QUADRATIC FORMS IN I^n OF DIMENSION $2^n + 2^{n-1}$

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ABSTRACT. For $n \geq 3$, we show that every anisotropic quadratic form in I^n of dimension $2^n + 2^{n-1}$ splits over a finite extension of the base field of degree not divisible by 4. The first new case is $n = 4$, were we obtain a classification of the corresponding quadratic forms up to odd degree base field extensions and get this way a strong upper bound on their essential 2-dimension. As well, we compute the reduced Chow group of the maximal orthogonal grassmannian of the quadratic form and conclude that its canonical 2-dimension is $2^n + 2^{n-2} - 2$.

Let F be a field (of any characteristic) and let $I = I(F)$ be the Witt group of classes of even-dimensional non-degenerate quadratic forms over F defined as in [4, §8] (and denoted $I_q(F)$ there). For $n \geq 2$, we write $I^n = I^n(F)$ for the subgroup in $I(F)$ generated by the n-fold Pfister forms. We refer to [4, 9.B] for other equivalent definitions of $I^n(F)$ (denoted $I^n_q(F)$ there).

Any element of I is represented by an anisotropic quadratic form. By the Arason-Pfister Hauptsatz, the smallest possible dimension of a nonzero anisotropic quadratic form in I^n is 2^n (see [4, Theorem 23.7(1)] for the characteristic-free version). The quadratic forms in I^n of dimension 2^n are classified: as a consequence of the Arason-Pfister Hauptsatz and [4, Corollary 23.4)], they are exactly the forms similar to n-fold Pfister forms. In particular, any 2^n-dimensional quadratic form in I^n splits over a finite base field extension of degree dividing 2.

By [13, Proposition 11.5], the smallest exceeding 2^n possible dimension of an anisotropic quadratic form in I^n is $2^n + 2^{n-1}$. We note that for $n = 3$ the original proof is in [12] (characteristic $\neq 2$) and in [1] (arbitrary characteristic); for $n = 4$ in [6] (characteristic $\neq 2$) and in [5, Théorème 4.2.11] (characteristic 2); for $n \geq 5$ (in characteristic $\neq 2$) the original proof is in [15, Theorem 5.4].

For $n = 2$, quadratic forms in I^n of dimension $2^n + 2^{n-1}$ are the well-understood Albert forms. For $n \geq 3$, by [6, Conjecture 2] (characteristic $\neq 2$) and [5, Conjecture 4.3.1] (characteristic 2), quadratic forms in I^n of dimension $2^n + 2^{n-1}$ should be classified as products of an Albert bilinear form (i.e., a 6-dimensional symmetric bilinear form of determinant -1) by a Pfister form (of foldness $n - 2$). In particular, such forms should split over a finite base field extension of degree dividing 2 as well.

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However, the two above conjectures are so far proved for $n = 3$ only: the proof for characteristic $\neq 2$ of [12] is extended to characteristic 2 in [5, Proposition 4.1.2]. The main result of the present note is

Theorem 1. *For any $n \geq 3$ and in any characteristic, every quadratic form in I^n of dimension $2^n + 2^{n-1}$ splits over a finite base field extension of degree not divisible by 4.*

Proof. Let X be a connected component of the highest orthogonal grassmannian of a quadratic form q in $I^n(F)$ of dimension $2^n + 2^{n-1}$. Theorem 1 means that the *index* $i(X)$ of the variety X, defined as the g.c.d. of the degrees of closed points on X, divides 2. In other terms, taking into account Springer's Theorem [4, Corollary 18.5], $i(X) = 2$ provided that q is not split.

We write $\bar{X}$ for X over an algebraic closure of F and we write $\bar{\mathrm{CH}}(X)$ for the ring given by the image of the change of field homomorphism $\mathrm{CH}(X) \to \mathrm{CH}(\bar{X})$ of the Chow rings. Note that the kernel of the change of field homomorphism is the ideal of the elements of finite order. For this reason, $\bar{\mathrm{CH}}(X)$ is sometimes called the *reduced Chow group* of X.

By [4, Theorem 86.12], the ring $\mathrm{CH}(\bar{X})$ is generated by certain homogeneous elements $e_1, \dots, e_l$ of codimensions $1, \dots, l := 2^{n-1} + 2^{n-2} - 1$. It is convenient to define $e_i := 0$ for $i > l$. For any $i \geq 1$, the element e_i is characterized by the property that $(-1)^i 2e_i$ is the ith Chern class of the tautological vector bundle on $\bar{X}$ (see [4, Proposition 87.13]); in particular, $2e_i \in \bar{\mathrm{CH}}(X)$.

Since for any field extension K/F, the anisotropic part of the quadratic form q_K over the field K is either 0, or 2^n, or $2^n + 2^{n-1}$, it follows by [4, Corollary 88.6] (see also [4, Corollary 88.7]) that $e_i \in \bar{\mathrm{CH}}(X)$ for all i different from $k := 2^{n-1} - 1$ and l. By [4, (86.15)], for any $i \geq 1$, we have

$$e_i^2 - 2e_{i-1}e_{i+1} + 2e_{i-2}e_{i+2} - \cdots + (-1)^{i-1}2e_1e_{2i-1} + (-1)^i e_{2i} = 0 \in \mathrm{CH}(\bar{X}).$$

In particular,

$$2e_k e_l = 2e_{k+1}e_{l-1} - 2e_{k+2}e_{l-2} + \cdots \pm 2e_{m-1}e_{m+1} \pm e_m^2 \in \bar{\mathrm{CH}}(X),$$

where $m := (k + l)/2$. Therefore $2e \in \bar{\mathrm{CH}}(X)$, where $e \in \mathrm{CH}(\bar{X})$ is the product $e_1 \dots e_l$ of all the generators. Since e is the class of a 0-cycle of degree 1 (see [4, Corollary 86.10]), the variety X (over F) possesses a 0-cycle of degree 2. □

For $n = 4$, in view of [6, Proposition 4.1] and [5, Proposition 4.3.2], Theorem 1 provides a classification of the corresponding quadratic forms "up to odd degree extensions" which yields a strong upper bound on their *essential 2-dimension*. We provide details right below, starting with the classification result:

Theorem 2. *For a field F (of any characteristic), let q be a quadratic form in $I^4(F)$ of dimension $24 = 2^4 + 2^3$. Then there exists a finite field extension K/F of odd degree such that q_K is isomorphic to the tensor product of an Albert bilinear form by a Pfister form.*

Proof. By Theorem 1, we can find a finite field extension K/F of odd degree and a field extension L/K of degree dividing 2 such that q_L is split. The description of q_K then follows from [6, Proposition 4.1] (for characteristic $\neq 2$) and [5, Proposition 4.3.2] (for characteristic 2). □

To formulate the result on the essential 2-dimension, let us consider the functor I^4_{24}, associating to every extension field K of a fixed field F the set of isomorphism classes of quadratic forms in $I^4(K)$ of dimension 24. The essential 2-dimension of an element in $I^4_{24}(K)$ as well as the essential 2-dimension $\operatorname{ed}_2 I^4_{24}$ of the functor I^4_{24} are defined as in [3, §1].

Corollary 3. *One has* $\operatorname{ed}_2 I^4_{24} \leq 7$.

Proof. Writing F for the base field and taking $q \in I^4_{24}(K)$ for a field extension K/F, we find by Theorem 2 an odd degree field extension L/K such that q_L is isomorphic to the tensor product of the diagonal Albert bilinear form $\langle a_1, a_2, a_3, a_4, a_5, -a_1a_2a_3a_4a_5\rangle$ by the Pfister form $\langle\langle b_1, b_2]]$ for some nonzero $a_1, \dots, a_5, b_1 \in L$ and some $b_2 \in L$, where in characteristic $\neq 2$ the element b_2 is also nonzero. The subfield $F(a_1, \dots, a_5, b_1, b_2) \subset L$, whose transcendence degree over F is at most 7, is then a *field of definition* of q_L. It follows that $\operatorname{ed}_2 q = \operatorname{ed}_2 q_L \leq 7$ and so $\operatorname{ed}_2 I^4_{24} \leq 7$. □

Recall that the essential 2-dimension is a 2-local version of and constitutes a lower bound for the *essential dimension*, measuring, informally speaking, how many independent parameters are required to describe an isomorphism class of the corresponding type of objects; in particular, $\operatorname{ed}_2 I^4_{24} \leq \operatorname{ed} I^4_{24}$. For $n = 3$, since the description of the corresponding quadratic forms does not involve odd degree extensions, similar to the proof of Corollary 3 arguments show that $\operatorname{ed}_2 I^3_{12} \leq \operatorname{ed} I^3_{12} \leq 6$. In fact, in characteristic $\neq 2$, $\operatorname{ed}_2 I^3_{12} = \operatorname{ed} I^3_{12} = 6$ by [3, Theorem 7.1]: the lower bound $6 \leq \operatorname{ed}_2 I^3_{12}$ is obtained by constructing a nontrivial degree 6 cohomological invariant with coefficients in $\mathbb{Z}/2\mathbb{Z}$ for I^3_{12}.

For $n \geq 4$, assuming [6, Conjecture 2], one gets

$$\operatorname{ed}_2 I^n_{2^n+2^{n-1}} \leq \operatorname{ed} I^n_{2^n+2^{n-1}} \leq n + 3.$$

Finally, one has $\operatorname{ed}_2 I^n_{2^n} = \operatorname{ed} I^n_{2^n} = n + 1$ for any n. Indeed, as already mentioned, any $q \in I^n_{2^n}$ is isomorphic to $b \cdot \langle\langle b_1, b_2, \dots, b_n]]$ for some $n+1$ parameters $b, b_1, \dots, b_n$, ensuring that $n + 1$ is an upper bound for $\operatorname{ed} I^n_{2^n}$. On the other hand, associating in characteristic $\neq 2$ to q the symbol $(b, b_1, \dots, b_n)$ in the $(n + 1)$st Galois cohomology group with coefficients in $\mathbb{Z}/2\mathbb{Z}$, one gets a nontrivial degree $n + 1$ cohomological invariant showing that $n + 1$ is a lower bound for $\operatorname{ed}_2 I^n_{2^n}$ (see [11, Theorem 3.4]). The non-triviality of the cohomological invariant is shown in [2, §3]. The characteristic 2 case is treated similarly using cohomological invariants with values in étale motivic cohomology groups (cf. [14, §3] and especially [14, Proof of Lemma 3.1]); the non-triviality of the cohomological invariant follows from [7].

To conclude, let us return to the case of arbitrary $n \geq 2$. Let X be the highest orthogonal grassmannian of an anisotropic quadratic form $q \in I^n$. If $\dim q = 2^n$, then $i(X) = 2$ and therefore the ring $\bar{\operatorname{CH}}(X)$ contains $2\operatorname{CH}(\bar{X})$. By [4, Corollary 88.6], $\bar{\operatorname{CH}}(X)$ also contains the elements $e_1, \dots, e_{2^{n-1}-2}$ – the generators of the ring $\operatorname{CH}(\bar{X})$ with exception of the very last one $e_{2^{n-1}-1}$. Since $i(X) \neq 1$, we conclude that $\bar{\operatorname{CH}}(X)$ is exactly the subring in $\operatorname{CH}(\bar{X})$ generated by $2\operatorname{CH}(\bar{X})$ and $e_1, \dots, e_{2^{n-1}-2}$ (cf. [4, Example 88.10]).

Now let us assume that $\dim q = 2^n + 2^{n-1}$, where $n \geq 3$. Then $\bar{\mathrm{CH}}(X) \supset 2\,\mathrm{CH}(\bar{X})$ by Theorem 1. Besides, it has been shown in the proof of Theorem 1 that $\bar{\mathrm{CH}}(X) \ni e_i$ for all i except $i = k := 2^{n-1} - 1$ and $i = l := 2^{n-1} + 2^{n-2} - 1$.

Theorem 4. *For any $n \geq 3$ and any anisotropic quadratic form q in I^n of dimension $2^n + 2^{n-1}$, the ring $\bar{\mathrm{CH}}(X)$ of its highest grassmannian X is generated by $2\,\mathrm{CH}(\bar{X})$ and all e_i with $i \notin \{k,\ l\}$.*

Proof. Since $\bar{\mathrm{CH}}(X) \supset 2\,\mathrm{CH}(\bar{X})$, it suffices to show that the ring $\bar{\mathrm{Ch}}(X)$ is generated by e_i with $i \notin \{k,\ l\}$, where $\mathrm{Ch}(X) := \mathrm{CH}(X)/2\,\mathrm{CH}(X)$ and $\bar{\mathrm{Ch}}(X) := \mathrm{Im}(\mathrm{Ch}(X) \to \mathrm{Ch}(\bar{X}))$. By [4, Theorem 87.7] (originally proved in [16]), it suffices to show that nor e_k neither e_l is in $\bar{\mathrm{Ch}}(X)$.

By [4, Corollary 82.3] once again, the anisotropic part of q over the function field of its quadric Y has dimension 2^n. By [4, Corollary 88.7], we conclude that $e_k \notin \bar{\mathrm{Ch}}(X)$.

Finally, let us assume that $e_l \in \bar{\mathrm{Ch}}(X)$ and seek for a contradiction. By [4, Theorem 90.3] (originally proved in [16]), the *canonical* 2*-dimension* of the variety X equals k and does not change when the base field is extended to the function field of Y. It follows by [10, Theorem 3.2] that a shift of the *upper Chow motive* $U(X)$ with coefficients $\mathbb{Z}/2\mathbb{Z}$ is a direct summand of the motive of Y. On the other hand, by [4, Lemma 82.4], the complete motivic decomposition of the quadric Y consists only of shifts of the upper motive $U(Y)$. Moreover, since the variety $X_{F(Y)}$ has no 0-cycle of odd degree, the motives $U(Y)$ and $U(X)$ are not isomorphic, see [9, Corollary 2.15]. The contradiction obtained proves Theorem 4. □

Regarding the motives of the varieties X and Y from the above proof, each of them decomposes in a finite direct sum of indecomposable motives; moreover, by [9, Corollary 2.6] , such a decomposition is unique in the usual sense. The upper motive $U(X)$ (resp., $U(Y)$) is defined as the summand with nontrivial Ch^0 (unique in any decomposition given). By [9, Corollary 2.15], the motives $U(X)$ and $U(Y)$ are isomorphic if and only if each of the two varieties $X_{F(Y)}$ and $Y_{F(X)}$ possesses a 0-cycle of odd degree.

Let us also recall that *canonical dimension* of a smooth projective variety X is the minimum of dimension of the image of a rational self-map $X \dashrightarrow X$, c.f. [8]. Canonical 2-dimension, which appeared in the above proof, is its 2-local version also providing a lower bound for it. By [4, Theorem 90.3], Theorem 4 implies

Corollary 5. *For any anisotropic q as in Theorem 4, the canonical* 2*-dimension of its highest grassmannian is equal to $k + l = 2^n + 2^{n-2} - 2$.* □

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