# ON SPIN(2023)-TORSORS

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Abstract. The torsion index of a spin group $\mathrm{Spin}(d)$, describing the splitting behaviour of generic $\mathrm{Spin}(d)$-torsor $E$, is a 2-power $2^t$ with the torsion exponent $t$ determined by B. Totaro in 2005. The *critical exponent* $i_t$ is responsible for partial splitting behaviour of $E$ and takes values inside the doubleton $\{t-1,\ t\}$. For all $d \leq 16$, the value of $i_t$ is known to be high. The very first case of the low value, obtained very recently, is $d = 17$. In the present work, we develop a new method which allows one to show that $i_t = t-1$ for most $d$. In particular, it is shown that $i_t$ is low for every $d = 2^r + 1$ with $r \geq 4$ as well as for $d = 2023$, playing the role of a "randomly chosen" high dimension. For $d = 2023$, using an extension of the new method (applicable to arbitrary $d$), several exponents beyond the critical one are also determined.

## Contents



## 1. Introduction

We use notation and terminology of [5]. Given a generic $d$-dimensional quadratic form $q$ (over a field) of trivial discriminant and Clifford invariant, where $d = 2n+1$ or $d = 2n+2$ for some $n \geq 1$, we write $t$ for the *torsion exponent* of the algebraic group $\mathrm{Spin}(d)$, depending only on $n$ and determined in [9, Theorem 0.1], and we are interested to determine the *critical exponent* $i_t$ of $q$, i.e., the integer such that $2^{i_t}$ is the index of the $t$th (orthogonal) grassmannian of $q$, where the index of a variety is the g.c.d. of degrees of its closed points.

By definition, the quadratic form $q$ is given by a generic torsor $E$ under the spin group $\mathrm{Spin}(d)$. The integer $2^t$ is the g.c.d. of finite extensions of the base field of $E$ trivializing $E$, or, equivalently, splitting $q$. The integer $2^{i_t}$ provides similar information on partial trivialization of $E$ and partial splitting of $q$, see [5] for details.

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The possible values of $i_t$ are $t$ and $t-1$. We say that the critical exponent is high if $i_t = t$. Otherwise we say that it is low which means that $q$ acquires Witt index $t$ over a finite base field extension of degree not divisible by $2^t$.

The critical exponent is high for any $d \le 16$. It has been shown in [5, Theorem 4.1 and Corollary 4.4] that the critical exponent is low for $d = 17$ and for $d = 18$. With the help of computer calculations, the same has been shown for $d = 19$ and $d = 20$ in [6]. No other cases of low critical exponent were known so far.

As follows from [7, Lemma 2.3], if the critical exponent is low for $d = 2n+1$, then it is also low for $d = 2n+2$. For this reason, below we are assuming that $d$ is odd.

As the main result of the present work, we develop in the next section (§2) a new method which allows one to show that the critical exponent if low for most $n$, the precise statement being Theorem 2.10 (with a minor addition given by Proposition 2.11). For $n \ge 10$ which are not covered by these two results, the critical exponent is yet to be determined.

Proposition 2.11 deals with $n = 16$ and is added in order to cover all values of $n$ given by 2-powers. It is proved by the method of [5], where the case of $n = 8$ is treated. The 2-power higher than 16 are covered by Theorem 2.10 which is proved by entirely different means.

In the last section (§3), we develop an extension of the new method of §2 which allows one to determine several exponents beyond the critical one. We illustrate the power of the extension applying it to a "randomly chosen" high dimension $d = 2023$.

## 2. The critical exponent

The following result, based on a joint effort of [2], [6], and [7], reduces the determination of the critical exponent to an "elementary" computation. Let us write $X$ for the highest grassmannian of a split $d$-dimensional quadratic form and let us write $Y$ for the complete flag variety of the tautological vector bundle $\mathcal{T}$ on $X$. By [4, Example 3.3.5], the $\mathrm{CH}(X)$-algebra $\mathrm{CH}(Y)$ is generated by the Chern classes $x_1, \dots, x_n$ of the line bundles given by the successive quotients of the tautological (rank 1 up to rank $n$) bundles on $Y$. Moreover, the elementary symmetric polynomials in $x_1, \dots, x_n$ are equal to the Chern classes of $\mathcal{T}$, and these are the defining relations of the $\mathrm{CH}(X)$-algebra $\mathrm{CH}(Y)$. By the results of [3, §86], originally obtained in [10], the ring $\mathrm{CH}(X)$ is generated by the elements $e_1, \dots, e_n$, satisfying the condition $2e_i = (-1)^i c_i(\mathcal{T})$ and subject to the relations

$$e_i^2 - 2e_{i-1}e_{i+1} + 2e_{i-2}e_{i+2} - \cdots + (-1)^{i-1} 2e_1 e_{2i-1} + (-1)^i e_{2i} = 0,$$

where $i = 1, \dots, n$ and where $e_i := 0$ for $i > n$. The condition on $2e_i$ determines $e_i$ because the additive group of $\mathrm{CH}(X)$ is free of torsion.

Let $C_Y \subset \mathrm{CH}(Y)$ be the subring generated by $x_1, \dots, x_n$. Let us consider the element

$$e := \prod_{I \subset \{t+1, \dots, n\}} (e_1 - \sum_{i \in I} x_i) \in \mathrm{CH}^{2^{n-t}}(Y). \tag{2.1}$$

By [7, Proposition 4.4], the element $c := 2e$ is in $C_Y$.

**Proposition 2.2.** *The critical exponent is low if and only if $e \notin C_Y$. Equivalently, the critical exponent is low if and only if the class of $c$ in $C_Y/2C_Y$ is nontrivial.*

*Proof.* Let $X_t$ be the $t$th grassmannian of the split $d$-dimensional quadratic form used in the definition of $X$ and $Y$. By Lemma 2.3, the pull-back homomorphism $\mathrm{CH}(X_t) \to \mathrm{CH}(Y)$ with respect to the projection $Y \to X_t$ is injective; we identify $\mathrm{CH}(X_t)$ with its image in $\mathrm{CH}(Y)$.

By [6, Proposition 3.4], the element $e$ belongs to $\mathrm{CH}(X_t)$. By Lemma 3.2, the intersection of $C_Y \cap \mathrm{CH}(X_t)$ coincides with the subring $C \subset \mathrm{CH}(X_t)$ generated by the Chern classes of the tautological (rank $t$) vector bundle on $X_t$. In particular, $c = 2e \in C$.

By [6, Theorem 3.6], the critical exponent satisfies (and is determined by) the formula $\deg(C[e]) = 2^{i_t}\mathbb{Z}$, where $C[e] \subset \mathrm{CH}(X_t)$ is the $C$-subalgebra generated by $e$, and where deg is the degree homomorphism $\mathrm{CH}(X_t) \to \mathbb{Z}$ given by the push-forward with respect to the structure morphism of the projective variety $X_t$. Note that $\deg(C) = 2^t\mathbb{Z}$ because the index of the $t$th grassmannian of a generic $d$-dimensional quadratic form (without restrictions on its discriminant and Clifford invariant), given by a generic torsors under the orthogonal group $\mathrm{O}(d)$, equals $2^t$ (see, e.g., [9, Theorem 3.2]).

In the case where $e \in C_Y$, we conclude that $e \in C$. This implies that $\deg(C[e]) = \deg(C) = 2^t\mathbb{Z}$ and therefore $i_t = t$.

Now assume that $e \notin C_Y$. Equivalently, the element $c = 2e \in C$ is nontrivial modulo $2C$. By duality in $C$, explained in [8, Theorem 1.1], there is an element $c' \in C$ such that $\deg(c \cdot c')$ is an odd multiple of $2^t$. Then $\deg(e \cdot c')$ is an odd multiple of $2^{t-1}$ and we conclude that $i_t = t - 1$. □

The following general statement, contained in the case of a Borel subgroup $Q$ in [2, Proof of Lemma 2.2], has been used in the above proof:

**Lemma 2.3.** *Let $Q \subset P \subset G$ be two parabolic subgroups of a split reductive group $G$ over a field $F$. The pull-back homomorphism $\pi^*\colon \mathrm{CH}(G/P) \to \mathrm{CH}(G/Q)$ with respect to the projection $\pi\colon G/Q \to G/P$ is a split monomorphism.*

*Proof.* By [1, Proposition 20.5], for any extension field $K/F$, the map $G(K) \to (G/P)(K)$ of the sets of $K$-points is surjective. Applying this property to the function field of the variety $G/P$, one sees that the $P$-torsor given by the generic fiber of the quotient map $G \to G/P$ is trivial. In particular, the generic fiber of $\pi$ has a rational point. The class $x \in \mathrm{CH}(G/Q)$ of its closure in $G/Q$ satisfies $\pi_*(x) = 1$. By projection formula, for any $y \in \mathrm{CH}(G/P)$ we have $\pi_*(\pi^*(y) \cdot x) = y \cdot \pi_*(x) = y$. It follows that $\pi^*$ is a split monomorphism. □

The following approach opens up a way to see that the critical exponent is low for most (in particular, for infinitely many) values of $d$ (see Theorem 2.10). Assume that $n \geq 3$ (assuring that $t \geq 1$) and set $d' := 2t + 1$. Let $X'$ be the highest grassmannian of a split $d'$-dimensional quadratic form, and let $Y'$ be the complete flag variety of the tautological (rank $t$) vector bundle $\mathcal{T}'$ on $X'$. We write $C_{Y'}$ for the subring in $\mathrm{CH}(Y')$ generated by the Chern classes of the tautological vector bundles on $Y'$. Note that $C_{Y'} \cap \mathrm{CH}(X')$ is the subring $C_{X'} \subset \mathrm{CH}(X')$, generated by the Chern classes $c_1(\mathcal{T}'), \dots, c_t(\mathcal{T}')$ of $\mathcal{T}'$. We write $e'_1$ for the element in $\mathrm{CH}(X')$ satisfying the condition $2e'_1 = -c_1(\mathcal{T}')$. Note once again that this condition determines $e'_1$ because the group $\mathrm{CH}(X')$ is free of torsion.

**Proposition 2.4.** *The critical exponent is low for dimension $d = 2n + 1$ provided that $(e'_1)^{2^{n-t}} \notin C_{X'}$, where $t$ is the torsion exponent of* $\mathrm{Spin}(d)$.

*Proof.* By Proposition 2.2, in order to prove Proposition 2.4, it suffices to show that the inclusion $e \in C_Y$ implies the inclusion $(e'_1)^{2^{n-t}} \in C_{Y'}$.

There is a (unique) ring homomorphism $\pi\colon \mathrm{CH}(Y) \to \mathrm{CH}(Y')$, mapping $x_i$ to $x'_i$ and $e_i$ to $e'_i$ for every $i = 1, \dots, t$ and killing both $x_i$ and $e_i$ for $i = t+1, \dots, n$. Since the generators $x_{t+1}, \dots, x_n$, involved in formula (2.1), vanish under $\pi$, we have $\pi(e) = \pi(e_1)^{2^{n-t}}$. Since $\pi(e_1) = e'_1$ and $\pi(C_Y) = C_{Y'}$, the result follows. □

A control on the condition of Proposition 2.4, required for its applications, is worked out in [9, §5]. As in [9, §5], let us define the degree of a subset in $\{1, \dots, n\}$ to be the sum of its elements. The following statement is actually proven in [9, §5] but is not explicitly formulated there. It will be generalized in Lemma 3.4 below. We write $C_X$ for the subring in $\mathrm{CH}(X)$, generated by the Chern classes of the tautological vector bundle $\mathcal{T}$ on $X$.

**Lemma 2.5** ([9, §5])**.** *For any given integer $\alpha \geq 0$, one has $e_1^{2^\alpha} \notin C_X$ if and only if there is a set $I \subset \{1, \dots, n\}$ of degree $2^\alpha$ that can be written as a disjoint union of subsets of order at most 2 and of degree a power of 2.*

*Proof.* Note that $2e_1 = -c_1(\mathcal{T}) \in C_X$ and the additive group of $C_X$ is a free abelian group of finite rank and, in particular, free of torsion. As explained in [9, §4], the element $(2e_1)^{2^\alpha}$ is divisible by $2^{2^\alpha - 1}$ in $C_X$. In other terms, the element $c := 2e_1^{2^\alpha} \in \mathrm{CH}(X)$ belongs to $C_X \subset \mathrm{CH}(X)$. We have $e_1^{2^\alpha} \notin C$ if and only if $c$ is nonzero modulo 2 in $C_X$. Below we view $c$ as an element of $C_X/2C_X$. Note that the quotient $C_X/2C_X$ is the exterior algebra on $c_i := c_i(\mathcal{T})$, $i = 1, \dots, n$, i.e., the generators $c_1, \dots, c_n$ are subject to the relations $c_i^2 = 0$. In particular, the products $c_I := \prod_{i \in I} c_i$ with $I \subset \{1, \dots, n\}$ form a basis of the $\mathbb{Z}/2\mathbb{Z}$-vector space $C_X/2C_X$. (The same products viewed in $C_X$ also form a basis of the free abelian group $C_X$.)

By [9, Lemma 5.1], for the list

$$c_1;\ c_2;\ c_4,\ c_1c_3;\ c_8,\ c_1c_7,\ c_2c_6,\ c_3c_5;\ \dots \tag{2.6}$$

of elements of $C_X/2C_X$ of the form $c_{2^j}$ with $j \geq 0$ or $c_{2^j - i}c_{2^j + i}$ with $1 \leq i \leq 2^j - 1$, the element $c$ is equal to the sum over all subsets $S$ of the list with total degree $2^\alpha$ of the product of the elements in $S$. Any monomial in this sum that involves the same generator $c_i$ twice is zero and so can be omitted. Otherwise, the monomial is $c_I$ for some set $I \subset \{1, \dots, n\}$ of degree $2^\alpha$ and the coefficient at this monomial in the decomposition of $c$ equals the number (modulo 2) of ways of writing $I$ as a disjoint union of subsets of order at most 2 and of degree a power of 2.

With this information in hand, Lemma 2.5 follows from Lemma 2.7 right below. □

**Lemma 2.7** ([9])**.** *For any subset $I \subset \{1, \dots, n\}$ (of any degree), the number of ways of decomposing $I$ into a disjoint union of subsets as above is always 0 or 1 (not just modulo 2); that is, if $I$ can be decomposed into such subsets, then the decomposition is unique.*

*Proof.* The statement of Lemma 2.7 and its proof appear inside [9, Proof of Lemma 5.4]. □

We are ready to apply Proposition 2.4. As a warm up, we prove

**Lemma 2.8.** *The critical exponent is low for $d = 2023$.*

*Proof.* For $d = 2023$, we have $n = 1011$, $t = 993$, and $n - t = 18$. By Lemma 2.5, $(e_1')^{2^{18}} \notin C_{X'}$ if and only if there is a set $I \subset \{1, \ldots, 993\}$ of degree $2^{18}$ that can be written as a disjoint union of subsets of order at most 2 and of degree a power of 2. The union $I$ of the doubletons $\{2^9 \pm i\}$ with $i = 1, \ldots, 2^8$ satisfies the condition. □

In fact, the critical exponent is low for all $d$ in a large interval around 2023:

**Proposition 2.9.** *The critical exponent is low for $d = 2n + 1$ provided that $786 \leq n \leq 1024$.*

*Proof.* For every such $d$, the difference $n - t$ is constantly 18. Therefore it suffices to show that the critical exponent is low in the case of the minimal $n = 786$. For this we need to find a set $I \subset \{1, \ldots, t = n - (n - t) = 786 - 18 = 768\}$ of degree $2^{n-t} = 2^{18}$ that can be written as a disjoint union of subsets of order at most 2 and of degree a power of 2. Since $768 = 2^9 + 2^8$, the union $I$ of the doubletons $\{2^9 \pm i\}$ with $i = 1, \ldots, 2^8$ (used in the proof of Lemma 2.8) still suits. □

Here comes the main result of this text, which will be proved similarly. It shows that for $N$ large enough, the proportion of $n < N$ such that the critical exponent is low for $d = 2n + 1$ is over 91%. Indeed, for $s \to \infty$, the proportion of $n \in [2^s,\ 2^{s+1}]$ for which the critical exponent is low by Theorem 2.10 tends to

$$1 - 2^{-s}(2^s + 2^{s-1} - 2^{s+\frac{1}{2}}) = \sqrt{2} - 2^{-1} > 0.91.$$

**Theorem 2.10.** *The critical exponent is low for $d = 2n+1$ (and therefore for $d = 2n+2$ as well, see §1) provided that*

$$n \in [2^s + 3s - 3,\ 2^{s+\frac{1}{2}} - 2s - 1] \cup [2^s + 2^{s-1} + 2s,\ 2^{s+1}]$$

*for some positive integer $s$.*

*Proof.* Assume first that $n \in [2^s + 3s - 3,\ 2^{s+\frac{1}{2}} - 2s - 1]$. Applying [9, Theorem 0.1], let us show that $n - t = 2s - 1$. If we were in the second case of [9, Theorem 0.1], then $n$ would have the form $n = 2^s + b$ with some $0 \leq b \leq s - 3$ implying that $n \leq 2^s + s - 3$, a contradiction. Therefore, by [9, Theorem 0.1], $n - t$ is the integral part of $\log_2(1 + n(n+1)/2)$ which is equal to $2s - 1$, indeed.

Since the difference $n - t$ is constant for $n$ on the interval $[2^s + 3s - 3,\ 2^{s+\frac{1}{2}} - 2s - 1]$, it suffices to show that the critical exponent is low in the case of the minimal $n = 2^s + 3s - 3$. For this we need to find a set $I \subset \{1, \ldots, 2^s + s - 2\}$ of degree $2^{2s-1}$ that can be written as a disjoint union of subsets of order at most 2 and of degree a power of 2. The union $I$ of the singleton $\{2^s\}$ and the doubletons $\{2^{s-1} \pm i\}$ with $i = 1, \ldots, 2^{s-1} - 1$ suits.

Now assume that $n \in [2^s + 2^{s-1} + 2s,\ 2^{s+1}]$. Then we are in the first case of [9, Theorem 0.1] which tells us that $n - t = 2s$. Therefore it suffices to show that the critical exponent is low in the case of the minimal $n = 2^s + 2^{s-1} + 2s$. For this we need to find a set $I \subset \{1, \ldots, 2^s + 2^{s-1}\}$ of degree $2^{2s}$ that can be written as a disjoint union of subsets of order at most 2 and of degree a power of 2. The union $I$ of the doubletons $\{2^s \pm i\}$ with $i = 1, \ldots, 2^{s-1}$ suits. □

Note that Theorem 2.10 in particular states the critical exponent is low for $d = 2n+1$ with $n$ any 2-power starting from 32. The same has been shown for $d = 17$ and $n = 8$ in [5]. We use the method of [5] to resolve the missing case of $d = 33$ and $n = 16$:

**Proposition 2.11.** *The critical exponent is low for $d = 2n+1$ with $n = 16$.*

*Proof.* We use notation from the paragraph before Proposition 2.4. Note that $t = 10$ for $d = 2n+1$ with $n = 16$.

The modulo 2 Chow group $\mathrm{Ch}(Y) := \mathrm{CH}(Y)/2\,\mathrm{CH}(Y)$ has a $\mathbb{Z}/2\mathbb{Z}$-basis given by the products

$$x_1^{a_1} \dots x_{16}^{a_{16}} e_I \quad \text{with } a_i < i \text{ and } I \subset \{1, \dots, 8\}, \tag{2.12}$$

where $e_I := \prod_{i \in I} e_i$. The $\mathbb{Z}/2\mathbb{Z}$-subspace $C_Y \subset \mathrm{Ch}(Y)$ is generated by the part of the basis without $e_I$ (i.e., with $I = \emptyset$).

By [5, Remark 3.4], $e$, as a polynomial in $e_1$ over the ring

$$R := (\mathbb{Z}/2\mathbb{Z})[x_{11}, \dots, x_{16}],$$

contains monomials of 2-power degrees only. For $e_1$ viewed in $\mathrm{Ch}(Y)$ one has $e_1^{2^i} = e_{2^i}$ for any $i \geq 0$, where we set $e_i := 0$ for $i > n$. Every element of $R$, viewed in $C_Y \subset \mathrm{Ch}(Y)$, can be written (uniquely) as a sum of $x_1^{a_1} \dots x_{16}^{a_{16}}$ with $a_i < i$. The element $e \in \mathrm{Ch}(Y)$ is a unique linear combination of $e_1, e_2, e_4, e_8, e_{16}$ with coefficients in $C_Y$. We prove $e \notin C_Y$ by showing that the coefficient at $e_{16}$ is nonzero.

By [5, Remark 3.4], the coefficient at $e_1^{16}$ in the *polynomial* $e$ is a sum of some monomials

$$x_{11}^{a_{11}} \dots x_{16}^{a_{16}} \tag{2.13}$$

with 2-powers $a_{11}, \dots, a_{16}$ satisfying $a_{11} + \dots + a_{16} = 64 - 16 = 48$. Since

$$x_{16}^{16} = c_1(\mathcal{T})x_{16}^{15} + c_2(\mathcal{T})x_{16}^{14} + \dots + c_{15}(\mathcal{T})x_{16} + c_{16}(\mathcal{T}) \in C_Y \subset \mathrm{Ch}(Y)$$

and $c_i(\mathcal{T}) = 0 \in \mathrm{Ch}(X) \subset \mathrm{Ch}(Y)$ for $i > 0$, the power $x_{16}^{16}$ vanishes in $C$. By symmetry of the relations on $x_1, \dots, x_{16}$ in $C_Y$, for every $i = 1, \dots, 15$, the power $x_i^{16}$ also vanishes in $C_Y$. It follows that among monomials (2.13) only the one with $a_{11} = \dots = a_{16} = 8$ remains nonzero in $C$. This monomial actually appears (with coefficient 1 modulo 2) and belongs to basis (2.12). Consequently $e \notin C_Y$, and [5, Proposition 3.2] terminates the proof. □

## 3. Beyond the critical exponent

Returning to $d = 2023$, we would also like to determine several exponents $i_{t+1},\ i_{t+2},\ \dots$ following the critical one $i_t$. Recall that for arbitrary $d$ the sequence of all exponents $i_0, \dots, i_n$ is non-strictly increasing with $i_m = m$ for $m \leq t-2$ and $i_{n-2} = i_{n-1} = i_n = t$. For every $m = 0, \dots, n$, the integer $i_m$ is defined to be such that $2^{i_m}$ is the index (i.e., the g.c.d. of degrees of closed points) of the $m$th grassmannian of a generic $d$-dimensional quadratic form of trivial discriminant and Clifford invariant.

Proposition 3.1 below is a generalization of Proposition 2.2 which is still based on a joint effort of [2], [6], and [7]. For an arbitrary dimension $d = 2n+1$ and the corresponding torsion exponent $t$, it reduces the computation of $i_m$ for arbitrary $m \geq t-1$ to an "elementary" computation. For any $m = t-1, t, t+1, \dots, n$, let us define

$$e := \prod_{I \subset \{m+1,\dots,n\}} (e_1 - \sum_{i\in I} x_i) \in \mathrm{CH}^{2^{n-t}}(Y).$$

(For $m = t$, this formula coincides with (2.1). For $m < t-1$ the formula makes sense as well, but by [7, Corollary A.3] the result is always 0.) By [7, Corollary 4.6], for any integer $\alpha \geq 0$, the element $2e^{2^\alpha}$ is in $C_Y$. As a consequence, $c := 2^{m-t+1}e^{2^{m-t+1}-1} \in C_Y$.

**Proposition 3.1.** *For a given $m \geq t$, one has $i_m = t-1$ is and only if the element $c$ is nontrivial modulo $2C_Y$; otherwise $i_m = t$. For $m = t-1$, one has has $i_m = m-1 = t-2$ is and only if $e \notin C_Y$; otherwise $i_m = m = t-1$.*

*Proof.* We modify the lines of the proof of Proposition 2.2. Let $X_m$ be the $mt$th grassmannian of the split $d$-dimensional quadratic form used in the definition of $X$ and $Y$. By Lemma 2.3, the pull-back homomorphism $\mathrm{CH}(X_m) \to \mathrm{CH}(Y)$ with respect to the projection $Y \to X_m$ is injective; we identify $\mathrm{CH}(X_m)$ with its image in $\mathrm{CH}(Y)$.

By [6, Proposition 3.4], the element $e$ belongs to $\mathrm{CH}(X_m)$. We will proceed with the proof of Proposition 3.1 after the following

**Lemma 3.2.** *The intersection of $C_Y \cap \mathrm{CH}(X_m)$ coincides with the subring $C \subset \mathrm{CH}(X_m)$ generated by the Chern classes of the tautological (rank $m$) vector bundle on $X_m$.*

*Proof.* Since $C \subset C_Y$, the inclusion $C \subset C_Y \cap \mathrm{CH}(X_m)$ holds trivially. To prove the opposite inclusion, we proceed as follows.

The rings $C$ and $C_Y$ are identified with the Chow rings of the following two varieties: the variety of $m$-dimensional totally isotropic subspaces and the variety of complete flags of totally isotropic subspaces of a $(2n)$-dimensional non-degenerate alternating bilinear form (see [6, Remark 3.3]). Under this identification, the embedding $C \hookrightarrow C_Y$ becomes the pull-nack homomorphism of Lemma 2.3 with $G$ being the split symplectic group $\mathrm{Sp}(2n)$. It follows by Lemma 2.3 that the embedding is a split monomorphism. Therefore the quotient $C_Y/C$ is free of torsion.

Let us take any $a \in C_Y \cap \mathrm{CH}(X_m)$. Since $a \in \mathrm{CH}(X_m)$, by [11, Propositions 2.11 and 2.1] there exists a nonzero integer $r$ such that $ra \in C$. Since $a \in C_Y$ and $C_Y/C$ is free of torsion, we conclude that $a \in C$. □

Returning to the proof of Proposition 3.1, since $2e \in C_Y$, we conclude by Lemma 3.2 that $2e \in C$.

By [6, Theorem 3.6], the exponent $i_m$ satisfies (and is determined by) the formula $\deg(C[e]) = 2^{i_m}\mathbb{Z}$, where $C[e] \subset \mathrm{CH}(X_m)$ is the $C$-subalgebra generated by $e$, and where deg is the degree homomorphism $\mathrm{CH}(X_m) \to \mathbb{Z}$ given by the push-forward with respect to the structure morphism of the projective variety $X_m$. Note that $\deg(C) = 2^m\mathbb{Z}$ because the index of the $m$th grassmannian of a generic $d$-dimensional quadratic form (without restrictions on its discriminant and Clifford invariant), given by a generic torsors under the orthogonal group $\mathrm{O}(d)$, equals $2^m$ (see, e.g., [9, Theorem 3.2]).

Let us first treat the case of $m = t-1$. If $e \in C_Y$, we conclude that $e \in C$, $\deg(C[e]) = \deg(C) = 2^m\mathbb{Z}$, and $i_m = m = t-1$. Otherwise, the element $2e$ of $C$ is nontrivial modulo $2C$ and by duality in $C$, explained in [8], there is an element $c' \in C$ such that $\deg(c \cdot c')$ is an odd multiple of $2^m$. Then $\deg(e\cdot c')$ is an odd multiple of $2^{m-1}$ and we conclude that $i_m = m-1 = t-2$.

Now we treat the case of $m \geq t$, starting with the assumption that $c \notin 2C_Y$, which is equivalent to the assumption that $c \notin 2C$. Then again we can find an element $c' \in C$ such that $\deg(c\cdot c')$ is an odd multiple of $2^m$. It follows that $\deg(e^{2^{m-t+1}-1}\cdot c')$ is an odd multiple of $2^{t-1}$ and we conclude that $i_m = t-1$.

Finally, assume that $c \in 2C$. Note that for any integer $l \geq 1$, $2^b e^l \in C$, where $b$ is the sum of base-2 digits of $l$. Since $b \leq m-t$ for $l < 2^{m-t+1}-1$, we conclude that $2^{m-t}C[e] \subset C$. Therefore $i_m \geq m-(m-t) = t$ meaning that $i_m = t$. □

Since, as we already know, $i_t = t-1 = 992$ for $d = 2023$, the possible values of the exponents beyond $i_t$ for this dimension are $t-1 = 992$ and $t = 993$. We are not able to determine the largest $m$ with $i_m = t-1$. The upper bound on such $m$ for general $d$, resulting from Proposition 3.1 by the reason of dimension of the variety $X_m$, is not exact as demonstrates the following example:

**Example 3.3.** The upper bound, resulting from [7, Theorem 3.2], is given by the maximal $m$ such that
$$\dim X_m = \frac{m(m-1)}{2} + m(d-2m) \geq 2^{n-t+1} - 2^{n-m}.$$
For $d = 31$ and $n = 15$ (for which $t = 9$), this upper bound on the largest $m$ with $i_m = t-1$ is 14. However, since we always have $t = i_n = i_{n-1} = i_{n-2}$, the actual value of $m$ is at most $n-3 = 12$.

For $d = 2023$, the upper bound on the largest $m$ with $i_m = t-1$, resulting from Proposition 3.1, is 1000. Therefore $i_m = t$ for all $m \geq 1001$. We are going to generalize the technique used in the proof of Lemma 2.8 to show that $i_m = t-1$ for $m$ in the closed interval $[t = 993,\ 996]$. (The value of $i_m$ with $m$ from 997 to 1000 remains undetermined.)

We first extend Lemma 2.5 on the power $e_1^{2^\alpha}$ to an arbitrary power $e_1^l$ of $e_1$. Let us write a given integer $l \geq 1$ as a sum $2^{\alpha_1} + \cdots + 2^{\alpha_b}$ of $b$ distinct 2-powers for some appropriate $b \geq 1$ (equal to the sum of the base-2 digits of $l$). Then, clearly, $c := 2^b e_1^l \in C_X$, where, as in the proof of Lemma 2.5, $C_X$ stands for the subring in $\mathrm{CH}(X)$, generated by Chern classes of the tautological vector bundle $\mathcal{T}$. The following lemma, generalizing Lemma 2.5 as well as [9, Lemma 5.4], controls vanishing of $c$ in $C_X/2C_X$:

**Lemma 3.4.** *For $c \in C_X$ as right above, one has $c \notin 2C_X$ if and only if there is a set $I \subset \{1,\dots,n\}$ of degree $l$ that can be written as a disjoint union of subsets of order at most 2 and of degree a power of 2.*

*Proof.* We rephrase the proof of [9, Lemma 5.3] in order to show that the the class of $c$ in the quotient $C_X/2C_X$ is equal to the sum over all subsets $S$ of list (2.6) with total degree $l$ of the product of the elements in $S$. This follows from [9, Lemma 5.1] once we show that for each subset $S$ of list (2.6) with total degree $l$, the number of ways of partitioning $S$ into subsets with total degrees $2^{\alpha_1},\dots,2^{\alpha_b}$ is odd. Clearly, this question depends only on the degrees of the elements of $S$, which are all powers of 2; that is, it suffices to show that for any nonnegative integers $a_1,\dots,a_r$ such that $2^{a_1}+\cdots+2^{a_r} = l$, the number of partitions of the set $S = \{1,\dots,r\}$ into subsets $S = \coprod_{j=1}^b S_j$ such that $\sum_{i\in S_j} 2^{a_i} = 2^{\alpha_j}$ for $j = 1,\dots,b$ is odd.

By [9, Lemma 5.2], the number of subsets $S_b$ such that $\sum_{i\in S_b} 2^{a_i} = 2^{\alpha_b}$ is congruent modulo 2 to $\binom{l}{2^{\alpha_b}}$ and thus to 1 (see [3, Lemma 78.6]). The total number of partitions as above is the product of this odd number of subsets $S_b$ with the analogous number of partitions of $S \setminus S_b$ (for any choice of $S_b$), a set with total degree $l - 2^{\alpha_b}$ rather than $l$. By induction on $b$, the latter number of partitions is odd. Therefore the number of partitions we consider is also odd.

By Lemma 2.7, for any subset $I \subset \{1, \ldots, n\}$ (of any degree), the number of ways of decomposing $I$ into a disjoint union of subsets as above is always 0 or 1. This proves Lemma 3.4. □

We now get an extension of Lemma 2.8:

**Proposition 3.5.** *For $d = 2023$ (where $n = 1011$ and $t = 993$) we have $i_m = t - 1$ for $m = 994,\ 995,\ 996$.*

*Proof.* We start with a general conclusion concerning an arbitrary $d = 2n + 1$ and the corresponding torsion exponent $t$, generalizing Proposition 2.4. By Proposition 3.1, for a given $m \geq t$ one has $i_m = t - 1$ is and only if the element

$$c := 2^{m-t+1} e^{2^{m-t+1}-1} \in C_Y \subset \mathrm{CH}(Y)$$

is nontrivial modulo $2C_Y$. Setting to 0 the generators $x_{m+1}, \ldots, x_n$ as well as $e_{m+1}, \ldots, e_n$ (like in the proof of Proposition 2.4) of the ring $\mathrm{CH}(Y)$ (and "keeping unchanged" the remaining generators), we transform $e$ to $(e'_1)^{2^{n-m}}$ and therefore $c$ to

$$c' := 2^{m-t+1} (e'_1)^{2^{n-t+1}-2^{n-m}} \in C_{X'} \subset \mathrm{CH}(X'),$$

where the variety $X'$ is the highest grassmannian of a split quadratic form of dimension $d' := 2m+1$. We conclude that $i_m = t-1$ provided that $c' \notin 2C_{X'}$. The condition we came to is controlled by Lemma 3.4 and is satisfied if and only if there is a set $I \subset \{1, \ldots, m\}$ of degree

$$l := 2^{n-t+1} - 2^{n-m} = 2^{n-t} + 2^{n-t-1} + \cdots + 2^{n-m}$$

that can be written as a disjoint union of subsets of order at most 2 and of degree a power of 2.

Returning to $d = 2023$, to prove Proposition 3.5, it suffices to show that $i_{996} = 992$. So, we set $m = 996$. Recall that $t = 993$ for this $d$. Note that the sum

$$2^9 + 2^8 + 2^7 + 2^6 + 2^5 = 992$$

does not exceed $m$. The union $I$ of the doubletons $\{2^9 \pm i\}$ with $i$ running from 1 to $2^8 + 2^7 + 2^6 + 2^5$ satisfies the above condition. □

**Remark 3.6.** For any $d$ and the corresponding $t$, the method of the proof of Proposition 3.5 does not allow one to determine the avant-critical exponent $i_{t-1}$ because $(e_1)^{2^{n-t+1}}$ (and therefore $(e'_1)^{2^{n-t+1}}$) turns out to vanish. However, as already mentioned in [7, §3], $i_{t-1} = t - 1$ for, asymptotically, 100% of dimensions $d$. This follows from Proposition 3.1 by the reason that $\dim(X_{t-1}) < 2^{n-t+1}$ for the majority of $d$, see [7, Proposition A.4].

By a fatal coincidence, $d = 2023$ falls into the 0% and the avant-critical exponent for it remains undetermined.

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