F]$  is not divisible by  $p$ , canonical  $p$ -dimension of  $\mathcal{R}_{L/F}X$  does not change under the base change  $F'/F$ . Moreover,  $(\mathcal{R}_{L/F}X)_{F'} \simeq \mathcal{R}_{L'/F'}(X_{L'})$  because

$L' \simeq L \otimes_F F'$ . Finally, observe that the index of a  $p$ -primary central simple  $L$ -algebra is not changed under the base change  $L'/L$ . In particular,  $D_{L'}$  is still a division algebra. Moreover, this central division  $L'$ -algebra is still balanced (now with respect to the subfield  $F' \subset L'$ ).

For the rest of the proof we will assume that the degree  $[E : F]$  is a  $p$ -power. We follow the proof of Theorem 1.1, given in [11, Theorem 11.2], and only indicate the changes.

The first place where a change is needed is where we compute the index of  $D$  over the field  $L' = F' \otimes_F L$ , where  $F'$  is the function field of the variety  $\mathcal{R}_{L/F} \mathbf{SB}(p^{n-1}, D)$ . In the case of Galois  $L/F$ , the Weil transfer disappears because  $\mathcal{R}_{L/F} \mathbf{SB}(p^{n-1}, D)_L$  is the product of the generalized Severi-Brauer varieties given by the conjugate algebras. This happens because of the isomorphism

$$L \otimes_F L \simeq L \times \cdots \times L$$

of  $L$ -algebras, where  $L$  acts on  $L \otimes L$  on the left. Note that this isomorphism is also  $L$ -linear for the right action of  $L$  on the tensor product if one lets  $L$  act on the factors of  $L \times \cdots \times L$  by means of the  $F$ -automorphisms of  $L$ .

Under the assumptions of Theorem 2.3 however,  $L \otimes_F L$  is identified with  $L_1 \times \cdots \times L_r$ , where each  $L_i$  is  $LL_0$  for an appropriate (depending on  $i$ ) choice of  $L_0$  as in the definition of balanced algebra. Since  $[L_i : L] > 1$  for at least one value of  $i$ , we cannot avoid the Weil transfer here and may want to use the index reduction formulas for the Weil transfer of generalized Severi-Brauer varieties from [15] (see also [14]). But this is not really necessary. Indeed, since  $D$  is balanced, for any  $i$  the  $L_i$ -algebra in question is Brauer-equivalent to a tensor power of  $D_{L_i}$ , its index will divide  $p^{n-1}$  if we extend  $L$  to the function field of  $\mathbf{SB}(p^{n-1}, D)$ . The Weil transfer becomes rational over this extension of  $L$  so that the index of  $D$  won't be affected by passing to the function field of the Weil transfer.

We have explained why the index of  $D_{L'}$  is indeed  $p^{n-1}$ . The final adjustment we need to make, in order to adapt the proof of Theorem 1.1 to the setting of Theorem 2.3, is in the choice of the degree  $p$  Galois field subextension  $\tilde{L}/F$  of  $L/F$ . Recall that the Galois group  $G$  of  $E/F$  is a  $p$ -group. Let  $H$  be its subgroup corresponding to the field  $L$ . By [3, Theorem 4.2.1],  $H$  is contained in a normal subgroup  $\tilde{H}$  of  $G$  such that  $[G : \tilde{H}] = p$ . We take for  $\tilde{L}$  the field corresponding to  $\tilde{H}$  and we have to compute canonical  $p$ -dimension of the variety  $\mathcal{R}_{L/F}(X)_{\tilde{L}}$ .

The variety  $\mathcal{R}_{L/F}(X)_{\tilde{L}}$  is isomorphic to the product  $\prod_{\tilde{g} \in \tilde{G}} \tilde{g}(\mathcal{R}_{L/\tilde{L}}X)$ , where  $\tilde{G} = G/\tilde{H}$  is the Galois group of  $\tilde{L}/F$ . Since  $D$  is balanced, this product is equivalent to its factor  $\mathcal{R}_{L/\tilde{L}}X$  given by  $\tilde{g} = 1$ . In particular, the canonical  $p$ -dimension of  $\mathcal{R}_{L/F}(X)_{\tilde{L}}$  is given by the dimension of  $\mathcal{R}_{L/\tilde{L}}X$ . After this is established, the remainder of the proof of Theorem 1.1 goes through unchanged.  $\square$

### 3. INCOMPRESSIBILITY OF PRODUCTS

Here is our basic result on  $p$ -incompressibility of products:

**Theorem 3.1.** *Let  $Y$  be a  $p\mathcal{R}\mathbf{SB}$ -variety and let  $Z$  be a projective homogeneous  $F$ -variety. The product  $Y \times Z$  is  $p$ -incompressible provided that the varieties  $Y_{F(Z)}$  and  $Z_{F(Y)}$  are so.*

*Proof.* Let  $Y := \mathcal{R}_{L/F} \mathbf{SB}(p^i, D)$ , where  $L/F$  is a finite separable field extension of degree  $p^r$  with  $r \geq 0$ ,  $D$  is a balanced central division  $L$ -algebra of degree  $p^n$  with  $n \geq 0$ , and  $i = 0, \dots, n$ . Let  $Z$  be a projective homogeneous  $F$ -variety such that the varieties  $Y_{F(Z)}$  and  $Z_{F(Y)}$  are  $p$ -incompressible. We are going to prove that the product  $Y \times Z$  is  $p$ -incompressible.

By Theorem 2.3, the assumption requiring that  $Y_{F(Z)}$  is  $p$ -incompressible is equivalent to the assumption that  $D_{L(Z)}$  is a division algebra.

Since canonical  $p$ -dimension is not changed under base field extensions of degree prime to  $p$  (see [12, Proposition 1.5(2)]), we may assume that there exists a finite  $p$ -primary Galois field extension  $E/F$  containing  $L$  and such that  $G_E$  is of inner type, where  $G$  is the semisimple algebraic group over  $F$  acting on  $Z$ . This assumption allows us to apply results of [7] and [8, §6].

For any  $j = 0, \dots, n$ , we set  $X^j := \mathbf{SB}(p^j, D)$  and  $Y^j := \mathcal{R}_{L/F} X^j$ . We also set  $X := X^i$  so that  $Y = \mathcal{R}_{L/F} X$ .

We induct on  $n$ . For  $n = 0$  the statement is trivial. From now on, we assume that  $n > 0$ .

For  $i = n$  the statement is trivial. From now on, we assume that  $i < n$ .

Let  $F'$  be the function field of the variety  $Y^{n-1}$ . Let  $L' := F' \otimes_F L$ . Since  $D$  is balanced, the index of the central simple  $L'$ -algebra  $D_{L'} := D \otimes_L L' = D \otimes_F F'$  is  $p^{n-1}$  so that there exists a central division  $L'$ -algebra  $D'$  such that the algebra of  $(p \times p)$ -matrices over  $D'$  is isomorphic to  $D_{L'}$ . We set  $X'^j := \mathbf{SB}(p^j, D')$ ,  $Y'^j := \mathcal{R}_{L'/F'} X'^j$ ,  $X' := X'^i$ , and  $Y' := Y'^i$ . By [4] (see also [1, Theorem 7.5]) and [9, Theorem 3.8], the motive of the variety  $X_{L'}$  decomposes into a direct sum

$$M(X_{L'}) \simeq M(X') \oplus M(X')(p^{i+n-1}) \oplus \\ M(X')(2p^{i+n-1}) \oplus \dots \oplus M(X')((p-1)p^{i+n-1}) \oplus N,$$

where  $N$  is a direct sum of some shifts of the upper motives  $U(X'^j)$  of some varieties  $X'^j$  with  $j < i$ . Therefore, by Lemma 2.1 and [5, Theorem 5.4] (which we use only to determine the shifts), the motive of the variety  $Y_{F'} = \mathcal{R}_{L'/F'}(X_{L'})$  decomposes into a direct sum

$$M(Y_{F'}) \simeq M(Y') \oplus M(Y')(p^{r+i+n-1}) \oplus \\ M(Y')(2p^{r+i+n-1}) \oplus \dots \oplus M(Y')((p-1)p^{r+i+n-1}) \oplus N \oplus N',$$

where now  $N$  is a direct sum of shifts of  $U(Y'^j)$  with  $j < i$ , and  $N'$  is a direct sum of corestrictions of motives over fields  $K$  with  $F \subsetneq K \subset E$ . It follows that

$$M(Y \times Z)_{F'} \simeq M(Y' \times Z_{F'}) \oplus M(Y' \times Z_{F'})(p^{r+i+n-1}) \oplus \\ M(Y' \times Z_{F'})(2p^{r+i+n-1}) \oplus \dots \oplus M(Y' \times Z_{F'})((p-1)p^{r+i+n-1}) \oplus N \oplus N',$$

with  $N'$  of the same shape as before and with  $N$  being a direct sum of shifts of

$$U(Y'^j) \otimes M(Z), \quad j < i.$$

We claim that the variety  $Y' \times Z_{F'}$  is  $p$ -incompressible by the induction hypothesis. To check the claim, we check that the varieties  $Z_{F'(Y')}$  and  $Y'_{F'(Z)}$  are  $p$ -incompressible. To

check that  $Z_{F'(Y')}$  is  $p$ -incompressible, we check that  $Z$  over a larger field  $F'(Y' \times Y_{F'})$  is so. The field  $F'(Y' \times Y_{F'}) = F'(Y)(Y')$  is purely transcendental over  $F(Y)$  and  $Z_{F(Y)}$  is  $p$ -incompressible (this is the place in the proof of Theorem 3.1, where the assumption on  $Z_{F(Y)}$  is used). Since canonical  $p$ -dimension of projective homogeneous varieties does not change under purely transcendental field extensions,<sup>1</sup>  $Z$  over  $F'(Y' \times Y_{F'})$  is  $p$ -incompressible.

To check that  $Y'_{F'(Z)}$  is  $p$ -incompressible, we check that  $D'_{L'(Z)}$  is a division algebra. Since  $L' = L(Y^{n-1})$ , we have  $L'(Z) = L(Y^{n-1} \times Z) = L(Z)(Y^{n-1})$ . Since  $D_{L(Z)}$  is a balanced division algebra over  $F(Z)$ ,  $D_{L(Z)(Y^{n-1})}$  has index  $p^{n-1}$ . Therefore  $D'_{L'(Z)}$  is a division algebra.

The claim being proved, it follows that

$$(3.2) \quad M(Y \times Z)_{F'} \simeq U(Y' \times Z_{F'}) \oplus U(Y' \times Z_{F'})(p^{r+i+n-1}) \oplus \\ U(Y' \times Z_{F'})(2p^{r+i+n-1}) \oplus \cdots \oplus U(Y' \times Z_{F'})((p-1)p^{r+i+n-1}) \oplus N,$$

with  $N$  having the property that no summand of its complete decomposition is isomorphic to a shift of  $U(Y' \times Z_{F'})$ .

We want to show that the variety  $Y \times Z$  is  $p$ -incompressible. Let  $l$  be the number of those summands in the complete decomposition of the motive  $U(Y \times Z)_{F'}$ , which are isomorphic to a shift of  $U(Y' \times Z_{F'})$ . We have  $1 \leq l \leq p$  and it suffices to show that  $l = p$ . Indeed,  $l = p$  implies that the complete decomposition of  $U(Y \times Z)_{F'}$  contains the summand  $U(Y' \times Z_{F'})$  of (3.2) with the maximal shift  $(p-1)p^{r+i+n-1}$ . Therefore

$$\mathrm{cd}_p(Y \times Z) = \dim U(Y \times Z) = \dim U(Y \times Z)_{F'} \geq \\ \dim U(Y' \times Z_{F'}) + (p-1)p^{r+i+n-1} = \dim(Y \times Z).$$

(We refer to [6, Theorem 5.1] for the relation between canonical  $p$ -dimension and dimension of the upper motive of a projective homogeneous variety, used here.)

Our next claim is:  $l$  divides  $p$  (therefore  $l = 1$  or  $l = p$ , and we will only need to show that  $l \neq 1$ ). To prove the claim, we consider the complete motivic decomposition of  $Y \times Z$ . It contains several shifts of  $U(Y \times Z)$  (it contains one non-shifted  $U(Y \times Z)$  and - as we hope - no other shifts of this motive, but we do not know whether the hope comes true by now). Let  $N$  be any of the remaining (indecomposable) summands. We affirm that no summand of the complete decomposition of  $N_{F'}$  is isomorphic to a shift of  $U(Y' \times Z_{F'})$ . Clearly, this affirmation implies the claim that  $l$  divides  $p$ .

To prove the affirmation, let us note that  $N$  can be of two alternative types. The first type is given by corestriction to  $F$  of a motive over a field  $K$  with  $F \subsetneq K \subset E$ . For such  $N$ , any indecomposable summand of  $N_{F'}$  is a corestriction to  $F'$  of a motive over  $K' := K \otimes_F F'$  (see [7, Proposition 3.1]) which is never isomorphic to a shift of the upper motive of a projective homogeneous  $F'$ -variety.

The second type of  $N$  is a shift of  $U(T)$ , where  $T$  is a projective homogeneous variety with  $\deg \mathrm{Ch}_0(T_{F(Y \times Z)}) = 0$  (and with  $\deg \mathrm{Ch}_0(Y \times Z)_{F(T)} = \mathbb{F}_p$ ). Over  $F'$  we still have

<sup>1</sup>To see that canonical  $p$ -dimension of projective homogeneous varieties does not change under a purely transcendental base field extension, one may use the characterization [8, Corollary 6.2] of the canonical  $p$ -dimension in terms of algebraic cycles together with the fact that a purely transcendental base field extension does not affect the Chow groups.

$\deg \text{Ch}_0(T_{F'(Y \times Z)}) = 0$  (and  $\deg \text{Ch}_0(Y \times Z)_{F'(T)} = \mathbb{F}_p$ ) because the field extension

$$F'(Y \times Z)/F(Y \times Z)$$

is purely transcendental. Any indecomposable summand of  $N_{F'}$ , which is not corestriction from some  $F \subsetneq K \subset E$ , is a shift of  $U(S)$ , where  $S$  is a projective homogeneous  $F'$ -variety with  $\deg \text{Ch}_0(T_{F'(S)}) = \mathbb{F}_p$ . It follows that  $\deg \text{Ch}_0(S_{F'(Y \times Z)}) = 0$  (otherwise we would be in contradiction with  $\deg \text{Ch}_0(T_{F'(Y \times Z)}) = 0$ ) implying that no shift of  $U(S)$  is isomorphic to  $U(Y' \times Z_{F'})$ , see [9, Corollary 2.15] for the criterion of isomorphism for upper motives.

We proved the affirmation and the claim. This means that we only need to show that  $l \neq 1$  to finish the proof of Theorem 3.1. So, we assume that  $l = 1$  and we look for a contradiction. By [8, Proposition 2.4], the complete decomposition of  $U(Y \times Z)_{F'}$  contains as a summand the motive  $U(Y' \times Z_{F'})$  shifted by the difference

$$\dim U(Y \times Z) - \dim U(Y' \times Z_{F'}).$$

Therefore,  $l = 1$  implies that the above difference is 0, and we come to

$$\text{cd}_p(Y \times Z) = \dim U(Y \times Z) = \dim U(Y' \times Z_{F'}) = d,$$

where  $d := \dim(Y' \times Z_{F'}) = \dim Y' + \dim Z$ .

By [8, Proposition 6.1], there exist  $\alpha \in \text{Ch}^d(Y \times Z)_{F'(Y \times Z)}$  and  $\beta \in \text{Ch}_d(Y \times Z)$  with  $\deg(\alpha \cdot \beta) \neq 0 \in \mathbb{F}_p$ . In the last formula, we consider both cycles over a common field extension of their fields of definition, before we multiply them. We use this convention below (in similar formulas on degree of products) as well.

Since  $\text{cd}_p(Y_{F'}) = \dim Y' =: d'$ , we can find  $\alpha' \in \text{Ch}^{d'}(Y_{F'(Y)})$  and  $\beta' \in \text{Ch}_{d'}(Y_{F'})$  with  $\deg(\alpha' \cdot \beta') \neq 0$ . Using these  $\alpha'$  and  $\beta'$  and a rational point  $\mathbf{pt} \in Z_{F(Z)}$ , we get the cycles

$$\alpha' \times [\mathbf{pt}] \in \text{Ch}^d(Y \times Z)_{F'(Y \times Z)} \quad \text{and} \quad \beta' \times [Z] \in \text{Ch}_d(Y \times Z)_{F'},$$

having the same property as  $\alpha$  and  $\beta$ :

$$\deg \left( (\alpha' \times [\mathbf{pt}]) \cdot (\beta' \times [Z]) \right) \neq 0.$$

It follows by [8, Lemma 6.5] that one can "mix up" the old cycles with the new ones and get the relation

$$\deg \left( (\alpha' \times [\mathbf{pt}]) \cdot \beta \right) \neq 0.$$

Since  $\alpha' \times [\mathbf{pt}] = (\alpha' \times [Z]) \cdot ([Y] \times [\mathbf{pt}])$ , the last degree relation can be rewritten as  $\deg(\alpha' \cdot \beta'') \neq 0$ , where  $\beta'' \in \text{Ch}_{d'}(Y_{F(Z)})$  is the push-forward of the product  $([Y] \times [\mathbf{pt}]) \cdot \beta$  along the projection  $(Y \times Z)_{F(Z)} \rightarrow Y_{F(Z)}$ . Since the field extension  $F'(Y)/F(Y)$  is purely transcendental, there exists  $\alpha'' \in \text{Ch}^{d'}(Y_{F(Y)})$  mapped to  $\alpha'$  under the change of field homomorphism. Changing notation, we write  $\alpha''$  for the image of  $\alpha''$  in  $\text{Ch}^{d'}(Y_{F(Z)(Y)})$ .

The cycles  $\alpha'' \in \text{Ch}^{d'}(Y_{F(Z)(Y)})$  and  $\beta'' \in \text{Ch}_{d'}(Y_{F(Z)})$  thus constructed have the property  $\deg(\alpha'' \cdot \beta'') \neq 0$ . It follows by [8, Corollary 6.2] that  $\text{cd}_p(Y_{F(Z)}) \leq d'$ . Since

$$(3.3) \quad d' = \dim Y' = p^i(p^{n-1} - p^i) < p^i(p^n - p^i) = \dim Y,$$

the relation  $\text{cd}_p(Y_{F(Z)}) \leq d'$  obtained contradicts the assumption on  $p$ -incompressibility of the variety  $Y_{F(Z)}$ .  $\square$

The necessary condition for  $p$ -incompressibility of a product of projective homogeneous varieties, showing up in the following lemma, turns out to be sufficient in the case of the special varieties we are interested in (see Corollary 3.5):

**Lemma 3.4.** *For any projective homogeneous  $F$ -varieties  $X$  and  $Y$ ,*

$$\mathrm{cd}_p(X \times Y) \leq \mathrm{cd}_p(X) + \mathrm{cd}_p(Y_{F(X)}).$$

*In particular, a necessary condition for  $p$ -incompressibility of  $X \times Y$  is  $p$ -incompressibility of  $X_{F(Y)}$  and  $Y_{F(X)}$ .*

*Proof.* Again, we are using the characterization of canonical  $p$ -dimension of projective homogeneous varieties given in [8, Corollary 6.2]. Since canonical  $p$ -dimension is not changed under base field extensions of degree prime to  $p$  (see [12, Proposition 1.5(2)]), we may assume that the condition of [8, Corollary 6.2] on the projective homogeneous variety is satisfied for  $X$  and  $Y$ : the semisimple algebraic groups acting on them become of inner type over a finite  $p$ -primary extension of  $F$ . We set  $x := \mathrm{cd}_p(X)$  and  $y := \mathrm{cd}_p(Y_{F(X)})$ . We find  $\alpha_X \in \mathrm{Ch}^x(X_{F(X)})$  and  $\beta_X \in \mathrm{Ch}_x(X)$  with  $\deg(\alpha_X \cdot \beta_X) \neq 0$ . Similarly, we find  $\alpha_Y \in \mathrm{Ch}^y(Y_{F(X)(Y)})$  and  $\beta_Y \in \mathrm{Ch}_y(Y_{F(X)})$  with  $\deg(\alpha_Y \cdot \beta_Y) \neq 0$ . Let  $\beta'_Y \in \mathrm{Ch}_{\dim X+y}(X \times Y)$  be an element mapped to  $\beta_Y$  under the surjection

$$\mathrm{Ch}_{\dim X+y}(X \times Y) \twoheadrightarrow \mathrm{Ch}_y(Y_{F(X)})$$

given by the pull-back along the morphism  $Y_{F(X)} \rightarrow X \times Y$  induced by the generic point of  $X$ . We set

$$\alpha := \alpha_X \times \alpha_Y \in \mathrm{Ch}^{x+y}(X \times Y)_{F(X \times Y)} \quad \text{and} \quad \beta := (\beta_X \times [Y]) \cdot \beta'_Y \in \mathrm{Ch}_{x+y}(X \times Y).$$

We have the relation  $\deg(\alpha \cdot \beta) \neq 0$  showing that  $\mathrm{cd}_p(X \times Y) \leq x + y$ .  $\square$

**Corollary 3.5.** *For two products  $X$  and  $Y$  of  $p\mathcal{RSB}$ -varieties over  $F$ , their product  $X \times Y$  is  $p$ -incompressible if and only if the varieties  $X_{F(Y)}$  and  $Y_{F(X)}$  are  $p$ -incompressible.*

*Proof.* The ‘‘only if’’ part being served by Lemma 3.4, we only prove the ‘‘if’’ part. We write  $X$  as a product  $X_1 \times \cdots \times X_r$  of  $p\mathcal{RSB}$ -varieties and induct on  $r$ . The case of  $r = 1$  follows from Theorem 3.1. For  $r > 1$ , set  $X' := X_2 \times \cdots \times X_r$ . To show that  $X \times Y = X_1 \times (X' \times Y)$  is  $p$ -incompressible, by Theorem 3.1, it suffices to check that  $(X_1)_{F(X' \times Y)}$  and  $(X' \times Y)_{F(X_1)}$  are  $p$ -incompressible. The assumption on  $p$ -incompressibility of  $X_{F(Y)}$  implies (by Lemma 3.4)  $p$ -incompressibility of  $(X_1)_{F(X' \times Y)}$ . The variety  $(X' \times Y)_{F(X_1)}$  is  $p$ -incompressible by induction hypothesis.  $\square$

Here is the most convenient statement to check  $p$ -incompressibility of a general product of  $p\mathcal{RSB}$ -varieties. Basically, it reduces the problem to application of index reduction formulas.

**Corollary 3.6.** *A product  $X_1 \times \cdots \times X_r$  of  $p\mathcal{RSB}$ -varieties over  $F$  is  $p$ -incompressible if and only if  $(X_i)_{F(X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_r)}$  for every  $i = 1, \dots, r$ , is  $p$ -incompressible.  $\square$*



## 4. PARTICULAR PRODUCTS

**4a. Products of Severi-Brauer varieties.** Applying Corollary 3.6 to a product of Severi-Brauer varieties (Weil transfer do not show up here), we get a version of [10, Theorem 2.1]. Note that the proof is different from the original one: it does not involve  $K$ -theory. Also note that the proof of Corollary 3.6 (which includes the proof of Theorem 3.1) becomes shorter and simpler when adapted to this particular case:

**Corollary 4.1.** *Let  $D_1, \dots, D_r$  be  $p$ -primary central division  $F$ -algebras. The product of their Severi-Brauer varieties is  $p$ -incompressible if and only if each  $D_i$  remains division algebra over the function field of the product of the Severi-Brauer varieties of the remaining algebras.  $\square$*

**Remark 4.2.** By the index reduction formula for Severi-Brauer varieties, the condition on  $D_i$  means that the index of every product of  $D_i$  by tensor powers of the remaining algebras is at least the index of  $D_i$ . With this, one may see that  $D_1, \dots, D_r$  satisfy the condition of Corollary 4.1 if and only if their nonzero Brauer classes (put in some/any order increasing the degrees) form a minimal basis in the sense of [10, Remark 2.9].

**Remark 4.3.** In general, the property of being  $p$ -incompressible for a projective homogeneous variety  $X$  is weaker than the property of having indecomposable motive. However, for a generically split  $X$  (i.e., for  $X$  such that the motive of  $X_{F(X)}$  is split, i.e., is a direct sum of shifts of the motive of a point), the above two properties are equivalent. Indeed,  $M(X)$  for a generically split  $X$  is a direct sum of shifts of  $U(X)$  so that  $M(X) = U(X)$  is and only if  $\dim U(X) = \dim X$ .

A product of Severi-Brauer varieties is a generically split projective homogeneous variety. Therefore a product of Severi-Brauer varieties is  $p$ -incompressible if and only if its motive is indecomposable.

**4b. Weil transfer of products of Severi-Brauer varieties.**

**Corollary 4.4.** *Let  $L/F$  be a  $p$ -primary separable field extension and let  $X$  be a product over  $L$  of Severi-Brauer varieties of some balanced  $p$ -primary division  $L$ -algebras. Then  $\text{cd}_p(\mathcal{R}_{L/F}X) = [L : F] \cdot \text{cd}_p(X)$ . In particular,  $\mathcal{R}_{L/F}X$  is  $p$ -incompressible provided that  $X$  is so.*

*Proof.* Taking a minimal basis of the subgroup in  $\text{Br}(L)$  generated by the algebras (note that any  $L$ -algebra representing an element of this subgroup is balanced), consider the product  $X'$  of their Severi-Brauer varieties. The variety  $X'$  is  $p$ -incompressible and equivalent to  $X$ . Moreover, its Weil transfer  $\mathcal{R}_{L/F}X'$  is equivalent to  $\mathcal{R}_{L/F}X$ . Therefore we reduced the proof of the first statement of Corollary 4.4 to the proof of the second one. The second statement follows directly from Corollary 3.6.  $\square$

**4c. Weil transfer of products of generalized Severi-Brauer varieties.** The second conclusion of Corollary 4.4 also holds for generalized Severi-Brauer varieties. Again, the result is an immediate consequence of Corollary 3.6:

**Corollary 4.5.** *Let  $L/F$  be a  $p$ -primary separable field extension and let  $X$  be a product over  $L$  of some generalized Severi-Brauer varieties of some balanced  $p$ -primary division  $L$ -algebras. Then  $\mathcal{R}_{L/F}X$  is  $p$ -incompressible if (and only if)  $X$  is so.  $\square$*

## 5. QUADRATIC EXTENSIONS

Let  $L/F$  be a separable quadratic field extension and let  $D$  be a central division  $L$ -algebra of a 2-primary index. In this section, we determine the canonical 2-dimension of the variety  $\mathcal{R}_{L/F} \mathbf{SB}(D)$  without imposing any restrictions on the conjugate algebra  $g(D)$ , where  $g$  is the non-trivial element of  $G := \text{Gal}(L/F)$ . We will provide two different recipes; which of them has to be applied depends on a property of the group  $A$  generated by the Brauer classes of  $D$  and  $g(D)$ . Since the case of cyclic  $A$  has been already treated, we assume that  $A$  is not cyclic. Therefore  $A/2A \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$  with  $g$  acting by exchanging the summands.

Let  $\alpha \in A \setminus 2A$  be an element with smallest index and let  $D_\alpha$  be a central division  $L$ -algebra representing  $\alpha$ . If the image of  $\alpha$  in  $A/2A$  is not invariant under the action of  $G$ , then the images of  $\alpha$  and  $g(\alpha)$  form a minimal basis of  $A/2A$  in the sense of [10, Remark 2.9]. It follows that the variety  $(\mathcal{R}_{L/F} \mathbf{SB}(D_\alpha))_L \simeq \mathbf{SB}(D_\alpha) \times \mathbf{SB}(g(D_\alpha))$  is 2-incompressible, therefore the variety  $\mathcal{R}_{L/F} \mathbf{SB}(D_\alpha)$  is 2-incompressible as well. Since  $\alpha$  and  $g(\alpha)$  generate  $A$ , the varieties  $\mathcal{R}_{L/F} \mathbf{SB}(D)$  and  $\mathcal{R}_{L/F} \mathbf{SB}(D_\alpha)$  are equivalent. So,  $\text{cd}_2 \mathcal{R}_{L/F} \mathbf{SB}(D) = \text{cd}_2 \mathcal{R}_{L/F} \mathbf{SB}(D_\alpha)$  and we get that

$$(5.1) \quad \text{cd}_2 \mathcal{R}_{L/F} \mathbf{SB}(D) = \dim \mathcal{R}_{L/F} \mathbf{SB}(D_\alpha) = 2(\text{ind } \alpha - 1).$$

Now we assume that the image of  $\alpha$  in  $A/2A$  is invariant under  $G$ . This means that this image is equal to the element  $(1, 1) \in \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ . Let  $\beta \in A$  be an element with smallest index having the property that its image in  $A/2A$  is outside of the subgroup generated by the image of  $\alpha$ . Let  $D_\beta$  be a central division  $L$ -algebra representing  $\beta$ . We claim that in this case

$$(5.2) \quad \text{cd}_2 \mathcal{R}_{L/F} \mathbf{SB}(D) = \dim \mathcal{R}_{L/F} \mathbf{SB}(D_\beta) = 2(\text{ind } \beta - 1).$$

Since the image of  $\beta$  in  $A/2A$  is  $(1, 0)$  or  $(0, 1)$ , it is not invariant under  $G$ . Moreover,  $A$  is generated by  $\beta$  and  $g(\beta)$ . It follows that the variety  $\mathcal{R}_{L/F} \mathbf{SB}(D)$  is equivalent to  $\mathcal{R}_{L/F} \mathbf{SB}(D_\beta)$ . To prove the claim, it suffices to prove that the variety  $\mathcal{R}_{L/F} \mathbf{SB}(D_\beta)$  is 2-incompressible. It is so because the  $F(\mathcal{R}_{L/F} \mathbf{SB}(D_\alpha))$ -variety

$$\mathcal{R}_{L/F} \mathbf{SB}(D_\beta)_{F(\mathcal{R}_{L/F} \mathbf{SB}(D_\alpha))} \simeq \mathcal{R}_{L(\mathcal{R}_{L/F} \mathbf{SB}(D_\alpha)) / F(\mathcal{R}_{L/F} \mathbf{SB}(D_\alpha))} \mathbf{SB}((D_\beta)_{L(\mathcal{R}_{L/F} \mathbf{SB}(D_\alpha))})$$

is 2-incompressible. Indeed,  $g(D_\beta)_{L(\mathcal{R}_{L/F} \mathbf{SB}(D_\alpha))}$  is Brauer-equivalent to a tensor power of  $(D_\beta)_{L(\mathcal{R}_{L/F} \mathbf{SB}(D_\alpha))}$  so that Theorem 1.1 applies. On the other hand,  $(D_\beta)_{L(\mathcal{R}_{L/F} \mathbf{SB}(D_\alpha))}$  is still a division algebra by the Schofield–van den Bergh index reduction formula [16] (see also [13]), since

$$\text{ind}(D_\beta \otimes D_\alpha^{\otimes i}) \geq \text{ind } D_\beta$$

for any  $i$  by the minimality of  $\text{ind } \beta = \text{ind } D_\beta$ .

We finish this section by examples where the above recipes apply.

**Example 5.3.** Let  $l$  be a field of characteristic  $\neq 2$ , let  $L$  be the rational function field over  $l$  in variables  $x, y, x', y'$ , let  $g$  be the  $l$ -automorphism of  $L$  exchanging  $x$  with  $x'$  and  $y$  with  $y'$ , and let  $F$  be the subfield of  $L$  consisting of the elements fixed by  $g$ . The variety  $\mathcal{R}_{L/F} \mathbf{SB}(D)$ , where  $D$  is the quaternion  $L$ -algebra  $(x, y)$ , is 2-incompressible by formula (5.1) with  $D_\alpha = D$ . Note that in this example the algebras  $D$  and  $g(D)$  are, informally speaking, “completely independent”.

In the next example,  $D$  and  $g(D)$  will be “partially dependent”. To get it, we replace  $F$  and  $L$  by the rational function fields

$$F(s_1, s_2, t_1, t_2) \quad \text{and} \quad L(s_1, s_2, t_1, t_2)$$

in some variables  $s_1, s_2, t_1, t_2$ . For  $D := C \otimes (x, y)$  with  $C := (s_1, t_1) \otimes (s_2, t_2)$ , the variety  $\mathcal{R}_{L/F} \text{SB}(D)$  is 2-incompressible by formula (5.2) with  $D_\alpha = (x, y) \otimes (x', y')$  and  $D_\beta = D$ . Note that the non-zero elements of the group  $A$  in this situation are the Brauer classes of the index 8 conjugate algebras  $C \otimes (x, y)$  and  $C \otimes (x', y')$  and the index 4 invariant algebra  $(x, y) \otimes (x', y')$ ; besides,  $2A = 0$ .

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